Principle and application of Quantum Technology
Homework 3 Due: May 9, 2024

Ex 1 10% Consider a two-state system with the Hamiltonian given by $H = E_1 |\,1\rangle\,\langle 1\,| + E_2 |\,2\rangle\,\langle 2\,| + \gamma e^{i\omega t}\,|\,1\rangle\,\langle 2\,| + \gamma e^{-i\omega t}\,|\,2\rangle\,\langle 1\,|$. Suppose that initially, the system is in $|\,1\rangle$, find the probability of finding it in the state of $|\,2\rangle$ at a later time t.

Ex. 2

(a) 5% Given any two vectors, $\mathbf{a}$ and $\mathbf{b}$, show that

$$(\sigma\cdot\mathbf{a})\,(\sigma\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} + i\sigma\cdot(\mathbf{a}\times\mathbf{b})\,.$$

(b) 5 % Suppose two spin-1/2 particles are known to be in the spin-singlet state. Let $S_a^{(1)}$ be the component of spin for one of the particles along $\hat{\mathbf{a}}$ direction and $S_b^{(2)}$ be the component of spin for the remaining particle along $\hat{\mathbf{b}}$ direction. Show that

$$\langle\, S_a^{(1)} S_b^{(2)}\,\rangle = -\frac{\hbar^2}{4}\,\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}\;.$$

Ex.3 10% Page 97, problem 4-1.

Ex.4 10% Page 98, problem 4-3.

Ex.5 Wigner function

In Quantum mechanics, one can not define the probability density $\mathrm{P}(\mathbf{P},\mathbf{R})$ for finding particles with position $\mathbf{R}$ and momentum $\mathbf{P}$. There is, however, a quantum analogy of the classical distribution function $\mathrm{P}(\mathbf{P},\mathbf{R})$. This is known as the Wigner function, which is defined as the transformation of the density operator ρ by

$$\mathrm{D}(\mathbf{P},\mathbf{R}) \equiv \frac{1}{(2\pi\hbar)^3}\int d^3\mathbf{r}\, e^{-i\mathbf{P}\cdot\mathbf{r}/\hbar}\langle\mathbf{R}-\mathbf{r}/2|\rho|\mathbf{R}+\mathbf{r}/2\rangle.$$

However, unlike the classical distribution function, the function $\mathrm{D}(\mathbf{P},\mathbf{R})$ is not positive definite and may become negative. To understand it, let us consider a pure state such that $\rho = |\psi\rangle\langle\psi|$ for the following questions:

(a) 5% Show that

$$\begin{aligned}\int d^3\mathbf{P}\,\mathrm{D}(\mathbf{P},\mathbf{R}) &= |\psi(\mathbf{R})|^2,\\ \int d^3\mathbf{R}\,\mathrm{D}(\mathbf{P},\mathbf{R}) &= |\phi(\mathbf{P})|^2,\end{aligned}$$

where $\phi(\mathbf{P}) \equiv \int d^3\mathbf{R}/(\mathbf{2}\pi\hbar)^{3/2}\, \psi(\mathbf{R}) \exp(-i\mathbf{P}\cdot\mathbf{R}/\hbar)$. In other words, just as $\mathrm{P}(\mathbf{P},\mathbf{R})$ does, appropriate integrations of $\mathrm{D}(\mathbf{P},\mathbf{R})$ can reproduce probability densities both in $\mathbf{P}$ and $\mathbf{R}$ spaces.

(b) 5% Show that if the particle is free, the way that $\mathrm{D}(\mathbf{P},\mathbf{R})$ evolves is the same as that of the classical distribution function $\mathrm{P}(\mathbf{P},\mathbf{R})$.

Ex 6 10% Quantum state tomography of qubit

Quantum state tomography is the process by which a quantum state is reconstructed by using measurements on an ensemble of identical quantum states. Specifically, as the quantum state can be represented by the density operator, constructing the density operator is equivalent to do the quantum state tomography. Take the qubit as example, the density operatior ρ can be generally expressed as $\rho = \frac{1}{2}(1 + a\sigma_x + b\sigma_y + c\sigma_z)$. In most experiments, what we can extract from the measured signal about the quantum state of a qubit is the population in σ_z. Devise the experimental procedure for quantum state tomography of the qubit.

Ex.7 10% Page 157, problem 7-5.

