2012

Rudiments of Conformal field theory

* The conformal group

As we have shown that at the critical point ($T = T_c$), the system is scale invariant in the large length scale. So that a number of scaling laws follow.

Specifically, the infra-red form - is invariant under the scale transformation

$$x^\mu \to \lambda x^\mu \qquad \mu = 1, 2, 3. \quad ①$$

provided the field that describes the order parameter

$$\phi \to \lambda^\Delta \phi \quad \cdots \quad ②$$

The scale transformation is global.

Similar to the global U(1) phase invariance in the Gizburg-Landau theory:

$$F = \alpha |\psi(\vec{r})|^2 + \frac{1}{2m}|D\psi|^2 + \frac{\beta}{2}|\psi|^4$$

$\psi(\vec{r}) \to \psi(\vec{r})e^{i\phi}$ $\quad$ F is invariant,

there may exist larger invariant group.

In the case of the Gizburg-Landau theory, the global U(1) invariance is enlarged by noting that ψ couples to the vector potential due to its charge:

$$F \Rightarrow \alpha |\psi(\vec{r})|^2 + \frac{1}{2m} \left| \left(\frac{P}{i} - \frac{q}{c} A \right) \psi \right|^2 + \frac{\beta}{2} |\psi|^4$$

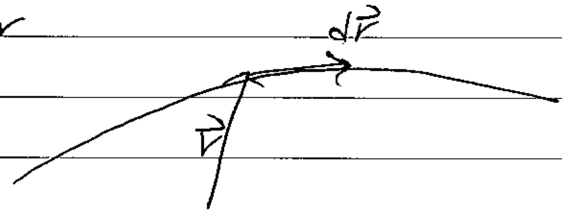
As a result, global U(1) invariance becomes local U(1) / gauge invariance:

$$\psi(r) \to \psi(r)\, e^{i\phi(r)}$$
$$\vec{A} \to A + \frac{q}{c} \nabla \phi(\vec{r}) \quad \cdots (3)$$

By the same reasoning, one expects that the global scale invariance of the critical point can be enlarged to be "local scale invariance". This was first considered by Polyakov in 1970. The enlarged group is the conformal group. It turns out that by imposing the conformal symmetry, on the microscopic models, various models can be ~~further~~ classified completely in 2D.

The conformal invariance is a local symmetry and can be best defined via the metric tensor:

$$\underbrace{ds^2}_{(d\vec{r})^2 \,=\, \text{local arc length}} = \underbrace{g_{\mu\nu}(\vec{r})}_{\text{metric tensor}} dx^\mu dx^\nu \quad -- \ (4)$$



In the flat 3+1 dimension (space for special relativity),

$g_{\mu\nu} = \eta_{\mu\nu}$ diagonal $(-1, 1, 1, 1)$

By definition, a conformal transformation is an invertible mapping $\vec{r} \to \vec{r}\,'$ such that

$$g'_{\mu\nu}(\vec{r}\,') = \Lambda(\vec{r})\, g_{\mu\nu}(\vec{r}) \quad --- \ (5)$$

i.e. it only changes the metric tensor up to a scale.

Physically, this happens when the transformation is locally equivalent to a translation, rotation & a dilation.

The word "conformal" means that the angle between two arbitrary curves crossing at some point is not affected.

Consider an infinitesimal transformation

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}(\vec{r})$$

$$dx'^{\mu} = dx^{\mu} + \sum_{\nu} \frac{\partial \epsilon^{\mu}}{\partial x_{\nu}} dx^{\nu}$$

$\therefore$ Eqs. (4) & (5) imply (to $O(\epsilon)$)

$$g'_{\mu\nu} = g_{\mu\nu} + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) + O(\epsilon^2)$$

Therefore, $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \propto g_{\mu\nu}$

If we write $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(\vec{r}) g_{\mu\nu}$ --- (6),

by taking $\sum \mu = \nu$, (trace), one gets

$$f(\vec{r}) = \frac{2}{d} \partial_\alpha \epsilon^\alpha \quad \text{--- (6-1)}$$

in d-dim Euclidean space $g_{\mu\nu} = (1, 1, \ldots 1)$

$$\therefore \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} (\partial_\alpha \epsilon^\alpha) g_{\mu\nu} \quad \text{--- (7)}$$

To solve eq. (7), one takes an extra derivative $\frac{\partial}{\partial x_\beta}$ on eq. (6), and obtains

$$\partial_\beta \partial_\mu \epsilon_\nu + \partial_\beta \partial_\nu \epsilon_\mu = (\partial_\beta f) g_{\mu\nu} \quad \text{--- (8)}$$

permuting (β, ν) & (β, μ)

$$\partial_\mu \partial_\beta \epsilon_\nu + \partial_\mu \partial_\nu \epsilon_\beta = (\partial_\mu f) g_{\beta\nu} \quad \text{--- (9)}$$

$$\partial_\nu \partial_\mu \epsilon_\beta + \partial_\nu \partial_\beta \epsilon_\mu = (\partial_\nu f) g_{\beta\mu} \quad \text{--- (10)}$$

$$(9) + (10) - (8) \Rightarrow 2 \partial_\mu \partial_\nu \epsilon_\beta = (\partial_\mu f) g_{\nu\beta} + (\partial_\nu f) g_{\mu\beta} - (\partial_\beta f) g_{\mu\nu} \quad \text{--- (11)}$$

Contracting μ & ν

$\Rightarrow \quad 2\partial^2 \epsilon_\beta = (2-d)\, \partial_\beta f$ - - - (12)

Taking $\sum \partial_\beta \Rightarrow \quad (d-1)\, \partial^2 f = 0$ - - (13)

Clearly, $d=1$, eq (13) imposes not contraint.
For
$d>2$ (we shall come back $d=2$ later)
$\partial^2 f = 0$

∂_α (12) $\Rightarrow \quad (2-d)\, \partial_\alpha \partial_\beta f = 2\partial^2 (\partial_\alpha \epsilon_\beta)$ - - (14)

∂^2 (6) $\Rightarrow \quad \partial^2(\partial_\alpha \epsilon_\beta) + \partial^2(\partial_\beta \epsilon_\alpha) = (\partial^2 f)\, g_{\alpha\beta}$ - - (15)

Exchanging α, β in (14), one gets
& using (15)

$$(2-d)\, \partial_\alpha \partial_\beta f = (\partial^2 f)\, g_{\alpha\beta} = 0$$

$$\therefore \quad \partial_\alpha \partial_\beta f = 0 \quad \& \quad \partial^2 f = 0$$

$\therefore$ f is at most linear in X^α

$$f(\vec{r}) = A + B_\alpha X^\alpha.$$

$\because f = \frac{2}{d} \partial_\alpha \epsilon^\alpha$. $\therefore$ ϵ_μ is at most quadratic in X^α!

Let $\quad \epsilon_\mu = a_\mu + b_{\mu\nu} X^\nu + c_{\mu\nu\rho} X^\nu X^\rho$ - - (16)
$(= c_{\mu\rho\nu})$

Clearly, a_μ = translation

To satisfy eqs. (6) & (7) for every $\vec{r}$, one

Can treat each term in (16) separately.

For $b_{\mu\nu}x^\nu$, eq. (9) implies

$$b_{\nu\mu} + b_{\mu\nu} = \frac{2}{d}(b^\alpha{}_\alpha)\, g_{\mu\nu} \quad \text{--- (17)}$$

which implies the symmetric part of $b_{\mu\nu}$ is proportional to $g_{\mu\nu}$ and there is no constraint on the anti-symmetric part.

$$\therefore\ b_{\mu\nu} = \underbrace{\lambda g_{\mu\nu}}_{\substack{\text{local scale transform} \\ \epsilon_\mu = \lambda x_\mu}} + a_{\mu\nu} \quad \text{--- (18)}$$

$a_{\mu\nu} = -a_{\nu\mu}$ (representing rotation)

$\delta\vec{r} = \delta\vec{\theta}\times\vec{r}$ $\quad \delta\vec{\theta} = \delta\theta\hat{z}$

$\delta x = -y\delta\theta$ $\quad \delta y = x\delta\theta$

$= \epsilon_{xy}\, y\,\delta\theta$ $\quad = -\epsilon_{yx}\, x\,\delta\theta$

For the quadratic term, $C_{\mu\nu\rho}x^\nu x^\rho$,

Eq. (6+) implies: $\quad f = \frac{4}{d}\sum_{\alpha,\rho} C^\alpha{}_{\alpha\rho}\, x^\rho$

(11) $\quad 2(C_{\beta\mu\nu} + C_{\beta\nu\mu})$

$$= \underbrace{\frac{4}{d}C^\alpha{}_{\alpha\mu}}_{b_\mu} g_{\nu\beta} + \frac{4}{d}C^\alpha{}_{\alpha\nu}\, g_{\mu\beta} - \frac{4}{d}C^\alpha{}_{\alpha\beta}\, g_{\mu\nu}$$

$\therefore\ C_{\mu\nu\rho} = g_{\mu\rho}b_\nu + g_{\mu\nu}b_\rho - g_{\nu\rho}b_\mu$

$C_{\mu\nu\rho}x^\nu x^\rho = g_{\mu\rho}x^\rho(\vec{b}\cdot\vec{r}) + g_{\mu\nu}x^\nu(\vec{b}\cdot\vec{r}) - g_{\nu\rho}x^\nu x^\rho b_\mu$

$\therefore$ It represents the so-called special conformal transformation in the infinitesimal form:

$$X'^{\mu} = X^{\mu} + 2(\vec{b}\cdot\vec{r})X^{\mu} - r^2 b^{\mu} \quad \cdots \quad (19)$$

For finite transformation, it becomes

$$X'^{\mu} = \frac{X^{\mu} - b^{\mu} r^2}{1 - 2\vec{b}\cdot\vec{r} + b^2 r^2} \leftarrow N(r)$$

$$r'^2 = \frac{1}{(1-2\vec{b}\cdot\vec{r}+b^2r^2)^2}\left(\vec{r}-\vec{b}r^2\right)^2$$

$$= \frac{r^2 - 2\vec{r}\cdot\vec{b}\,r^2 + b^2 r^2}{(1-2\vec{b}\cdot\vec{r}+b^2r^2)^2} = \frac{r^2}{1-2\vec{b}\cdot\vec{r}+b^2r^2}$$

$$\therefore \frac{X'^{\mu}}{r'^2} = \frac{X^{\mu} - b^{\mu} r^2}{r^2} = \frac{X^{\mu}}{r^2} - b^{\mu}$$

Therefore, it is nothing but an inversion

$X^{\mu} \to \frac{X^{\mu}}{r^2}$ followed by a translation.

$\left(z \to \frac{1}{z},\ z' = \frac{1}{z} = \frac{z^*}{|z|^2}\right)$

$\therefore$ Angles are preserved!

Furthermore, $\therefore dX'^{\mu} = \frac{1}{N(r)} dX^{\mu} - \frac{X^{\mu} - b^{\mu} r^2}{N^2} dN(r)$

$$- \frac{b^{\mu}}{N(r)} 2\vec{r}\cdot d\vec{r}$$

One can show $ds'^2 = \sum_{\mu} dx'^{\mu} dx'_{\mu} = (1-2\vec{b}\cdot\vec{r}+b^2r^2)^{-2} ds^2$

(Ex.)

$$\therefore \Lambda(\vec{r}) = \left(1 - 2\vec{b}\cdot\vec{r} + b^2 r^2\right)^{2} \quad \cdots (20)$$

In summary, for $d > 2$ ($d \geq 3$), the finite transformations that lead to $g'_{\mu\nu} = \Lambda g_{\mu\nu}$ are

(i) translation $\quad x'^{\mu} = x^{\mu} + a^{\mu}$

(ii) dilation $\quad x'^{\mu} = \lambda x^{\mu}$

(iii) rotation $\quad x'^{\mu} = M^{\mu}{}_{\nu} x^{\nu}$

(i)–(iii): Poincaré' group

(iv) specifial conformal transformation (SCT) $\quad x'^{\mu} = \dfrac{x^{\mu} - b^{\mu} r^2}{1 - 2\vec{b}\cdot\vec{r} + b^2 r^2}$

The corresponding generators are

translation: $\hat{P}_{\mu} = -i\partial_{\mu}$

dilation: $\hat{D} = -i x^{\mu}\partial_{\mu}$ $\quad (\lambda = 1 + \epsilon,\ \epsilon x^{\mu} \to \epsilon x^{\mu}(-i\partial_{\mu}))$

rotation: $\hat{R}_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$

SCT: $\hat{K}_{\mu} = -i(2x_{\mu}x^{\nu}\partial_{\nu} - r^2\partial_{\mu})$

It can be shown that commutators of these generators form an algebra and (conformal algebra) for $d \geq 3$

are the same as those for $SO(d+1, 1)$.
Therefore, for $d \geq 3$, # of generators is finite and the conformal group is isomorphic to the group $SO(d+1, 1)$

Constraints of conformal invariance in d dim

In addition to coordinate transformations, fields also change under these transformations.

As we have learned from the theory of critical phenomenon, at critical points, there exist scaling fields $\phi_i(\vec{r})$ such that

if $\vec{r}' = \lambda \vec{r}$

$$\phi_i(\vec{r}) \to \phi_i'(\vec{r}') = \lambda^{-\Delta_i} \phi_i(\vec{r}) \quad \text{-- (20)}$$

where in general, ϕ_i' may have different functional form than that of ϕ_i. Δ_i = scaling dimension of ϕ_i

The concept of scaling fields is generalized to be quasi primary such that one defines a theory with conformal invariance if

(i) There is a set of fields $\{A_i\}$ with i specify different fields.

