Principle and application of Quantum Technology
Homework 2 Due: April 18, 2024

Ex.1 10%

Given a density operator $\hat{\rho}$, it may represent many different ensembles. As an example, consider $\hat{\rho} = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$. $\hat{\rho}$ can be re-written as $\hat{\rho} = \frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|b\rangle\langle b|$ by suitable linear superpositions of $|0\rangle$ and $|1\rangle$ to form $|a\rangle$ and $|b\rangle$. Find $|a\rangle$ and $|b\rangle$ in terms of suitable linear superpositions of $|0\rangle$ and $|1\rangle$.

Ex. 2 10% The Hadamard gate can be written as $H = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$. Consider the n-qubit state $\psi_0\rangle \equiv |000\cdots 0\rangle$. Show that $H \otimes H \otimes \cdots \otimes H|\psi_0\rangle = \frac{1}{\sqrt{2^n}}\sum_x |x\rangle$, where x is any integer between 0 and $2^n - 1$ expressed in binary representation.

Ex.3 10% Page 99, problem 4-5.

Ex.4 20% Consider a two-state system characterized by the Hamiltonian

$$\mathbf{H} = \Delta\left[2|1><1| - |2><2| + 2(|1><2| + |2><1|)\right],$$

where Δ is a real number with the dimension of energy. $|1>$ and $|2>$ are normalized and are also orthogonal to each other. Consider a collection of such two-state systems. Suppose at $t = 0$, the percentage of systems in $|1\rangle$ is p_0, the remaining systems are all in the state of $|2\rangle$. Find the density matrix $\rho(t)$ for $t > 0$.

Ex 5 Consider a harmonic oscillator characterized by the Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$.

(a) 10% For the nth eigenstate $|n\rangle$, find $\langle n|x^4|n\rangle$.

(b) 10% Let $|0\rangle$ be the ground state. Show that

$$\langle 0|e^{ikx}|0\rangle = \exp\left(-k^2\langle 0|x^2|0\rangle/2\right).$$