Ex.8 Generation of entangled photons

The propagation of light in a medium depends on the electromagnetic properties of the medium. In a nonconducting and non-magnetic dielectric materials, the electromagnetic field of the light induces oscillating electric dipoles for bound charges in the medium, which in turn generate secondary waves. The secondary wave, when combined with the original electric field $\vec{E}$, yields the electric displacement $\vec{D}$ that accounts for effects of free and bound charge within the medium. For homogeneous and isotropic dielectric material, we have $\vec{D} = \epsilon\vec{E}$ with ϵ being the permittivity of the material and $\vec{D} = \epsilon_0\vec{E} + \vec{P}$, where ϵ_0 is the vacuum permittivity and $\vec{P}$ is the induced polarization density such that $\vec{P} = (\epsilon - \epsilon_0)\vec{E}$. In general medium, the propagation of light may become anisotropic. Assuming that the medium is the same as its being specified in the beginning except for being electrically anisotropic along one direction, the electric displacement $\vec{D}$ is related to the electric field $\vec{E}$ by $D_x = \epsilon_o E_x$, $D_y = \epsilon_o E_y$, and $D_z = \epsilon_e E_z$, where x, y and z are three mutually-orthogonal principal axes. As a result, the corresponding refraction indices for light ray with $\vec{D}$ lying in xy plane and being along z direction are different and are denoted by $n_o = \sqrt{\mu_0\epsilon_o}$ and $n_e = \sqrt{\mu_0\epsilon_e}$ respectively. The material is known as the uniaxial medium.

(a) In an uniaxial medium, the propagation of the wave (specified by $\vec{k}$) may explicitly depend on the direction of $\vec{k}$.

(i) 5% For simplicity, we shall first focus on the propagation in the xz plane. For a given propagation direction of light wave, $\vec{k} = k(\sin\theta, 0, \cos\theta)$, find possible values of refraction indices and the direction (termed as the optical axis) along which there is only one value of refraction index.

(ii) 5% The polarization of the light, i.e., the direction for the electric field,can be classified as either being perpendicular to the plane (termed as the ordinary light ray) or being in the same plane formed by $\vec{k}$ and the optical axis (termed as extraordinary light ray). Find the the corresponding polarization of the light for different refraction indices and indicate which kind of the light ray it belongs to. Compute $\tan\alpha$, where α is the angle between $\vec{E}$ and $\vec{D}$.

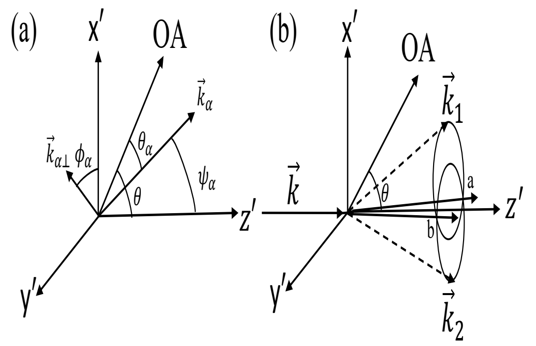

FIG. 1: Propagation of light from A to B through an interface between an isotropic medium 1 and an anisotropic medium 2.

(b) 10% One of usage for optically anisotropic material is to generate entangled photon pairs. To generate entangled photons, we consider a nonlinear medium. In this medium, in addition to the usual linear dependence of the polarization density on the electric field, $\vec{P} = (\epsilon - \epsilon_0)\vec{E}$, there is a nonlinear contribution to the polarization given by $P_i^{NL} = \sum_j \sum_k \chi_{ijk}^{(2)} E_j E_k$. Here i, j and k are the component indices (x, y or z) for $\vec{P}$ and $\vec{E}$, and $\chi_{ijk}^{(2)}$ is the second order nonlinear susceptibility of the medium and is a constant. The presence of non-vanishing $\chi_{ijk}^{(2)}$ implies that during the propagation of a light wave, there is a certain probability that the light wave can split into two light waves. In the following, we shall investigate entanglement of the two splitted light waves.

Consider the splitting of a light wave with angular frequency ω and wavevector $\vec{k}$ into two light waves with angular frequencies being ω_1, ω_2 and corresponding wavevectors being $\vec{k}_1$, $\vec{k}_2$ respectively. Assuming that $\omega, \omega_1 \geq \omega_2$, find all possible relations between that these frequencies and wavevectors. These relations are known as the phase matching conditions. By viewing light as composed by photons, explain the meaning of these relations at the level of individual photon. What are the corresponding relations when a photon with frequency ω and wavevector $\vec{k}$ is split into two photons with frequencies being ω_1, ω_2 and corresponding wavevectors being $\vec{k}_1$, $\vec{k}_2$ respectively?