(in general, it contains derivatives of all relevant fields and is infinite)

(ii) There is a subset $\{\phi_j\} \subset \{A_i\}$, called "quasi-primary" so that eq (20) is generalized to

$$\phi_i(\vec{r}) \to \phi_i'(\vec{r}') = \underbrace{\left|\frac{\partial \vec{r}'}{\partial \vec{r}}\right|}_{\text{Jacobian of the transformation } \vec{r}' \to \vec{r}}^{-\Delta_i/d} \phi_i(\vec{r})$$

The theory is then covariant under the transformation with

$$\langle \phi_1(\vec{r}_1)\, \phi_2(\vec{r}_2) \cdots \phi_n(\vec{r}_n) \rangle$$

$$= \left|\frac{\partial \vec{r}\,'}{\partial \vec{r}}\right|^{\Delta_1/d}_{\vec{r}=\vec{r}_1} \left|\frac{\partial \vec{r}\,'}{\partial \vec{r}}\right|^{\Delta_2/d}_{\vec{r}=\vec{r}_2} \cdots \left|\frac{\partial \vec{r}\,'}{\partial \vec{r}}\right|^{\Delta_n/d}_{\vec{r}=\vec{r}_n}$$

$$\times \langle \phi_1(\vec{r}_1{}')\, \phi_2(\vec{r}_2{}') \cdots \phi_n(\vec{r}_n{}') \rangle \quad \cdots (21)$$

One can see eq. (20) is a special example when $\vec{r}\,' = \lambda \vec{r}$. $\left|\frac{\partial \vec{r}\,'}{\partial \vec{r}}\right| = \lambda^d$

$$\phi_i'(\vec{r}\,') = \lambda^{-\Delta_i} \phi_i(\vec{r})$$

For special conformal transformations,

$$\vec{r}\,' = \frac{\vec{r} - \vec{b} r^2}{1 - 2\vec{b}\cdot\vec{r} + b^2 r^2}$$

$\because$ $\Lambda(r) = (1 - 2\vec{b}\cdot\vec{r} + b^2 r^2)^2$ $\therefore$ $\left|\frac{\partial \vec{r}\,'}{\partial \vec{r}}\right| = \frac{1}{(1 - 2\vec{b}\cdot\vec{r} + b^2 r^2)^d}$

(local rescaling factor)

$\cdots (22)$

(iii) Rest of $\{A_i\}$ can be expressed as linear combination of quasi-primary fields & their derivatives

(iv) There is a vacuum invariant under global conformal group.

Eq. (21), when including SCT, has severe contraints on forms of correlation functions of quasi-primary fields. These are consequences of requiring the microscopic theory to be locally conformal invariant.

First, for <u>two-point correlation functions</u>

$$\because \langle \phi_1(\vec{r}_1)\phi_2(\vec{r}_2)\rangle = \left|\frac{\partial \vec{r}_1{}'}{\partial \vec{r}_1}\right|^{\Delta_1/d} \left|\frac{\partial \vec{r}_2{}'}{\partial \vec{r}_2}\right|^{\Delta_2/d} \langle \phi_1(\vec{r}_1{}')\phi_2(\vec{r}_2{}')\rangle \quad \text{--- (23)}$$

by taking $\vec{r}\,' = \lambda \vec{r}$, one obtains

$$\langle \phi_1(\vec{r}_1)\phi_2(\vec{r}_2)\rangle = \lambda^{\Delta_1+\Delta_2} \langle \phi_1(\lambda\vec{r}_1)\phi_2(\lambda\vec{r}_2)\rangle \quad \text{--- (24)}$$

Since rotational & translational invariance requires

$$\langle \phi_1(\vec{r}_1)\phi_2(\vec{r}_2)\rangle = f(|\vec{r}_1-\vec{r}_2|), \quad \underset{\parallel}{f(\lambda r)}$$

eq (24) requires $f(\vec{r}) = \lambda^{\Delta_1+\Delta_2} f(\lambda\vec{r})$ for any $\vec{r}$ & λ.

Setting $\lambda = \frac{1}{r}$ (for a given $\vec{r}$), one gets

$$f(\vec{r}) = \frac{\text{Const.}}{r^{\Delta_1+\Delta_2}}$$

$$\therefore \langle \phi_1(\vec{r}_1)\phi_2(\vec{r}_2)\rangle = \frac{C_{12}}{|\vec{r}_1-\vec{r}_2|^{\Delta_1+\Delta_2}} \quad \text{--- (25)}$$

For SCT, $\left|\frac{\partial \vec{r}\,'}{\partial \vec{r}}\right| = \frac{1}{(1-2\vec{r}\cdot\vec{b}+b^2r^2)^d}$, eqs. (23) & (25)

yield
$$\frac{C_{12}}{|\vec{r}_1-\vec{r}_2|^{\Delta_1+\Delta_2}} = \frac{1}{N(r_1)^{\Delta_1}}\frac{1}{N(r_2)^{\Delta_2}}\frac{C_{12}}{|\vec{r}_1{}'-\vec{r}_2{}'|^{\Delta_1+\Delta_2}} \quad \text{--- (26)}$$

No.
Date.

Now, $\because\ \vec{r}_i' = \dfrac{\vec{r}_i - \vec{b}\,r_i^2}{1 - 2\vec{b}\cdot\vec{r}_i + b^2 r_i^2}$

$$\therefore\ \vec{r}_i' - \vec{r}_j' = \frac{(\vec{r}_i - \vec{b}r_i^2)(1-2\vec{b}\cdot\vec{r}_j + b^2 r_j^2) - (\vec{r}_j - \vec{b}r_j^2)(1-2\vec{b}\cdot\vec{r}_i + b^2 r_i^2)}{(1-2\vec{b}\cdot\vec{r}_i + b^2 r_i^2)\ \underbrace{(1-2\vec{b}\cdot\vec{r}_j + b^2 r_j^2)}_{N(r_j)}}$$

Note that $(\vec{r}_i - \vec{b}r_i^2)^2 = r_i^2(1 - 2\vec{r}_i\cdot\vec{b} + b^2 r_i^2)$

$$2(\vec{r}_i - \vec{b}r_i^2)\cdot(\vec{r}_j - \vec{b}\,r_j^2)$$

$$= 2\vec{r}_i\cdot\vec{r}_j - 2\vec{b}\cdot\vec{r}_j\, r_i^2 - 2\vec{b}\cdot\vec{r}_i\, r_j^2 + 2b^2 r_i^2 r_j^2$$

$$= 2\vec{r}_i\cdot\vec{r}_j + r_i^2(1 - 2\vec{b}\cdot\vec{r}_j + b^2 r_j^2)$$
$$+ r_j^2(1 - 2\vec{b}\cdot\vec{r}_i + b^2 r_i^2) - r_i^2 - r_j^2$$

$\therefore\ |\vec{r}_i' - \vec{r}_j'|^2$

$$= \frac{1}{N^2(r_i)N^2(r_j)}\Big\{ r_i^2 N(r_i)\cancel{N^2(r_j)} + r_j^2 \cancel{N(r_j)} N^2(r_i)$$
$$- \left[2\vec{r}_i\cdot\vec{r}_j + r_i^2 \cancel{N(r_j)} + r_j^2 \cancel{N(r_i)} - r_i^2 - r_j^2\right)$$
$$\times N(r_j)N(r_i)\Big\}$$

$$= \frac{|\vec{r}_i - \vec{r}_j|^2}{N(r_i)N(r_j)}$$

$$\therefore\ |\vec{r}_i' - \vec{r}_j'| = \frac{|\vec{r}_i - \vec{r}_j|}{(1-2\vec{b}\cdot\vec{r}_i + r_i^2)^{\frac{1}{2}}(1 - 2\vec{b}\cdot\vec{r}_j + r_j^2)^{\frac{1}{2}}} \cdots (27)$$

Using eq. (27), eq. (26) becomes

$$\frac{C_{12}}{|\vec{r}_1 - \vec{r}_2|^{\Delta_1+\Delta_2}} = \frac{C_{12}}{N(r_1)^{\frac{-\Delta_2+\Delta_1}{2}} N(r_2)^{\frac{\Delta_2-\Delta_1}{2}}}\ \frac{1}{|\vec{r}_1 - \vec{r}_2|^{\Delta_1+\Delta_2}}$$

Clearly, one sees that

if $\Delta_1 \neq \Delta_2$. $C_{12} = 0$ $\langle \phi_1(\vec{r}_1)\phi_2(\vec{r}_2)\rangle = 0$

only $\Delta_1 = \Delta_2 \equiv \Delta$, C_{12} can be non-zero.

In this case $\langle \phi_1(\vec{r}_1)\phi_2(\vec{r}_2)\rangle = \dfrac{C_{12}}{|\vec{r}_1 - \vec{r}_2|^{2\Delta}}$ ⋯ (28)

Note that if $\phi_1 = \phi_2$, $2\Delta = d+2-\eta = p$ that was defined before as

$$G(r) \sim r^{-p} f(r/\xi) \text{ near } T_c$$

In addition, if $\Delta_1 \neq \Delta_2$, $\langle \phi_1(\vec{r}_1)\phi_2(\vec{r}_2)\rangle = 0$ implies ϕ_1 & ϕ_2 are independent from each other!

Three-point correlation functions:

Similarly, invariance under translations, rotations and dilations implies

$$\langle \phi_1(\vec{r}_1)\phi_2(\vec{r}_2)\phi_3(\vec{r}_3)\rangle = \sum_{a,b,c} \frac{C_{123}^{abc}}{r_{12}^a\, r_{23}^b\, r_{13}^c}$$

wher $r_{ij} = |\vec{r}_i - \vec{r}_j|$ and a, b, c are any real #s satisfying $a+b+c = \Delta_1+\Delta_2+\Delta_3$. (Hence a summation over a, b, c is performed)

Eqs. (21) & (27) imply (SCT)

$$\frac{C_{123}^{abc}}{r_{12}^a\, r_{23}^b\, r_{13}^c} = \frac{C_{123}^{abc}}{N(r_1)^{\Delta_1} N(r_2)^{\Delta_2} N(r_3)^{\Delta_3}} \cdot \frac{[N(r_1)N(r_2)]^{\frac{a}{2}}[N(r_2)N(r_3)]^{\frac{b}{2}}[N(r_1)N(r_3)]^{\frac{c}{2}}}{r_{12}^a \cdot r_{23}^b \cdot r_{13}^c}$$

Hence only when $a+c = 2\Delta_1$

$$a+b = 2\Delta_2$$

$$b+c = 2\Delta_3, \quad C_{123}^{abc} \neq 0$$

$\therefore a = \Delta_1+\Delta_2-\Delta_3, \quad b = -\Delta_1+\Delta_2+\Delta_3, \quad c = \Delta_1-\Delta_2+\Delta_3.$

$$\therefore \langle \phi_1(\vec{r}_1)\phi_2(\vec{r}_2)\phi_3(\vec{r}_3)\rangle = \frac{C_{123}}{r_{12}^{\Delta_1+\Delta_2-\Delta_3}\, r_{23}^{\Delta_2+\Delta_3-\Delta_1}\, r_{13}^{\Delta_3+\Delta_1-\Delta_2}} \quad \cdots (28)\text{-}1$$

is also completely specified upto a global factor C_{123}.

The SCT can't specified N-point correlation functions for $N \geq 4$. In this case, Eq. (27) implies the cross-rations of the form $\frac{r_{ij}\, r_{kl}}{r_{ik}\, r_{jl}}$ are invariant under SCT (i.e. full conformal group).

Since cross-ratios require at least four-points, this happen at $N \geq 4$.

The N-point correlation functions now depends on these rations with arbitrary dependence. For instance,

$$\langle \phi_1(\vec{r}_1)\phi_2(\vec{r}_2)\phi_3(\vec{r}_3)\phi_4(\vec{r}_4)\rangle = f\left(\frac{r_{12}r_{34}}{r_{13}r_{24}}, \frac{r_{14}r_{23}}{r_{12}r_{34}}\right)$$

$$\times \prod_{i<j}^{4} r_{ij}^{\Delta/3-\Delta_i-\Delta_j}, \quad \Delta = \sum_{i=1}^{4}\Delta_i \quad \cdots (28)\text{-}2$$

Here f is an arbitrary function of ratios. 15.
In general, for $N \geq 4$, one forms all possible pairs (i,j), $\prod_{i<j} r_{ij}^{-(\Delta_i+\Delta_j)}$ (C^N_2 pairs)

Since each $\vec{r}_i$ appears in $(N-1)$ r_{ij} $(j \neq i)$ pairs, (generate $(N-1)\Delta$)
to get $\lambda^{-\Delta_i}$ correctly, one needs to compensate $(N-2)\Delta$ by introducing $r_{ij}^{\frac{N-2}{C^N_2}\Delta}$ with $\Delta = \sum_{i=1}^{N} \Delta_i$.

$$\therefore \langle \phi_1(\vec{r}_1)\phi_2(\vec{r}_2) \dots \phi_N(\vec{r}_N) \rangle = \prod_{i<j} r_{ij}^{-(\Delta_i+\Delta_j)+\frac{2(N-2)}{N(N-1)}\Delta}$$
$$\times f\Big(\underbrace{\frac{r_{ij}r_{kl}}{r_{ik}r_{jl}}, \dots}_{\frac{N(N-3)}{2} \text{ independent cross-ratios (see order)}}\Big)$$

(e.g. $N=4$, $\frac{4\times 1}{2} = 2$ independent cross-ratios)

Conformal invariance in 2D

The conformal invariance in 2D is very different from that in other dimensions.
As we shall see, in 2D, there exists infinite varieties of coordinate transformations that are locally conformal (although not well-defined everywhere). While we have

seen that for other dimensions, there are only dilation + special conformal transformations, translations & rotations that are locally conformal.