(c) 10% Consider splitting of light waves in an uniaxial medium. If we denote the ordinary light ray by **o** and the extraordinary light ray by **e**. There are 8 possible ways for a light wave to split: $\mathbf{o} \to \mathbf{o}+\mathbf{o}$, $\mathbf{o} \to \mathbf{e}+\mathbf{o}$, $\mathbf{o} \to \mathbf{o}+\mathbf{e}$, $\mathbf{o} \to \mathbf{e}+\mathbf{e}$, $\mathbf{e} \to \mathbf{o}+\mathbf{o}$, $\mathbf{e} \to \mathbf{e}+\mathbf{o}$, $\mathbf{e} \to \mathbf{o}+\mathbf{e}$, and $\mathbf{e} \to \mathbf{e}+\mathbf{e}$. Assuming that both the refraction indices n_o and n_e increase as the frequency increases and considering the collinear case when $\vec{k}$, $\vec{k}_1$, and $\vec{k}_2$ are along the same direction, indicate which splittings are not possible.

(d) 15% pt In general, the phase matching conditions need to be solved numerically. Suppose that for $n_e < n_o$, the collinear phase-matching conditions for $\mathbf{e} \to \mathbf{e}+\mathbf{o}$ are realized for an uniaxial medium with $k = K_p$, $k_1 = K_e$, $k_2 = K_o$, $\omega = \Omega_p$, $\omega_1 = \Omega_e$, and $\omega_2 = \Omega_o$. Here the directions of wavevectors $\vec{k}$, $\vec{k}_1$ and $\vec{k}_2$ are all along the z' direction as shown in Fig. 1(a). x' and y' are the corresponding coordinate systems associated with the z' axis, the angle between the optical axis (OA) and z' is θ, and OA lies in x'-z' plane. Using the same direction of incident $\vec{k}$ and $\omega = \Omega_p$, the non-collinear splitting of the **e** light ray into $\mathbf{e}+\mathbf{o}$ can form two cones as shown in Fig. 1(b). Here $\omega_1 = \omega_2 = \Omega$, $k_1 = k_2$, and the refraction index for the **e** light ray is denoted generally as $n_e(\omega, \theta)$. In the plane that is perpendicular to $\vec{k}$ (incident **e** light ray), two circles representing $\vec{k}_1$ (**e** light ray) and $\vec{k}_2$ (**o** light ray) intersect at points a and b with line $\overline{ab}$ being in parallel to y'-axis. Assuming that all relevant wavevectors $\vec{k}_\alpha$, specified by θ_α and ϕ_α indicated in Fig. 1(a), are close to z'-axis, one can treat $|(\Omega - \Omega_e)/\Omega_e| \ll 1$, the perpendicular components $|\vec{k}_{\alpha,\perp}|/k_\alpha \ll 1$ and the difference of the angle $|\theta_\alpha - \theta| \ll 1$. By considering the z'-component of $\vec{k}_\alpha$ and the angle θ_α to the second order, $O(k_{\alpha,\perp}^2)$ and $O((\theta_\alpha - \theta)^2)$, one finds that the perpendicular components $\vec{k}_{2,\perp} = (q'_x, q'_y)$ satisfy $D(q'_x - E)^2 + Dq'^2_y = F$. Evaluate D, E, and F in terms of Ω, Ω_e, Ω_o, K_e, K_o, $N_e(\omega, \theta) = \frac{1}{n_e(\Omega_e,\theta)}\frac{dn_e(\Omega_e,\theta)}{d\theta}$ and the group velocities $u_o = \frac{d\omega_2}{dk_2}$ and $u_e = \frac{d\omega_1}{dk_1}$

for the **o** and **e** light rays. Estimate the angle between the axis of the cone and $\vec{k}$ and the angle of the cone in terms of D, E, F, and K_e.