At $d=2$, eq. (7) becomes

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \sum_\alpha \partial_\alpha \varepsilon^\alpha g_{\mu\nu}$$

Therefore, one obtains the Cauchy-Riemann equation used in complex analysis :

$$\partial_x \varepsilon_y + \partial_y \varepsilon_x = 0 \quad (\mu \neq \nu)$$
$$\partial_x \varepsilon_x = \partial_y \varepsilon_y \quad (\mu = \nu)$$

In other words, if we define $z = x+iy$, $\bar{z} = x-iy$, we get (x,y) $(\bar{z}, z) \in C^2$

$\varepsilon_x + i\varepsilon_y (x,y)$ is only a function of z

$\varepsilon_x - i\varepsilon_y$ is " " $\bar{z}$

$\therefore$ we define $\varepsilon = \varepsilon_x + i\varepsilon_y$ & $\bar{\varepsilon}(\bar{z}) = \varepsilon_x - i\varepsilon_y$

$$\therefore \lim_{\Delta x \to 0} \frac{\varepsilon(x+\Delta x, y) - \varepsilon(x,y)}{\Delta x} = \lim_{\Delta y \to 0} \frac{\varepsilon(x, y+\Delta y) - \varepsilon(x,y)}{i\Delta y}$$

$$= \frac{\partial \varepsilon_x}{\partial x} + i\frac{\partial \varepsilon_y}{\partial x} = -i\frac{\partial \varepsilon_x}{\partial y} + \frac{\partial \varepsilon_y}{\partial y} \text{ is satisfied}$$

$$\therefore = \frac{d\varepsilon}{dz} \quad \text{it means } \varepsilon = \varepsilon(z) = \varepsilon_x + i\varepsilon_y$$

Similarly, $\frac{d\bar{\varepsilon}}{d\bar{z}} = \frac{\partial \varepsilon_x}{\partial x} - i\frac{\partial \varepsilon_y}{\partial x} = +i\frac{\partial \varepsilon_x}{\partial y} + \frac{\partial \varepsilon_y}{\partial y}$, $\varepsilon_x - i\varepsilon_y = \bar{\varepsilon}(\bar{z})$

Therefore, ξ is holomorphic, $\bar{\xi}$ is anti-holomorphic.

In other words, any analytic coordinate

(for $f(z) = u + iv$, $\nabla^2 u = \nabla^2 v = 0$ too!)

transformation $z \to f(z)$, $\bar{z} \to \bar{f}(\bar{z})$

is locally conformal! (note that $\bar{f} \neq f^*$!)

To go to complex coordinates, we treat

z & $\bar{z}$ as independent variables.

$$\because ds^2 = dx^2 + dy^2 = (dx + idy)(dx - idy)$$
$$= dz\, d\bar{z}$$

$$\therefore g_{zz} = g_{\bar{z}\bar{z}} = 0, \quad g_{z\bar{z}} = g_{\bar{z}z} = 1/2$$

Formally, one can use $g_{\alpha\beta} = \frac{\partial z^\alpha}{\partial x^\mu} \frac{\partial z^\beta}{\partial x^\nu} g^{\mu\nu}$, $z^0 = z$, $z' = \bar{z}$

to get $g^{zz} = \frac{\partial z}{\partial x^\mu} \frac{\partial z}{\partial x^\nu} \delta^{\mu\nu} = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 0$

$g^{\bar{z}\bar{z}} = 0$, $g^{z\bar{z}} = \frac{\partial z}{\partial x}\frac{\partial \bar{z}}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial \bar{z}}{\partial y} = 2 = g^{\bar{z}z}$

$\Rightarrow g_{z\bar{z}} = g_{\bar{z}z} = 1/2$

Using z & $\bar{z}$, one has $\partial_z = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z}$

$$= \frac{1}{2}(\partial_x - i\partial_y)$$

$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, $\partial_x = \partial_z + \partial_{\bar{z}}$, $\partial_y = i(\partial_z - \partial_{\bar{z}})$

$\nabla^2 = \partial_x^2 + \partial_y^2 = 2\partial_z \partial_{\bar{z}}$ $\therefore \nabla^2 \phi = 0 \Rightarrow \phi = \phi(z)$ or $\phi(\bar{z})$

In the complex coordinates, one has

$$ds^2 \to \left|\frac{\partial f}{\partial z}\right|^2 dz\, d\bar{z}, \quad g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}\left|\frac{\partial f}{\partial z}\right|^2$$

Conformal algebra in 2D

Local conformal transformations are generated by considering $z \to z' = z - \varepsilon z^{n+1}$, $\varepsilon \to 0$, $\bar{z}' = \bar{z} - \bar{\varepsilon}\bar{z}^{n+1}$

this leads to $\delta f(z) = f(z - \varepsilon z^{n+1}) - f(z)$

$$= -\varepsilon z^{n+1} \frac{d}{dz} f$$

$\therefore$ $l_n = -z^{n+1}\frac{d}{dz}$, $\bar{l}_n = -\bar{z}^{n+1} d_{\bar{z}}$, $n =$ integers

are corresponding generators.

Since any analytic functions in the region $0<|z|<\infty$ (holomorphic or anti-holomorphic) can be expanded in series of z^n (Laurent series), any transformations $z' = z + \varepsilon(z)$, $\varepsilon(z)$ can be expressed as

$$\varepsilon(z) = \sum_{n=-\infty}^{\infty} C_n z^{n+1}$$

Therefore $\sum_{n=-\infty}^{\infty} \{l_n, \bar{l}_n\}$ generates the whole local conformal transformations.

It's easy to see $l_n, \bar{l}_n$ form a local conformal algebra

$$[l_m, l_n] = -z^{m+1}\frac{d z^{n+1}}{dz}\frac{d}{dz} = z^{n+1}\left(\frac{d z^{m+1}}{dz}\right)\frac{d}{dz}$$

$$= (n-m) z^{m+n+1}\frac{d}{dz} = (m-n)\, l_{n+m}$$

$[\bar{l}_m, \bar{l}_n] = (m-n)\, \bar{l}_{n+m}$. $[l_m, \bar{l}_n] = 0$.

However, not all the transformations (ϵz^{n+1}) are well-defined globally!

In particular, $l_n = -z^{n+1}\partial_z$ is non-singular as $z \to 0$ only for $n \geq -1$; while for $z \to \infty$, one defines $w = -1/z$, $l_n = -\left(\frac{-1}{w}\right)^{n+1} \frac{dz}{(dw)}^{-1} \partial_w$

$$= -\left(\frac{-1}{w}\right)^{n-1} \partial_w$$

is non-singular as $w \to 0$ only for $n \leq 1$

Therefore, only $n = 0, \pm 1$ are globally defined.

This applies to anti-holomorphic transformations as well.

$\{l_0, l_{\pm1}\}$ / $\{\bar{l}_0, \bar{l}_{\pm1}\}$ actually forms a subalgebra (closed)

The finite form of these transformations is

$$z \to \frac{az+b}{cz+d}, \quad \bar{z} \to \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}$$

which defines the global (projective) conformal group in 2D.

Here $a, b, c, d \in \mathbb{C}$, $\underline{ad - bc = 1}$

Since $(a, b, c, d) \to (-a, -b, -c, -d)$ leaves the transformation invariant, this is the group $SL(2, \mathbb{C})/\mathbb{Z}_2$ (modular group)
(special linear..)

By ~~rewriting~~

$$z' = \frac{az+b}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c^2}\frac{1}{z+d/c},$$

one sees that it includes

(i') translation, $z_1 = z + d/c$ (generators: $l_{-1}, \bar{l}_{-1}$)

(ii') Inversion, $z_2 = 1/z_1$

(iii') Rotation & dilation $z_3 = \frac{bc-ad}{c^2} z_2$
$\hookrightarrow$ ($i(l_0-\bar{l}_0)$ & $l_0+\bar{l}_0$ are generators)

~~(iv) translation.~~

In particular, $z' = \frac{z}{cz+1}$ leads to the special conformal transformation (SCT) we discussed for other dimensions: (generator, l_1, $\bar{l}_1$)

$$z' = \frac{z}{|1+cz|^2}(1+c^*z^*)$$

$$|1+cz|^2 = 1+|c|^2|z|^2 + (cz+c^*z^*) = 1 + 2\vec{b}\cdot\vec{r} + b^2r^2$$

$\vec{b} \equiv (c_x, -c_y)$, $\vec{r} = (x, y)$

$$zc^*z^* = c^*|z|^2 = -\vec{b}\, r^2$$

$$\Rightarrow \quad \vec{r}\,' = \frac{\vec{r} - \vec{b}r^2}{1 - 2\vec{b}\cdot\vec{r} + b^2r^2}$$

(global)

Hence SCT is a subset of 2D conformal group. For other dimensions, there exists only a global conformal group. Strictly speaking, in 2D, the only true conformal group is the projective transformations as they are well-defined globally.

However, only in 2D, it is possible to extending the transformations to include local conformal transformations so that local conformal algebra can be imposed on the theory. As we shall see, it will lead to strong contraints on possible models in 2D

Correlation functions in 2D

Primary fields (similar to the quasi-primary fields in other dimensions) generalize notions of vectors and tensors under the rotation group of the conformal group: ϕ is a primary field if, $\omega = f(z)$, $\bar{\omega} = \bar{f}(\bar{z})$ (any f & $\bar{f}$!)

$$\phi'(\omega, \bar{\omega}) = \left(\frac{d\omega}{dz}\right)^{-h} \left(\frac{d\bar{\omega}}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}) \quad \text{---} (29)$$

or $\phi \to \left(\frac{d\omega}{dz}\right)^{h} \left(\frac{d\bar{\omega}}{d\bar{z}}\right) \phi(\omega(z), \bar{\omega}(\bar{z}))$

(which is similar to the transformation of tensor $A'_{\mu\nu\cdots} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \cdots A_{\alpha\beta\cdots}$)

ϕ is termed as a primary field of conformal weight $(h, \bar{h})$. Not all fields are primary. The rest are called secondary fields. Note that primary fields are quasi-primary which is defined only for $SL(2, C)$

Under infinitesimal transformations, $z \to \underbrace{z+\epsilon(z)}_{\omega}$, $\bar z \to \underbrace{\bar z+\bar\epsilon(\bar z)}_{\bar\omega}$, the change of ϕ at the same point is

$$\delta_{\epsilon,\bar\epsilon}\phi \equiv \left(\frac{d\omega}{dz}\right)^{h}\left(\frac{d\bar\omega}{d\bar z}\right)^{\bar h}\phi(\omega(z),\bar\omega(\bar z)) - \phi(z,\bar z)$$

$$= (1+\partial_z\epsilon)^{h}(1+\partial_{\bar z}\bar\epsilon)^{\bar h}\phi(z+\epsilon,\ \bar z+\bar\epsilon) - \phi(z,\bar z)$$

$$\cong (1+h\partial_z\epsilon)(1+\bar h\partial_{\bar z}\bar\epsilon)(\phi(z,\bar z)+\epsilon\partial_z\phi+\bar\epsilon\partial_{\bar z}\phi)$$
$$- \phi(z,\bar z))$$

$$= (h\partial_z\epsilon+\epsilon\partial_z)\phi + (\bar h\partial_{\bar z}\bar\epsilon+\bar\epsilon\partial_{\bar z})\phi \quad \cdots (30)$$

(see over)

Consider two-point correlation functions

$$G_2 = \langle \phi_1(z_1,\bar z_1)\,\phi_2(z_2,\bar z_2)\rangle$$

$$\delta_{\epsilon\bar\epsilon}G_2 = \langle \delta\phi_{1\epsilon,\bar\epsilon}\,\phi_2\rangle + \langle \phi_1\,\delta_{\epsilon\bar\epsilon}\phi_2\rangle$$

For a theory being invariant under global conformal group, $\delta_{\varepsilon,\bar\varepsilon}G_2 = 0$ for $\varepsilon,\bar\varepsilon = 1,\ z/\bar z,\ z^2, \bar z^2$

$$\therefore \Big[(\epsilon_{z_1}\partial_{z_1}+h_1\partial_{z_1}\epsilon_{z_1}) + (\epsilon_{z_2}\partial_{z_2}+h_2\partial_{z_2}\epsilon_{z_2})$$
$$+(\bar\epsilon_{\bar z_1}\partial_{\bar z_1}+\bar h_1\partial_{\bar z_1}\bar\epsilon_{\bar z_1}) + (\bar\epsilon_{\bar z_2}\partial_{\bar z_2}+\bar h_2\partial_{\bar z_2}\bar\epsilon_{\bar z_2})\Big]G_2 = 0$$

Taking $\epsilon,\bar\epsilon = 1$, $[\partial_{z_1}+\partial_{z_2}+\partial_{\bar z_1}+\partial_{\bar z_2}]\,G_2(z_i,\bar z_i) = 0$

$$\therefore G_2 = G_2(z_1-z_2,\ \bar z_1-\bar z_2)$$

Taking $\epsilon(z)=z$, $\bar\epsilon(\bar z)=\bar z$

$$[\underbrace{z_1\partial_{z_1}+h_1+z_2\partial_{z_2}}_{\to\ z_{12}\frac{d}{dz_{12}}\ (z_{12}=z_1-z_2)}+h_2+\bar z_1\partial_{\bar z_1}+\bar h_1+\bar z_2\partial_{\bar z_2}+\bar h_2]\,G_2 = 0$$

Which implies

$$G_2 = \frac{C_{12}}{(z_1-z_2)^{h_1+h_2}(\bar{z}_1-\bar{z}_2)^{\bar{h}_1+\bar{h}_2}}$$

Finally, by taking $\epsilon(z)=z^2$, $\bar{\epsilon}(\bar{z})=\bar{z}^2$,

$$(z_1^2\partial_{z_1} + 2z_1h_1 + z_2^2\partial_{z_2} + 2h_2z_2)\,G_2 = 0$$

implies $$0= \frac{-(h_1+h_2)z_1^2}{(z_1-z_2)^{h_1+h_2+1}} + \frac{2h_1z_1}{(z_1-z_2)^{h_1+h_2}} + \frac{(h_1+h_2)z_2^2}{(z_1-z_2)^{h_1+h_2+1}} + \frac{2h_2z_2}{(z_1-z_2)^{h_1+h_2}}$$

$$\therefore -(h_1+h_2)z_1^2 + 2h_1z_1(z_1-z_2) + (h_1+h_2)z_2^2 + 2h_2z_2(z_1-z_2) = 0$$

$$z_1^2(h_1-h_2) + z_2^2(h_1-h_2) + 2z_1z_2(h_2-h_1) = 0$$

$$\therefore h_1 = h_2 \equiv h.$$

Similarly, $\bar{h}_1 = \bar{h}_2 \equiv \bar{h}$

$$\therefore G_2(z_i,\bar{z}_i) = \frac{C_{12}}{z_{12}^{2h}\,\bar{z}_{12}^{2\bar{h}}} \quad \cdots \quad (31)$$

Eq. (31) is a consequence of <u>global conformal</u> transformation (not involved with local!)

When $h=\bar{h}$, i.e, $z_{12}, \bar{z}_{12}$ play the same role, i.e. no angle involved ($s = h-\bar{h} \neq 0 \Rightarrow$ angle is involved, $\Rightarrow$ spin)

$$G_2 = \frac{C_{12}}{|z_{12}|^{2(h+\bar{h})}} = \frac{C_{12}}{r_{12}^{2\Delta}}$$

which is Consistent with eq. (28). In general, eq. (31)

implies $G_2 = \frac{C_{12}}{r^{2(h+\bar{h})} e^{i2(h-\bar{h})\theta}}$ which describes fields with spins and generalizes eq. (28)

Note that the form of eq. (31) is determined essentially by translational, rotational & SCT invariance.
Similarly, eq. (28)-1 is generalized to

$$G_3(z_i, \bar{z}_i) = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_1+h_3-h_2}}$$

$$\times \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_1+\bar{h}_3-\bar{h}_2}} \quad \cdots (32)$$

Eq. (28)-2 is generalized to

$$G_4(z_i, \bar{z}_i) = f(\eta, \bar{\eta}) \prod_{i<j}^{4} z_{ij}^{h/3-h_i-h_j} \bar{z}_{ij}^{\bar{h}/3-\bar{h}_i-\bar{h}_j} \quad \cdots (33)$$

Here # of independent cross-ratios is reduced due to the fact that z_1, z_2, z_3, z_4 lie on the same plane and additional linear relations exist:

if $\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}}$, $\frac{z_{14}z_{23}}{z_{13}z_{24}} = \frac{(z_1-z_4)(z_2-z_3)}{z_{13}z_{24}} = \frac{(z_1-z_3)(z_2-z_4) - (z_1-z_2)(z_3-z_4)}{z_{13}z_{24}}$

$= 1-\eta$

Similarly, $\frac{z_{12}z_{36}}{z_{14}z_{23}} = \frac{\eta}{1-\eta}$. Hence, only one cross-ratio (η) is required for z_i & another $\bar{\eta}$ for $\bar{z}_i$! Note that by imposing all local conformal transformation, f can be fixed! (we shall come back ~~to this point later~~)

Energy – momentum tensor

For microscopic models, either models in statistical mechanics or zero-temperature quantum many-body systems, correlation functions (or Green's functions) are determined by path integrals over weight specified by an action S

$$\langle \phi_i(\vec{r}_1) \ldots \phi_n(\vec{r}_n) \rangle = \int \mathcal{D}\phi \, \phi_1(\vec{r}_1) \ldots \phi_n(\vec{r}_n) \, e^{-S[\phi]} / Z$$

$$Z = \int \mathcal{D}\phi \, e^{-S[\phi]}$$

As we have seen, for free bosons, one have $S = \frac{1}{2}\int d^2\vec{r}\left[(\nabla\phi)^2 + m^2\phi^2\right]$

In the quantum field theory, the integrand is often termed as $\mathcal{L}[\phi, \partial\phi]$, $\therefore S = \int d^d\vec{r}\, \mathcal{L}$ in general.

S has the dimension of energy. Therefore, under the transformation, $x^\mu \to x^\mu + \epsilon^\mu$

$$\delta S \equiv -\int d^d\vec{r}\; T^{\mu\nu}\, \partial_\mu \epsilon_\nu \quad \text{--- (33)}$$

$$= \frac{-1}{2}\int d^d\vec{r}\; T^{\mu\nu}\, (\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu)$$

(dimensionless)

$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu$ is symmetric and is the strain field (exactly the same concept used in the elastic theory of crystals).

Hence $T^{\mu\nu}$ is symmetric and is known as the energy-momentum tensor. (see below)

Note that eq. (33) can be integrated by part, one gets $\delta S = \int d^d\vec{r}\; \partial_\mu T^{\mu\nu}\, \varepsilon_\nu$ for any ε_ν. If the system is symmetric under the transformation ε, $\delta S = 0$ for any ε_ν. In this case, we conclude $\partial_\mu T^{\mu\nu} = 0$ i.e. $\nabla \cdot \overleftrightarrow{T} = 0$. $\therefore$ $\overleftrightarrow{T}$ is a conserved current. "Symmetries imply existence of conserved currents" is known as the Noether's theorem in classical field theory.

The meaning of $T^{\mu\nu}$ can be found by considering variation of S due to $\delta x^\mu (= \xi^\mu)$

Given a classical scalar field ϕ, variation of S versus ϕ leads to the Euler–Lagrange eq. for ϕ

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \quad \cdots (34)$$

Under the change δx^μ

$$x'^\mu = x^\mu + \delta x^\mu$$

$$\phi'(x') = \phi'(x+\delta x) = \phi'(x) + [\partial_\mu \phi'(x)]\, \delta x^\mu$$

The exact variation of $\phi \equiv \delta\phi = \phi'(x') - \phi(x)$. Which includes both the change due to x' and the functional form ϕ'.

For later uses, one also defines the
[illegible]

defines $\bar{\delta}\phi(x) = \phi'(x) - \phi(x)$ which is due to change of functional form.

Now. $\delta S = \int d^d x' \, \mathcal{L}'(x') - \int d^d x \, \mathcal{L}(x)$

$$\because \mathcal{L}'(x') = \mathcal{L}'(x) + \frac{d\mathcal{L}}{dx^\mu}\delta x^\mu = \mathcal{L}(x) + \mathcal{L}'(x) - \mathcal{L}(x) + \frac{d\mathcal{L}}{dx^\mu} dx^\mu$$

$$= \mathcal{L}(x) + \bar{\delta}\mathcal{L}(x) + \frac{d\mathcal{L}}{dx^\mu} dx^\mu$$

& $\frac{\partial x'^{\nu}}{\partial x^{\mu}} = \delta_{\mu\nu} + \partial_\mu \delta x^\nu$; $\det(\mathbb{1}+A) \simeq 1 + \mathrm{Tr}\,A$ for small A

$$\left|\frac{\partial x'}{\partial x}\right| \simeq 1 + \partial_\mu \delta x^\mu + O(\delta x^2)$$

$$\therefore \delta S = \int d^dx \left[\bar{\delta}\mathcal{L}(x) + \frac{d\mathcal{L}}{dx^\mu}\delta x^\mu + (\partial_\mu \delta x^\mu)\mathcal{L}\right]$$

$$= \int d^dx \left[\bar{\delta}\mathcal{L}(x) + \frac{d}{dx^\mu}(\mathcal{L}\,\delta x^\mu)\right]$$

Now at fixed x, $\bar{\delta}\mathcal{L}(x)$ is due to $\bar{\delta}\phi = \phi'(x) - \phi(x)$.

$$\therefore \bar{\delta}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\bar{\delta}\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\partial_\mu \bar{\delta}\phi$$

$$\underset{\text{eq (34)}}{=} \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\bar{\delta}\phi\right]$$

Hence

$$\delta S = \int d^dx \frac{d}{dx^\mu}\left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\bar{\delta}\phi + \mathcal{L}\,\delta x^\mu\right]$$

For scalar fields (under the coordinate transformations considered), $\underline{\phi'(x') = \phi(x)}$

$$\therefore \phi'(x') = \phi'(x) + (\partial_\mu \phi'(x))\delta x^\mu$$

$$= \bar{\delta}\phi(x) + \phi(x) + (\partial_\mu \phi'(x))\delta x^\mu$$

$$\therefore \bar{\delta}\phi(x) = -(\partial_\mu \phi'(x))\delta x^\mu = -(\partial_\mu \phi(x))\delta x^\mu + O(\delta^2)$$

$$\therefore \delta S = \int d^dx \frac{d}{dx^\mu}\left\{\left[\mathcal{L}\delta^\mu_\nu - (\partial_\nu \phi)\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\right]\delta x^\nu\right\}$$

$$= -\oint_S ds^\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\partial_\nu \phi - \mathcal{L}\delta^\mu_\nu\right]\delta x^\nu \quad \text{--- (34)-1}$$

$$= -\int d^dx \,(\partial_\mu \underset{0}{T^{\mu\nu}})\delta x^\nu + \int d^dx \,T^{\mu\nu}\partial_\mu \delta x^\nu$$

One finds

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial^\nu \phi - \mathcal{L} \delta^{\mu\nu} \quad \text{--- (34)-2}$$

which is the canonical energy momentum tensor

$\therefore \partial_\mu T^{\mu\nu} = 0$ (35) For $(d-1+1)$ dimension, one

gets $\partial_t T^{0\nu} = \partial_j T^{j\nu}$

$\therefore T^{0\nu}$ is the conserved charge

$$P^\nu \equiv \int d^{d-1}x \, T^{0\nu} = \int d^{d-1}x \, \frac{\partial \mathcal{L}}{\partial \dot\phi} \partial^\nu \phi - \mathcal{L} \delta^{0\nu}$$

is the 4-momentum!

In particular, $P^0 = \int d^{d-1}x \, \frac{\partial \mathcal{L}}{\partial \dot\phi} \dot\phi - \mathcal{L} \; (= T^{00})$ (34)-3

is the Hamiltonian

<u>Classical conservation laws</u>

Eq. (35) is the consequence of arbitrary constant ε^μ (independent of x^μ), i.e. translational invariance (c.f. eq. (33))

For scale invariance, $\varepsilon^\mu = \lambda x^\mu$, eq. (33) implies (after partial integration) for any λ

$$\partial^\mu (T_{\mu\nu} x^\nu) = 0 \quad \text{(36)}$$

$$= (\partial^\mu T_{\mu\nu}) x^\nu + T_{\mu\nu} \partial^\mu x^\nu = 0 \quad \therefore T^\mu{}_\mu = 0$$

$\therefore$ $T_{\mu\nu}$ is traceless.

For 2D, it's more convenient to use complex numbers. For this purpose, we shall carefully distinguish lower & upper indices and use μ, ν for (x, y) and (α, β) for $(z, \bar{z})$.