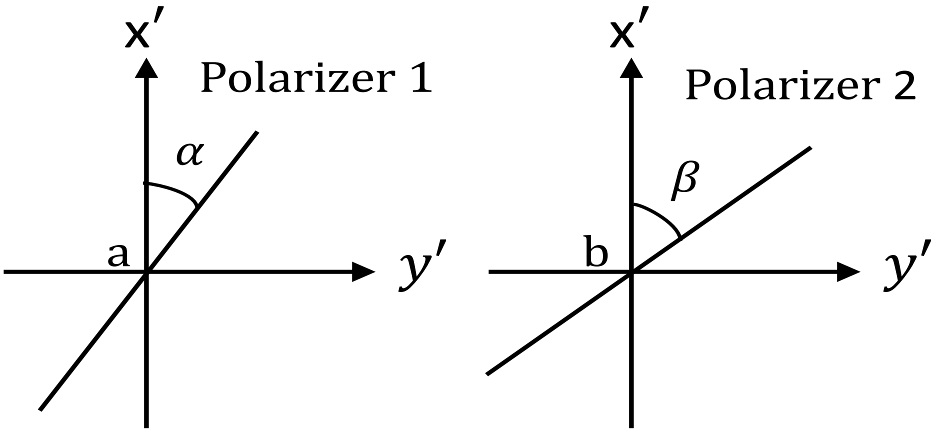

FIG. 2: (a) Orientation of a general $\vec{k}_\alpha$ relative to the optical axis and the coordinate system x', y', and z'. Here $\vec{k}_{\alpha\perp}$ is the projection of $\vec{k}_\alpha$ is x'-y' plane. (b) Non-collinear splitting of the **e** light ray into $\mathbf{e} + \mathbf{o}$ that form two cones. Here $\overline{ab}$ is in parallel to y'-axis.

(d) 20 % Following problem (c), two split photons come out in directions that pass points a and b are polarized in either $\hat{x}'$ or $\hat{y}'$ directions. These two photons are the so-called entangled photon pair such that when one photon that passes a (called a-photon) is polarized in $\hat{x}'$, the other one that passes b (called b-photon) will be polarized in $\hat{y}'$; or when a-photon is polarized in $\hat{y}'$, b-photon will be polarized in $\hat{x}'$. Experimentally, the entangled photon-pair state can be prepared such that it is a superposition of the above two alternatives and can be represented as $\frac{1}{\sqrt{2}}(|\hat{x}'_a\rangle|\hat{y}'_b\rangle + |\hat{y}'_a\rangle|\hat{x}'_b\rangle)$, where $|\hat{x}'_a\rangle|\hat{y}'_b\rangle$ represents the state when a-photon is polarized in $\hat{x}'$ direction, b-photon is polarized in $\hat{y}'$ direction; similar meaning applies to $|\hat{y}'_a\rangle|\hat{x}'_b\rangle$. As shown in Fig. 2, suppose that we use two linear polarizers 1 and 2 that make angles α and β with respect to $\hat{x}'$ to perform the coincident measurement of the two photons that pass a and b respectively. Let the probability for coincident finding a-photon and b-photon by polarizers 1 and 2 be $P(\alpha, \beta)$, which is proportional to the product of light intensities measured by polarizers 1 and 2. Let us denote $\alpha + \pi/2$ and $\beta + \pi/2$ by $\alpha_\perp$ and $\beta_\perp$ respectively.

(i) By considering the total electric field projected by linear polarizers, find the probabilities $P(\alpha, \beta)$, $P(\alpha, \beta_\perp)$, $P(\alpha_\perp, \beta)$, and $P(\alpha_\perp, \beta_\perp)$.

(ii) Let us assign $\sigma_\alpha = 1$ when the polarizer 1 with the angle being α finds an a-photon and assign $\sigma_\alpha = -1$ when the polarizer 1 with the angle being $\alpha_\perp$ finds an a-photon. Similarly, $\sigma_\beta = 1$ or -1 is assigned when the polarizer 2 with the angle being β or $\beta_\perp$ finds a b-photon. If $E(\alpha, \beta)$ denotes the average of $\sigma_\alpha \sigma_\beta$, the quantity, $S \equiv |E(\alpha, \beta) - E(\alpha, \beta')| + |E(\alpha', \beta) + E(\alpha', \beta')|$, has an important meaning. For classical theories of light, $S \leq 2$. This is known as a variant form of the Bell's inequality [the Clauser, Horne, Shimony and Holt (CHSH) inequality]. Evaluate S for a special case when $\alpha = \frac{\pi}{4}$, $\alpha' = 0$, $\beta = -\frac{\pi}{8}$, $\beta' = \frac{\pi}{8}$.