Now.
$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} \equiv S\begin{pmatrix} z \\ \bar{z} \end{pmatrix}$$

$$\therefore \frac{dx^\nu}{dx^\alpha} \equiv S_\alpha^\nu$$

$$\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}}_{S^{-1}}\begin{pmatrix} x \\ y \end{pmatrix} \quad \therefore S_\nu^\alpha = \frac{dx^\alpha}{dx^\nu} = (S_\alpha^\nu)^{-1}$$

$$\sum_\alpha S_\alpha^\nu S_{\nu'}^\alpha = \delta_{\nu'}^\nu = 1 \text{ if } \nu = \nu', \; = 0 \; \nu \neq \nu'$$

$$\sum_\nu S_\nu^\alpha S_{\alpha'}^\nu = \delta_{\alpha'}^\alpha \qquad \text{i.e. both } \nu \;\&\; \alpha \text{ are complete}$$

$$\therefore \sum_\alpha A_{\cdots\alpha} B^{\cdots\alpha} = \sum_\nu A_{\cdots\nu} B^{\cdots\nu} \left(= \sum_{\alpha\nu\nu'} S_\alpha{}^\nu A_{\cdots\nu} S_{\nu'}^\alpha B^{\cdots\nu'} = \sum_\nu A_{\cdots\nu} B^{\cdots\nu} \right)$$

Using these conventions, one has

$$g_{\alpha\beta} = S_\alpha{}^\nu S_\beta{}^\mu g_{\nu\mu}$$
$$= S_T g S = \frac{1}{4}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

$g^{\alpha\beta}$ is defined to be inverse of $g_{\alpha\beta}$

$$\therefore g^{\alpha\beta} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \quad \therefore g^{z\bar{z}} = g^{\bar{z}z} = 2, \; g^{zz} = g^{\bar{z}\bar{z}} = 0$$

$$g_{zz} = g_{\bar{z}\bar{z}} = 0, \quad g_{\bar{z}z} = g_{z\bar{z}} = \tfrac{1}{2}$$

From $g_{\alpha\beta}$, $g^{\alpha\beta}$, one defines $A^\alpha = g^{\alpha\beta} A_\beta$, $A_\beta = g_{\alpha\beta} A^\beta$.

$$\therefore g_\alpha{}^\beta = \sum_\gamma g_{\alpha\gamma} g^{\gamma\beta} \Rightarrow g_\alpha{}^\beta = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore\ T_{\alpha\beta} = S^{T} \begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix} S$$

$$= \frac{1}{4} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} T_{00} - iT_{01} - iT_{10} - T_{11} & T_{00} + iT_{01} - iT_{10} + T_{11} \\ T_{00} - iT_{01} + iT_{10} + T_{11} & T_{00} + iT_{01} + iT_{10} + T_{11} \end{pmatrix}$$

$\because\ T_{10} = T_{01}$ $\quad \therefore\ T_{zz} = \frac{1}{4}\left(T_{00} - 2iT_{10} - T_{11}\right)$

$$T_{\bar{z}\bar{z}} = \frac{1}{4}\left(T_{00} + 2iT_{10} - T_{11}\right)$$

$$T_{z\bar{z}} = \frac{1}{4}\left(T_{00} + T_{11}\right) = T_{\bar{z}z}$$

$\therefore$ Traceless condition (eq. 36) implies

$$T_{z\bar{z}} = T_{\bar{z}z} = 0 \qquad -- (37)$$

~~$\therefore$ T is diagonal on the complex plane~~

In addition, from the conservation law, eq. 35

$$0 = \partial^{\mu} T_{\mu\nu} = g^{\mu\alpha} \partial_{\alpha} T_{\mu\nu}$$

$$= g^{\alpha\beta} \partial_{\alpha} T_{\beta\nu}$$

$$\Sigma^{\mu}{}_{\mu} = \Sigma^{\beta}{}_{\beta}$$

~~if we multiply $S^{\nu}{}_{\gamma}$ & sum over ν, we get~~

$$g^{\alpha\beta} \partial_{\alpha} T_{\beta\gamma} = 0 \qquad \because \text{only } g^{z\bar{z}} \,\&\, g^{\bar{z}z} \neq 0$$

Hence $\quad g^{z\bar{z}} \partial_{\bar{z}} T_{\bar{z}\gamma} + g^{zz} \partial_{\bar{z}} T_{z\gamma} = 0 \qquad (37)-1$

$\gamma = z \quad \therefore\ \partial_{\bar{z}} T_{zz} + \partial_{z} T_{\bar{z}z} = 0 \qquad \because\ T_{\bar{z}z} = 0, \ \therefore\ \partial_{\bar{z}} T_{zz} = 0$

Similarly, $\gamma = \bar{z} \Rightarrow \partial_z T_{\bar{z}\bar{z}} = 0$

$\therefore$ T_{zz} is a function of z $\equiv T(z)$

$T_{\bar{z}\bar{z}}$ 〃 〃 〃 $\bar{z}$ $\equiv T(\bar{z})$

Radial quantization & transformation generators

As we have seen, classically, symmetries lead to conservations of certain charges.

This is valid even classically.

As we shall see, when the system is quantized, the conserved charges generate the corresponding transformation.

As an example, we consider a symple mechanical system with the action

$$S = \int_{t_b}^{t_a} dt\, L(q(t), \dot{q}(t), t)$$

Under a symmetry transformation,

$$q(t) \to q'(t)$$

$\therefore$ $\delta_S q(t) \equiv q'(t) - q(t)$ is the change

Note that $\delta_S q(t) \neq \delta q(t)$ as $\delta q(t_b) = \delta q(t_a) = 0$

while $\delta_S q(t_b)$, $\delta_S q(t_a)$ need not to vanish

$\delta_S q(t)$ is generally a functional

of $q(t)$ & $\dot{q}(t)$: $\delta_S q(t) = \epsilon \Delta(q, \dot{q}, t)$

ϵ is an arbitrary small parameter.

Under this transformation, one gets

$$\delta_S S = \int_{t_a}^{t_b} dt \underbrace{\left[\frac{\partial L}{\partial q} - d_t \frac{\partial L}{\partial \dot{q}(t)} \right]}_{\substack{\parallel \\ 0}} \delta_S q(t) + \frac{\partial L}{\partial \dot{q}(t)} \delta_S q(t) \Big|_{t_a}^{t_b}$$

$$= \epsilon \frac{\partial L}{\partial \dot{q}} \Delta(q, \dot{q}, t) \Big|_{t_a}^{t_b}$$

By definition, this transformation is a symmetry.

$\because$ $\delta_S S = 0$ for any orbit $q(t)$.

(or $\delta_S S = \int_{t_a}^{t_b} \frac{d\Lambda}{dt} dt$, we shall not discuss this case)

$\therefore$ For any orbit $q(t)$, $\frac{\partial L}{\partial \dot{q}} \Delta(q, \dot{q}, t) \Big|_{t=t_b}$

$$= \frac{\partial L}{\partial \dot{q}} \Delta(q, \dot{q}, t) \Big|_{t=t_a}$$

$\therefore$ $Q \equiv \frac{\partial L}{\partial \dot{q}} \Delta(q, \dot{q}, t)$ is independent of t_b

and is the conserved quantity

$\therefore$ $\frac{dQ}{dt} = 0$

As an example, $q'(t) = q(t) + \varepsilon$; $\delta q_S = \varepsilon$ is a translation. $Q \equiv -\frac{\partial L}{\partial \dot{q}} = m\dot{q} = -p$

After quantization, we find $\delta_S \hat{q} = -i\varepsilon[\hat{Q}, \hat{q}(t)]$.

$[\hat{p}, \hat{q}] = -i$

Another example: consider a Lagrangian of ϕ which is symmetric under

$$\phi \to \phi + \delta_S \phi$$

$$\therefore \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta_S \phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\mu \delta_S \phi$$

$$= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta_S \phi \right)$$

Euler-Lagrangian eq.

$$\therefore \delta S = \int d^d x \frac{d}{dx^\mu} \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta_S \phi \right) \quad \cdots (38)\text{-}1$$

$$= \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta_S \phi \Big|_{\text{boundary}}$$

$$\therefore \Theta = \int \frac{\partial \mathcal{L}}{\partial \partial_0 \phi} \delta_S \phi \, d^{d-1}x \quad \left\{ \frac{\partial \mathcal{L}}{\partial \partial_i \phi} \delta_S \phi \right\} \; i = 1, \ldots, d-1$$

is the conserved current

$\because \pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$ is the momentum density on the field

If we impose the quantization $[\pi(x), \phi(x')] = i\delta^{d-1}(x-y)$ at equal time,

$$[\hat{\Theta}, \hat{\phi}(x)] = \int d^{d-1}y \, [\pi(y), \phi(x)] \, \delta_S \phi(y)$$

$$= i \delta_S \hat{\phi}(x)$$

We see that $\hat{\Theta}$ generates $\delta_S \hat{\phi}$:

$$\delta \hat{\phi}_S = -i [\hat{\Theta}, \hat{\phi}] \quad \cdots (39)$$

(at equal time)

Hence the effect of the commutator of the conserved charge $\hat{Q}_T$ due to the symmetry transform T on any operator $\hat{A}$ is to generate the change of $\hat{A}$ due to the transformation T!

$$-i[\hat{Q}_T, \hat{A}] = \delta \hat{A}_T \quad \cdots \quad (40)$$

This is the result in quantum field at $(d-1)+1$. For coordinate transformation, eq. (34)-1 implies. $\partial_\mu J^\mu = 0$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu} \partial_\nu \phi - \mathcal{L}\, \delta^\mu_\nu \quad \text{for a given } \nu.$$

The conserved charge Q in this case is $Q = \int d^{d-1}x\, J^0$

In the Euclidean space, one transforms $(\vec{r}, ict)$ to $(\vec{r}, \tau)$ by $t \to -i\tau$

$\therefore\ \Theta \to -iQ$ $\therefore$ For d-dim Euclidean fields, one has

$$\delta \hat{A}_\Theta = [\hat{\Theta}, \hat{A}] \quad \cdots \quad (41)$$

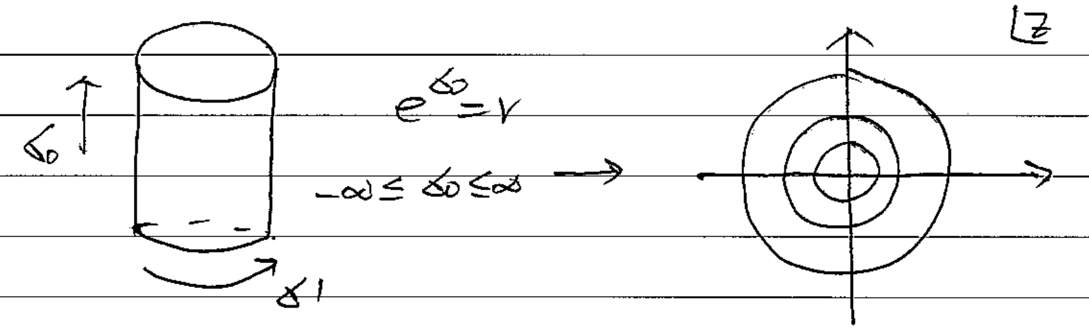
where ~~one~~ of dimensions is interpretated as t and eq. (41) is ~~at~~ evaluated equal coordinate of this dimension and $\hat{\Theta} = \int d^d x_{\perp} (\cdots)$ is evaluated at the remaining dimensions perpendicular to this dim.

<u>time-ordering</u> & <u>radial quantization</u>

For 2D, it's tempting to re-interpretate r as time and $0 \leq \theta < 2\pi$ as the remaining $d-1$ space dimension. However, the problem is $r \geq 0$, but $-\infty < t < \infty$. To include $t < 0$, one performs the so-called radial quantization in which one writes

$$z = e^{\sigma^0 + i\sigma^1} \qquad \sigma^0 = \text{"time"}$$
$$\sigma^1 = \text{"space"}$$

Hence one formally has the mapping



The "equal-time" commutator in eq. (4) is in fact "time ordered" with difference in time $\to 0$:

Consider a transformation $x'^{\mu} = x^{\mu} + w_a \frac{\delta x^{\mu}}{\delta w_a}$, w_a = infinitesimal parameters

which induces a change in $\Phi \to \Phi' = F(\Phi)$

resulting in $\delta\Phi = i\hat{G}_a\Phi \, w_a$ ← some matrices/operator

For any operator $\hat{A}$ that is function of Φ (e.g. $\phi(x_1)\phi(x_2) \ldots \phi(x_n)$), one gets

$$\langle \hat{A} \rangle = \frac{1}{Z} \int \mathcal{D}\phi \, [A + \delta A] \, e^{-(S+\delta S)}$$

$$\delta S = \int d^d x \int d\tau \, (\partial_\mu j_a^\mu) \, \omega_a$$

$$\delta \hat{A} = -i \sum_{i=1}^{n} \phi(x_1) \cdots G_a \phi(x_i) \cdots \phi(x_n) \, \omega_a(x_i)$$

$$= -i \int d^d x \, \omega_a(x) \sum_i i\,\delta(x - x_i) \left\{ \phi(x_1) \cdots G_a \phi(x_i) \cdots \phi(x_n) \right\}$$

$$\therefore \frac{\partial}{\partial x^\mu} \langle j_a^\mu(x) \, \phi(x_1) \cdots \phi(x_n) \rangle$$

$$= -i \sum_{i=1}^{n} \delta(x - x_i) \langle \phi(x_1) \cdots G_a \phi(x_i) \cdots \phi(x_n) \rangle \quad \text{--- (42)}$$

Let $Y = \phi(x_2) \cdots \phi(x_n)$ and $x_1^0 = t$ and integrate

Eq. (42) $\displaystyle\int_{t_- < t}^{t_+ > t} dx^0 \int d^{d-1} x, \quad \int \frac{\partial}{\partial x_j} \langle J_a^j(x) \cdots \rangle \, dx_j = 0$

we get

$$\therefore \langle \Theta_a(t_+) \phi(x_1) Y \rangle - \langle \Theta_a(t_-) \phi(x_1) Y \rangle$$

$$= -i \langle G_a \phi(x_1) Y \rangle \qquad \Theta_a = \int d^{d-1} x \, j_a^0(x) \quad \text{--- (43)}$$

$\displaystyle\int_{t_i < t}^{t_+ > t} dx^0 \int d^{d-1} x \, \delta(x - x_i) \neq 0$ only when $x_i = x_1$ $(x_i^0 = t)$

($\therefore$ only $i=1$ contributes)

$\therefore$ Eq. (43) $\Rightarrow \langle 0 | [\Theta_a, \phi(x_1)] Y | 0 \rangle = -i \langle 0 | G_a \phi(x_1) Y | 0 \rangle$

for any Y. $\quad \therefore [\Theta_a, \phi(x_1)] = -i G_a \phi(x_1) = \delta \phi(x_1)$

Conformal Ward identity at 2D & operator product.

At quantum mechanics level, conformal invariance has to be implemented according to eq. (41)

At 2D, following eq. (34)-1, we write

$$\delta S = \frac{-1}{2\pi}\int d^2x \, \partial^\mu \varepsilon^\nu T_{\mu\nu}$$

and consider some correlation function

$$\langle \phi_1 \cdots \phi_N \rangle = \frac{1}{Z}\int \mathcal{D}\phi \, e^{-S} \phi_1 \cdots \phi_N$$

The conformal invariance implies

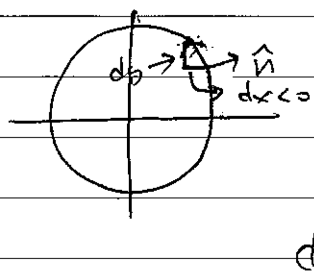
$$\sum_{i=1}^{N} \langle \phi_1 \cdots \delta\phi_i \cdots \phi_N \rangle + \langle -\delta S \, \phi_1 \cdots \phi_N \rangle = 0$$

$$\therefore \sum_{i=1}^{N} \langle \phi_1 \cdots \delta\phi_i \cdots \phi_N \rangle + \frac{1}{2\pi}\int d^2x \, \partial^\mu \varepsilon^\nu \langle T_{\mu\nu}(x) \phi_1 \cdots \phi_N \rangle = 0$$

— (44)

$\because \partial^\mu T_{\mu\nu} = 0$

$$\therefore \text{The 2nd term} = \frac{1}{2\pi}\oint dr \, \hat{n}_\mu \underbrace{\langle T_{\mu\nu}(x) \phi_1 \cdots \phi_N \rangle \varepsilon^\nu}_{= A^\mu}$$



$$dr\,\hat{n} = (-dy, dx) \rightarrow \varepsilon_{\mu\rho} dx^\rho$$

$$dr\,\hat{n}\cdot\vec{A} = -A_y dx + A_x dy = \frac{1}{2i}(dx + i\,dy)(A_x - iA_y) - \frac{1}{2i}(dx - i\,dy)(A_x + iA_y)$$

$\therefore$ Eq. 44 may be written as

$$\sum_{i=1}^{N} \langle \phi_1 \cdots \delta\phi_i \cdots \phi_N \rangle = \frac{1}{2\pi i}\Big[\oint dz\, A(z,\bar z)$$

$$+ \oint d\bar z\, A(z,\bar z) \Big] \quad -- \ 45$$

($d\bar z$ goes opposite to dz: $-\oint d\bar z = \oint dz$)

Since $\partial^\mu \varepsilon^\nu T_{\mu\nu} = \partial^\alpha \varepsilon^\beta T_{\alpha\beta}$, $\alpha, \beta = z\, \bar z$

$\varepsilon = \varepsilon(z)$ or $\bar\varepsilon(\bar z)$, $T_{z\bar z} = 0$, $T_{\bar z z} = 0$

$-2\pi T_{zz} = T(z)$, $-2\pi T_{\bar z \bar z} = \bar T(\bar z)$
↑(note the minus sign)

Eq. 45 becomes

$$\sum_{i=1}^{N} \langle \phi_1(z_1,\bar z_1) \cdots \delta\phi_i \cdots \phi_N(z_N,\bar z_N) \rangle$$

$$= \oint \frac{dz}{2\pi i}\, \epsilon(z) \langle T(z) \phi_1 \cdots \phi_N \rangle$$

$$+ \oint \frac{d\bar z}{2\pi i}\, \bar\epsilon(\bar z) \langle \bar T(\bar z) \phi_1 \cdots \phi_N \rangle \quad --- \ 45-1$$

Similarly, if one performs the same treatment as eqs 42 & 43, one gets, for $\phi(\omega, \bar\omega)$

$$\delta_{\varepsilon\bar\varepsilon}\, \phi(\omega,\bar\omega) = \frac{1}{2\pi i} \oint dz\, [\, T(z)\, \epsilon(z),\ \phi(\omega,\bar\omega)]$$

at $|z| = $ fixed. ("equal time") $\quad$ L-45-2

With $Q = \frac{1}{2\pi i} \oint \big(dz\, \varepsilon(z) T(z) + d\bar z\, \bar\varepsilon(\bar z) \bar T(\bar z) \big)$

Now, according to the derivation of eq. (43),
one replaces the commutator

$$[T(z)\epsilon(z), \Phi(\omega,\bar{\omega})]$$

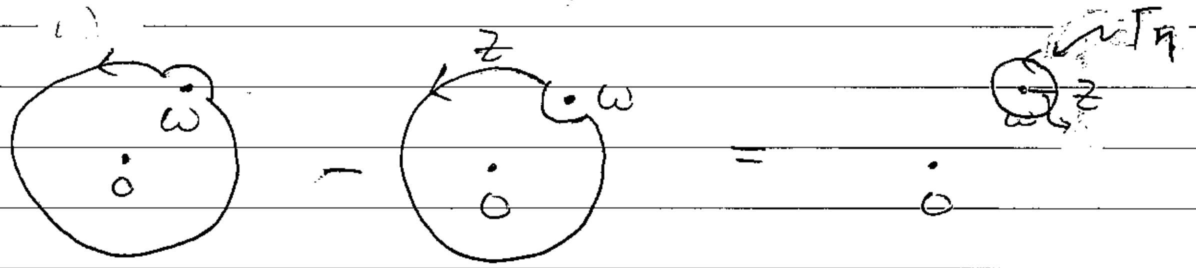
by $T(z)\epsilon(z)\Phi(\omega,\bar{\omega})\Big|_{|z|>|\omega|} - \Phi(\omega,\bar{\omega})T(z)\epsilon(z)\Big|_{|z|<|\omega|}$

$$\therefore \delta_{\epsilon,\bar{\epsilon}}\Phi(\omega,\bar{\omega}) = \frac{1}{2\pi i}\left(\oint_{|z|>|\omega|} - \oint_{|z|<|\omega|}\right)\Big(dz\,\epsilon(z)\,R\big(T(z)\Phi(\omega,\bar{\omega})\big) + d\bar{z}\,\bar{\epsilon}(\bar{z})\,R\big(\bar{T}(\bar{z})\Phi(\omega,\bar{\omega})\big)\Big) \quad \text{--- (46)}$$

where R is the radial ordering operator defined as

$$R[A(z)B(\omega)] = \begin{cases} A(z)B(\omega) & |z|>|\omega| \\ B(\omega)A(z) & |\omega|>|z| \end{cases}$$

The contour of z in eq. (46) can be chosen to avoid ω ($\because$ possible singularity may occur at $\omega=z$) in the following way:



$$\therefore \delta_{\epsilon,\bar{\epsilon}}\Phi(\omega,\bar{\omega}) = \frac{1}{2\pi i}\oint_{\Gamma_\eta}\Big\{dz\,\epsilon(z)R[T(z)\Phi(\omega,\bar{\omega})] + d\bar{z}\,\bar{\epsilon}(\bar{z})R[\bar{T}(\bar{z})\Phi(\omega,\bar{\omega})]\Big\} \quad \text{--- (47)}$$

By using eq. (30), one arrives at

$$h\partial\epsilon\,\phi(\omega,\bar\omega) + \epsilon(\omega)\partial_\omega\phi(\omega,\bar\omega)$$
$$+\bar h\bar\partial\bar\epsilon\,\phi(\omega,\bar\omega) + \bar\epsilon\bar\partial\phi(\omega,\bar\omega)$$
$$= \frac{1}{2\pi i}\oint_{\Gamma_\eta}\Big\{dz\,\varepsilon(z)R[T(z)\phi(\omega,\bar\omega)]$$
$$+ d\bar z\,\bar\varepsilon(\bar z)R[\bar T(\bar z)\phi(\omega,\bar\omega)]\Big\} \quad \cdots (48)$$

Using the fact that $\oint\frac{dz}{2\pi i}\frac{f(\omega)}{z-\omega} = f(z)$

↑ simple pole at ω

$$\oint\frac{dz}{2\pi i}\frac{f(\omega)}{(z-\omega)^2}$$
$$= \oint\frac{dz}{2\pi i}\frac{f(\omega)+(z-\omega)f'(\omega)+\cdots}{(z-\omega)^2}$$
$$= f'(\omega),$$

One concludes that as $z\to\omega$

$$R[T(z)\phi(\omega,\bar\omega)] = \frac{h}{(z-\omega)^2}\phi(\omega,\bar\omega) + \frac{1}{z-\omega}\partial_\omega\phi(\omega,\bar\omega)$$
$$+ \text{regular terms}\ \cdots$$

$$R[\bar T(\bar z)\phi(\omega,\bar\omega)] = \frac{\bar h}{(\bar z-\bar\omega)^2}\phi(\omega,\bar\omega) + \frac{1}{\bar z-\bar\omega}\partial_{\bar\omega}\phi(\omega,\bar\omega)$$
$$+ \text{regular terms}\ \cdots (49)$$

↑ we shall drop this notation

These are examples of so-called operator product expansions but they are for primary fields.

In general, singularities that occur when operators approach one another and are encoded in operator product expansion of the form

$$O_1(x) O_2(y) \sim \sum_i C_i(x-y)\, O_i(y),$$

where O_i's are a complete set of local operators. C_i's are numerical factors.

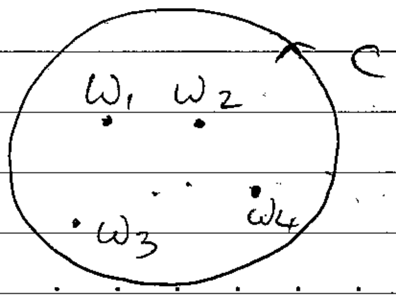
This is usually an asymptotic expansion but it has argued to be converge for a conformal theory.

By dimensional analysis, one has $C_i \sim \frac{1}{|x-y|^{d_1+d_2-d_i}}$

<u>Conformal Ward identities</u>

We can extend the above analysis for one field to many fields at $w_1, w_2, \ldots, w_N$.

By considering a contour C that includes all $w_1, w_2, \ldots, w_N$, the radial ordering of



$$R[T(z), (\phi_1, \ldots, \phi_N)]$$
$$= T(z)\phi_1 \cdots \phi_N$$

∴ Eq. 45-1 is correct. If we combine eq. 30 & eq. 45-1 (48 & 49), we get

$$\sum_{i=1}^{N} \oint \frac{dz}{2\pi i} \left[\frac{h_i}{(z-w_i)^2} + \frac{1}{z-w_i} \frac{\partial}{\partial w_i} \right]^{\epsilon(z)} \langle \phi_1(w_1, \bar{w}_1) \cdots \phi_N(w_N, \bar{w}_N) \rangle$$

$$+ \oint \frac{d\bar{z}}{2\pi i} \left[\frac{\bar{h}_i}{(\bar{z}-\bar{w}_i)^2} + \frac{1}{\bar{z}-\bar{w}_i} \frac{\partial}{\partial \bar{w}_i} \right]^{\bar{\epsilon}(\bar{z})} \langle \cdots \rangle$$

$$= \oint \frac{dz}{2\pi i} \epsilon(z) \langle T(z) \phi_1 \cdots \phi_N \rangle$$

$$+ \oint \frac{d\bar{z}}{2\pi i} \bar{\epsilon}(\bar{z}) \langle \bar{T}(\bar{z}) \cdots \rangle \quad \text{--- } 49\text{-}1$$

We get the conformal Ward identities:

$$\langle T(z) \phi_1(w_1, \bar{w}_1) \cdots \phi_N(w_N, \bar{w}_N) \rangle$$

$$= \sum_{i=1}^{N} \left[\frac{h_i}{(z-w_i)^2} + \frac{1}{z-w_i} \frac{\partial}{\partial w_i} \right] \langle \phi_1(w_1, \bar{w}_1) \cdots \phi_N(w_N, \bar{w}_N) \rangle \quad \text{L-50}$$

The central charge and the Virasoro algebra

Not all the fields satisfy eq. 28 as primary field. For instance, taking $\frac{\partial}{\partial w}$ 28, one gets

$$\partial_w \phi'(w, \bar{w}) = \frac{\partial}{\partial z} \left[\left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}) \right] \Big/ \frac{dz}{dw}$$

which does not lead to

$$\partial_w \phi'(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-h'} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}'} \partial_z \phi(z, \bar{z})$$

Hence $\partial_z \phi'$ is not a primary field

However, under a scale transformation, $\partial_z \phi$ scales if ϕ scales

This applies to the energy-momentum tensor $T(z)$ as well.

Since the theory is scale invariant,

$\delta S = \int T^{\mu\nu} \partial_\mu \varepsilon^\nu$ $\therefore$ $T_{\mu\nu}$ transforms covariantly under scale transformations
$(h=2, \bar{h}=2)$ $(T(\lambda z) = \lambda^2 T(z))$

$\therefore$ Translation invariance implies

$$\langle T(z) \rangle = 0, \quad \langle T(z) T(\omega) \rangle = \frac{c/2}{(z-\omega)^4}$$

The most general form of $T(z)T(\omega)$ as $\omega \to z$

is thus :
$$T(z)T(\omega) = \frac{c/2}{(z-\omega)^4} + \frac{2}{(z-\omega)^2} T(\omega) + \frac{1}{z-\omega} \partial T(\omega) + \text{regular} \quad \text{--- (57)}$$

c is known as the central charge or conformal anomaly (not present in classical field)
Eq. (57) implies that $T(z)$ is not a primary field as it does not follow eq. (49) due to the presence of the central charge term.

Similarly,
$$\bar T(\bar z)\bar T(\bar w) = \frac{\bar c/2}{(\bar z-\bar w)^4} + \frac{2}{(\bar z-\bar w)^2}\bar T(\bar w) + \frac{1}{\bar z-\bar w}\bar\partial \bar T(\bar w)$$
$$+\text{regular}.$$

Since according to eq. (49),

$$\delta_\epsilon T(z) = \frac{1}{2\pi i}\oint_{\Gamma_z} dz\, \varepsilon(z)\, T(z)\, T(\omega),$$

eq. (51) implies $\left(\varepsilon(z) = \varepsilon(\omega) + (z-\omega)\varepsilon'(\omega) + \frac{1}{2!}(z-\omega)^2\varepsilon''(\omega) + \frac{1}{3!}(z-\omega)^3\varepsilon'''(\omega) + \cdots \right)$

$$\delta_\epsilon T(z) = \epsilon(z)\partial T(z) + 2\partial\epsilon(z)\, T(z)$$
$$+ \frac{c}{12}\partial^3\epsilon(z) \quad \cdots\cdots (52)$$

$\therefore$ For $\epsilon(z) = 1, z, z^2$ (i.e. global conformal transformation), eq. (52) implies $T(z)$ transforms like a primary field, but is not and is quasi-primary!

Eq. (52) indicates that for finite transformation $z \to f(z)$, since $\delta_\epsilon T(z) = T'(z) - T(z)$

the first two terms $\to (\partial f)^2 T(f(z))$ $\left(T'(z) = \left(\frac{df}{dz}\right)^h T(f) \right)$

$$\therefore \underset{(T'(z))}{T(z)} = (\partial f)^2 T(f(z)) + \underbrace{\frac{c}{12}\{f; z\}}_{\text{independent of } T!}$$

$$\therefore\ T'(\omega) = \left(\frac{d\omega}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{\omega; z\}\right] \quad \cdots (13)$$

Since two successive transformation $z \to \omega \to u$ is equivalent to $z \to u$, one requires

$$T''(u) = \left(\frac{du}{d\omega}\right)^{-2}\left[T'(\omega) - \frac{c}{12}\{u, \omega\}\right]$$

$$= \left(\frac{du}{d\omega}\right)^{-2}\left[\left(\frac{d\omega}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{\omega; z\}\right] - \frac{c}{12}\{u; \omega\}\right]$$

$$\equiv \left(\frac{du}{dz}\right)^{-2}\left(T(z) - \frac{c}{12}\{u; z\}\right)$$

$$\therefore\ \{u; z\} = \{\omega; z\} + \left(\frac{d\omega}{dz}\right)^2 \{u; \omega\} \quad \cdots (13)\text{-}1$$

It turns out that

$$\{f; z\} = \frac{\partial_z f\, \partial_z^3 f - \frac{3}{2}(\partial_z^2 f)^2}{(\partial_z f)^2} \quad \left(= \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2\right)$$

satisfies eq (13) and is known as the Schwarzian derivative (which vanishes for any global conformal transformation $f = \frac{az+b}{cz+d}$)

Virasoro algebra

It's useful to ~~we~~ perform the Laurent expansion w.r.t. z_0 for T & $\bar{T}$,

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{1}{z^{n+2}} L_n, \quad \bar{T}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{1}{\bar{z}^{n+2}} \bar{L}_n \quad \cdots (54)$$

(Note that w.r.t. ω $\ T(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-\omega)^{n+2}} L_n, \ \bar{T}(\bar{z}) = \sum_n \frac{1}{(\bar{z}-\bar{\omega})^{n+2}} \bar{L}_n$)

We have

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z) \quad \bar{L}_n = \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{n+1} \bar{T}(\bar{z})$$

$\{L_n, \bar{L}_n\}$ are similiar to the classical generators $l_n, \bar{l}_n$ defined in eg. (28)-3 to generate local conformal transformation.

The exponent $n+2$ is choosen so that under the scale translation, $T(\lambda z) = \lambda^2 T(z)$.

$$\therefore \sum_n \frac{1}{\lambda^{n+2}} \frac{1}{z^{n+2}} L_n(\lambda z) = \sum_n \frac{1}{z^{n+2}} L_n(z)$$

$$\therefore L_n(\lambda z) = \lambda^n L_n(z)$$

$\therefore$ L_{-n}'s scaling dimension $= n$.

Similiar to $\{l_n, \bar{l}_n\}$, $\{L_n, \bar{L}_n\}$ also form an algebra that generalizes $\{l_n, \bar{l}_n\}$'s to the quantum level.

To calculate $[L_n, L_m]$, we note that

$$[L_n, L_m] = \left(\oint \frac{dz}{2\pi i} \oint \frac{d\omega}{2\pi i} - \oint \frac{d\omega}{2\pi i} \oint \frac{dz}{2\pi i}\right) z^{n+1} T(z)\, \omega^{m+1} T(\omega)$$

$$= \left(\oint \frac{dz}{2\pi i} \oint \frac{d\omega}{2\pi i} - \oint \frac{d\omega}{2\pi i} \oint \frac{dz}{2\pi i}\right) z^{n+1} \omega^{m+1} \left[\frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega} + \cdots\right]$$

The integrals can be performed by fixing ω first and deforming the z integration around ω just as eg. (47), the difference due to commutator the integrals done that lead to.

leads to an integral of z around ω

Since $\oint \frac{dz}{2\pi i} z^{n+1} \omega^{m+1} \left[\frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega} + \cdots \right]$

$$= \frac{1}{3!} \frac{d^3 z^{n+1}}{dz^3} \Big|_{z=\omega} \cdot \frac{c}{2} \cdot \omega^{m+1}$$

$$+ 2 \frac{dz^{n+1}}{dz} \Big|_{z=\omega} \omega^{m+1} T(\omega) + \omega^{n+1} \omega^{m+1} \partial T(\omega)$$

$$= \frac{c}{12}(n+1) n (n-1) \omega^{n+m-1} + 2(n+1) \omega^{n+m+1} T(\omega)$$

$$+ \omega^{n+m+2} \partial T(\omega),$$

we obtain

$$[L_n, L_m] = \oint \frac{d\omega}{2\pi i} \left\{ \frac{c}{12} (n+1) n (n-1) \omega^{n+m-1} \right.$$

$$\left. + 2(n+1) \omega^{n+m+1} T(\omega) + \omega^{n+m+2} \partial T(\omega) \right\}$$

By integration by parts, the last term

$$= \oint \frac{d\omega}{2\pi i} -(n+m+2) \omega^{n+m+1} T(\omega)$$

$$\therefore [L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12}(n^3 - n) \delta_{n+m,0} \quad \text{--- } (55)$$

Similarly, $[\bar{L}_n, \bar{L}_m] = (n-m) \bar{L}_{n+m} + \frac{\bar{c}}{12}(n^3 - n) \delta_{n+m,0}$ --- (56)

On the other hand, $[L_n, \bar{L}_m] = 0$ --- (57)

The two copies of an infinite dimensional algebra are called Virasoro algebra.

One sees that every conformally invariant quantum field theory determines a representation of this algebra with some value of c & $\bar{c}$.
When $c=\bar{c}=0$, they reduce to the classical algebra.

The global conformal group $SL(2,C)$ is generated by $L_{\pm 1}, L_0, \bar{L}_{\pm 1}, \bar{L}_0$

$$[L_{\mp 1}, L_0] = \mp L_{\mp 1}$$

$$[L_1, L_{-1}] = 2L_0$$

which forms a closed set and is a subalgebra of Virasoro algebra

Examples — Free massless boson in 2D

$$S = \frac{1}{2} g \int d^2x\, \partial_\mu \phi\, \partial^\mu \phi \quad (\text{massless})$$

To solve this system, it's useful to note that

$$\delta^2(\vec{r}) = \frac{1}{\pi} \partial_{\bar{z}} \frac{1}{z} = \frac{1}{\pi} \partial_z \frac{1}{\bar{z}} \quad \cdots \quad (57)\text{-}1$$

As we have shown in deriving eq (45),

$$\int d^2x\, \partial_\mu F^\mu = \oint dr\, \hat{n}_\mu F^\mu = \oint dx^\rho\, \varepsilon_{\mu\rho} F^\mu$$
$$= -\oint dx^\alpha\, \varepsilon_{\alpha\beta} F^\beta$$

No.
Date.

$$\varepsilon_{\alpha\beta} = \frac{1}{4}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}$$

$$\therefore \int d^2x\, \partial_\mu F^\mu = \oint dz\, \varepsilon_{\bar{z}z} F^{\bar{z}} + \oint d\bar{z}\, \varepsilon_{z\bar{z}} F^{z}$$

$$= \frac{-i}{2}\oint dz\, F^{\bar{z}} + \frac{i}{2}\oint d\bar{z} F^{z}$$

Therefore $\int d^2x\, \delta^2(\vec{r}) f(z)$

$$= \int d^2x\, f(z) \frac{1}{\pi} \partial_{\bar{z}} \frac{1}{z}$$

$F^{\bar{z}}$ (is holomorphic)

$$= \frac{1}{\pi}\int d^2x\, \partial_{\bar{z}} \left(\frac{f(z)}{z}\right)$$

$$= \frac{-i}{2\pi}\oint dz \frac{f(z)}{z}$$

$$= \frac{1}{2\pi i}\oint dz \frac{f(z)}{z} \underset{\text{Contour integral}}{=} f(0)$$

$\therefore\ \delta^2(\vec{r}) = \frac{1}{\pi}\partial_{\bar{z}} \frac{1}{z}$. Similarly, by considering $\delta^2(\vec{r}) f(\bar{z})$, one finds $\delta^2(\vec{r}) = \frac{1}{\pi}\partial_z \frac{1}{\bar{z}}$ as well. --- (5f)

By integration by parts, one can rewrite

$$S = \frac{1}{2}\int d^2x\, \phi(-g\,\partial_\mu\partial^\mu)\phi$$

Now, $\langle\phi(x)\phi(y)\rangle = (g\,\partial_\mu\partial^\mu)^{-1}$

i.e. $-g\partial^2\langle\phi(x)\phi(y)\rangle = \delta^2(x-y)$

Now, $\because -g\,\partial_\alpha \partial^\alpha = -g\,\partial_\alpha g^{\alpha\beta}\partial_\beta = -2g(\partial_z\partial_{\bar z} + \partial_{\bar z}\partial_z)$.

Using eq. (*), we find

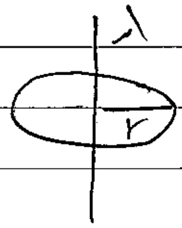
$$-2g(\partial_z\partial_{\bar z} + \partial_{\bar z}\partial_z)\frac{-1}{4\pi g}\ln z = \frac{1}{2}\frac{1}{\pi}\partial_{\bar z}\frac{1}{z} = \frac{1}{2}\delta^2(\vec r)$$

$$-2g(\partial_z\partial_{\bar z} + \partial_{\bar z}\partial_z)\frac{-1}{4\pi g}\ln \bar z = \frac{1}{2\pi}\partial_z\frac{1}{\bar z} = \frac{1}{2}\delta^2(\vec r)$$

$$\therefore \langle\phi(x)\phi(y)\rangle = \frac{-1}{4\pi g}(\ln z + \ln\bar z) + \text{const}$$

$$= \frac{-1}{4\pi g}\ln(x-y)^2 + \text{const}.$$

Note that this problem can be also solved in an elementary way by noting that it is equivalent to find electric potential $\phi(r)$ of a line charge at $\vec r = 0$. with $\lambda = \epsilon_0/g$



$$E\cdot 2\pi r = \frac{\lambda}{\epsilon_0} = \frac{1}{g}$$

$$E = \frac{1}{2\pi g}\frac{1}{r} \qquad \int_r^\infty E\,dr = \int_r^\infty -d\phi$$

$$= \phi(r) + \text{const}.$$

$$\therefore \phi(r) = \frac{-1}{2\pi g}\ln r + \text{const} = \frac{-1}{4\pi g}\ln r^2 + \text{const}$$

$$\therefore \langle\phi(x)\phi(y)\rangle = \frac{-1}{4\pi g}\ln(x-y)^2 + \text{const}.$$

In the complex notations, one writes --- (59)

$$\langle\phi(z,\bar z)\,\phi(\omega,\bar\omega)\rangle = \frac{-1}{4\pi g}\{\ln(z-\omega) + \ln(\bar z-\bar\omega)\} + \text{const}$$

$$\therefore \langle\partial_z\phi(z,\bar z)\,\partial_\omega\phi(\omega,\bar\omega)\rangle = -\frac{1}{4\pi g}\frac{1}{(z-\omega)^2}$$

$$\langle\partial_{\bar z}\phi(z,\bar z)\,\partial_{\bar\omega}\phi(\omega,\bar\omega)\rangle = -\frac{1}{4\pi g}\frac{1}{(\bar z-\bar\omega)^2} \quad \text{--- (60)}$$

$\therefore \langle \partial_z\phi\, \partial_\omega\phi\rangle$ only involves z & ω

we shall write $\partial_z\phi(z)$ & $\partial_\omega\phi(\omega)$

We now evaluate $T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi}\partial_\nu\phi - \mathcal{L}\delta_{\mu\nu}$

with $\mathcal{L} = \frac{g}{2}\partial_\rho\phi\partial^\rho\phi$

$$\therefore T_{\mu\nu} = g :(\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}\delta_{\mu\nu}\,\partial_\rho\phi\,\partial^\rho\phi):$$

normal ordering

$\langle T_{\mu\nu}\rangle = 0$

$$T_{\alpha\beta} = \zeta^\mu_\alpha\,\zeta^\nu_\beta\,T_{\mu\nu}$$

$$= g:(\partial_\alpha\phi\,\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}\,\partial_{\bar\alpha}\phi\, g^{\bar\alpha\bar\beta}\partial_{\bar\beta}\phi):$$

$$g_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0\end{pmatrix} \qquad g^{\alpha\beta} = \begin{pmatrix} 0 & 2 \\ 2 & 0\end{pmatrix}$$

$$\therefore T_{zz} = g:\partial_z\phi\,\partial_z\phi:$$

$$T_{\bar z\bar z} = g:\partial_{\bar z}\phi\,\partial_{\bar z}\phi:$$

$$T_{z\bar z} = g:\partial_z\phi\,\partial_{\bar z}\phi - \frac{1}{2}\times\frac{1}{2}(\partial_z\phi\, g^{z\bar z}\partial_{\bar z}\phi + \partial_{\bar z}\phi\, g^{\bar z z}\partial_z\phi): = 0$$

$$T_{\bar z z} = 0$$

$$\therefore T(z) = -2\pi T_{zz} = -2\pi g :\partial_z\phi\,\partial_z\phi: \quad \text{---(60)}$$

$$= -2\pi g \lim_{\omega\to z}\left(\partial_z\phi(z,\bar z)\partial_\omega\phi(\omega,\bar\omega) - \langle\partial_z\phi\,\partial_\omega\phi\rangle\right)$$

We shall need to compute the operator expansion of $T(z)\,\partial_\omega\phi(\omega,\bar\omega)$:

$$R[T(z)\,\partial_\omega\phi(\omega,\bar\omega)]$$

$$= -2\pi g\, R[:\partial_z\phi\,\partial_z\phi:\,\partial_\omega\phi(\omega,\bar\omega)] \quad \text{---(61)}$$

We shall omit $R[\cdots]$ and use the

Wick's theorem: time-order product

= normal ordered product

+ all possible contraction.

$$\begin{array}{c} \shortparallel \\ \langle\ \rangle \end{array}$$

$$R[\phi_1\phi_2\phi_3\phi_4\cdots]$$

$$= :\phi_1\phi_2\phi_3\phi_4\cdots: + :\overbrace{\phi_1\phi_2}\phi_3\phi_4\cdots:$$

$$+ :\phi_1\overbrace{\phi_2\phi_3}\phi_4\cdots: + :\phi_1\overbrace{\phi_2\phi_3\phi_4}\cdots: + \cdots$$

$\therefore$ $T(z)\partial_\omega\phi(\omega,\bar{\omega})$

$$= -2\pi g :\partial_z\phi(z,\bar{z})\partial_z\phi(z,\bar{z})\partial_\omega\phi(\omega,\bar{\omega}):$$

$\langle\cdots\rangle = 0$ ↳ regular omit

$$= -4\pi g\ \overbrace{\partial_z\phi(z,\bar{z})\partial_\omega\phi(\omega,\bar{\omega})}\ :\partial_z\phi(z,\bar{z}):$$

$+$ 2 possible contraction

$$\underset{(6.0)}{=} \frac{1}{(z-\omega)^2}\partial_z\phi(z,\bar{z})$$

$$\underset{\text{expand w.r.t. } \omega}{=} \frac{1}{(z-\omega)^2}\partial_\omega\phi(\omega,\bar{\omega}) + \frac{1}{(z-\omega)}\partial_\omega^2\phi(\omega,\bar{\omega}) + \cdots$$

$\therefore$ $\partial_\omega\phi(\omega,\bar{\omega})$ is a primary field with conformal dimension $h=1$

(expected as ϕ has no spin and no scaling dimension, $\therefore$ it's derivative has scaling dimension 1) need to be radial ordered

$$T(z)\,T(\omega) = 4\pi^2 g^2 \, \overbrace{:\partial_z\phi\,\partial_z\phi::\partial_\omega\phi\,\partial_\omega\phi:}$$

$$= 4\pi^2 g^2 :\partial_z\partial_z\phi\,\partial_\omega\phi\,\partial_\omega\phi: \leftarrow \text{regular}$$

$$+ 4\pi^2 g^2 \, \partial_z\phi(z)\partial_\omega\phi(\omega) \; \partial_z\phi(z)\partial_\omega\phi(\omega) \times 2$$

$$+ 4\pi^2 g^2 \, 4\times \partial_z\phi\,\partial_\omega\phi \; :\partial_z\phi\,\partial_\omega\phi:$$

~~$(\partial_z\phi\,\partial_z\phi$ has been subtracted in $:\partial_z\phi\,\partial_z\phi:)$~~

$$\because \partial_z\phi(z)\partial_\omega\phi(\omega) = -\frac{1}{4\pi g}\frac{1}{(z-\omega)^2}$$

$$\therefore T(z)\,T(\omega) = \frac{4\pi^2 g^2 \times 2 \times \frac{1}{16\pi^2 g^2}}{(z-\omega)^4} - 4\pi g \frac{:\partial_z\phi\,\partial_\omega\phi:}{(z-\omega)^2}$$

$$= \frac{1/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial_\omega T(\omega)}{(z-\omega)} \quad \text{which}$$

$z\to\omega$ & expand $\partial_z\phi = \partial_\omega\phi + (z-\omega)\partial_\omega^2\phi$ agrees with eq. (5-1)

$$:\partial_\omega^2\phi\,\partial_\omega\phi: = \frac{1}{2}\partial_\omega :\partial_\omega\phi\,\partial_\omega\phi:$$

$\therefore c = 1$. Similarly, one finds $\bar{c} = 1$.

Example — The Free massless Majorana fermion in 2D

$$S = \frac{1}{2} g \int d^2x \, \psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \psi$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^1 = i\begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix}$$

$$\gamma^0\partial_0 + \gamma^1\partial_1 = \begin{pmatrix} 0 & \partial_x - i\partial_y \\ \partial_x + i\partial_y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2\partial_z \\ 2\partial_{\bar z} & 0 \end{pmatrix}$$

$$\gamma^0(\gamma^0\partial_0 + \gamma^1\partial_1) = \begin{pmatrix} 2\partial_{\bar z} & 0 \\ 0 & 2\partial_z \end{pmatrix}$$

For Majorana fermions $\psi_\alpha^+ = \psi_\alpha$, one gets

$$S = g\int d^2x \left(\bar\psi \partial_z \bar\psi + \psi \partial_{\bar z} \psi\right) = \frac{1}{2}\left(2g\int \cdots\right) \quad \text{---} (61)$$

where one associates $\partial_{\bar z}$ with ψ, ∂_z with $\bar\psi$

so that classical equations of motions become

$$\frac{\delta S}{\delta \psi} = 0 \Rightarrow \partial_{\bar z}\psi = 0, \quad \psi = \psi(z)$$

$$\frac{\delta S}{\delta \bar\psi} = 0 \Rightarrow \partial_z \bar\psi = 0. \quad \bar\psi = \bar\psi(\bar z)$$

The correlation functions are inverse to (eq (61))

$$2g\begin{pmatrix} \partial_{\bar z} & 0 \\ 0 & \partial_z \end{pmatrix}.$$

$$\therefore 2g\begin{pmatrix} \partial_{\bar z} & 0 \\ 0 & \partial_z \end{pmatrix}\begin{pmatrix} \langle\psi(z,\bar z)\psi(\omega,\bar\omega)\rangle & \langle\psi(z,\bar z)\bar\psi(\omega,\bar\omega)\rangle \\ \langle\bar\psi(z,\bar z)\psi(\omega,\bar\omega)\rangle & \langle\bar\psi(z,\bar z)\bar\psi(\omega,\bar\omega)\rangle \end{pmatrix}$$

$$= \delta^2(\vec r)$$

Using eq. (17)-1, one gets

$$\langle\psi(z,\bar z)\psi(\omega,\bar\omega)\rangle = \frac{1}{2\pi g}\frac{1}{z-\omega}$$

$$\langle\bar\psi(z,\bar z)\bar\psi(\omega,\bar\omega)\rangle = \frac{1}{2\pi g}\frac{1}{\bar z-\bar\omega} \quad \text{---} (62)$$

$$\langle\psi(z,\bar z)\bar\psi(\omega,\bar\omega)\rangle = 0$$

$$\therefore \langle\partial_z\psi(z,\bar z)\psi(\omega,\bar\omega)\rangle = -\frac{1}{2\pi g}\frac{1}{(z-\omega)^2} \quad \text{---} (63)$$

$$\langle\partial_z\psi(z,\bar z)\partial_\omega\psi(\omega,\bar\omega)\rangle = -\frac{1}{\pi g}\cdot\frac{1}{(z-\omega)^3}.$$

To find the energy-momentum tensor in the complex variables, we go back to

$$T_\nu^{\ \mu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^\lambda} \partial_\nu \varphi^\lambda - \mathcal{L}\,\delta_\mu^\nu$$

$$\left(T^{\nu\mu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \partial^\nu \varphi - \mathcal{L}\,\delta^{\mu\nu} \right)$$

Applying the transformation matrix S, one can write $T_\alpha^{\ \beta} = \frac{\partial \mathcal{L}}{\partial \partial_\beta \varphi} \partial_\alpha \varphi - \mathcal{L}\, g_\alpha^\beta$

$$\therefore\ T_{\alpha\beta} = g_{\alpha\gamma} T^\gamma_{\ \beta}$$

$$= g_{\alpha\gamma} \frac{\partial \mathcal{L}}{\partial \partial_\gamma \varphi} \partial_\beta \varphi - g_{\alpha\beta} \mathcal{L}$$

$$\mathcal{L} = g(\bar{\varphi}\partial_z \bar{\varphi} + \varphi \partial_{\bar{z}} \varphi) \qquad g_{\alpha\gamma} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\therefore\ T_{zz} = \frac{1}{2} :\frac{\partial \mathcal{L}}{\partial \partial_{\bar{z}} \varphi} \partial_z \varphi: \ = \frac{g}{2} : \varphi \partial_z \varphi:$$

(there is no $\partial_{\bar{z}} \bar{\varphi}$, $\therefore$ no $\bar{\varphi}$ involved)

Similarly, $T_{\bar{z}\bar{z}} = \frac{1}{2} : \frac{\partial \mathcal{L}}{\partial \partial_z \bar{\varphi}} \partial_{\bar{z}} \bar{\varphi} : \ = \frac{g}{2} : \bar{\varphi} \partial_{\bar{z}} \bar{\varphi}:$

Note that

$$T_{z\bar{z}} = g_{z\bar{z}} \frac{\partial \mathcal{L}}{\partial \partial_{\bar{z}} \varphi} \partial_{\bar{z}} \varphi - \frac{1}{2}\mathcal{L}$$

$$= \frac{1}{2} g\, \varphi \partial_{\bar{z}} \varphi - \frac{1}{2}\mathcal{L} = -\frac{g}{2} \bar{\varphi} \partial_z \bar{\varphi} \neq 0$$

This is due to $T_{\mu\nu} \neq T_{\nu\mu}$ and can be removed by adding appropriate gradient terms to $T_{\mu\nu}$. We shall not worry about it here.

From T_{zz}, we get .

$$T(z) = -2\pi T_{zz} = -\pi g :\psi(z)\,\partial_z\psi(z):$$

$$T(z)\psi(w,\bar{w}) = -\pi g :\psi(z)\,\partial_z\psi(z): \psi(w,\bar{w})$$

$$= -\pi g\, \psi(z)\,\partial_z\psi(z)\,\psi(w,\bar{w})$$

$$+\pi g\, \partial_z\psi(z)\,\psi(z,\bar{z})\,\psi(w,\bar{w})$$

$$= \frac{1}{2}\frac{\psi(z)}{(z-w)^2} + \frac{1}{2}\frac{\partial_z\psi(z)}{z-w}$$

eqs. 62 & 63

$$= \frac{\frac{1}{2}\psi(w) + \frac{1}{2}(z-w)\,\partial_w\psi(w,\bar{w})}{(z-w)^2} + \frac{1}{2}\frac{\partial_w\psi(w)}{z-w} + \ldots$$

expand w.r.t. w

$$= \frac{\frac{1}{2}\psi(w,\bar{w})}{(z-w)^2} + \frac{\partial_w\psi(w,\bar{w})}{z-w}$$

∴ $h = 1/2$. Similarly, by considering $\bar{T}(\bar{z})$, one gets $\bar{h} = 1/2$.

Now, to calculate $T(z)T(w) = \pi^2 g^2 :\psi\partial_z\psi::\psi(w)\partial_w\psi(w):$, we can compute

(i) two contractions

$$= -\pi^2 g^2\left[\psi(z)\psi(w)\,\partial_z\psi(z)\,\partial_w\psi(w) - \psi(z)\,\partial_w\psi(w)\,\partial_z\psi(z)\psi(w)\right]$$

$$= \pi^2 g^2 \frac{1}{2\pi^2 g^2}\frac{1}{(z-w)(z-w)^3} + \pi^2 g^2 \frac{1}{2\pi g(z-w)^2}\,\frac{-1}{2\pi g}\,\frac{1}{(z-w)^2}$$

using eqs 62 & 63

$$= \frac{\frac{1}{4}}{(z-w)^4}$$

(ii) one contraction

$$= \pi^2 g^2 :\psi(z)\partial_w\psi(\omega): \partial_z\psi(z)\,\psi(\omega)$$

$$+\pi^2 g^2 :\partial_z\psi(z)\,\psi(\omega): \psi(z)\,\partial_w\psi(\omega)$$

$$-\pi^2 g^2\ \psi(z)\psi(\omega) :\partial_z\psi(z)\,\partial_w\psi(\omega):$$

$$-\pi^2 g^2 :\psi(z)\psi(\omega): \partial_z\psi(z)\,\partial_w\psi(\omega)$$

1st term $= -\frac{\pi g}{2} :\psi(z)\partial_w\psi(\omega) \frac{1}{(z-w)^2}$

$$= -\frac{\pi g}{2}\,\frac{:\psi(\omega)\partial_w\psi(\omega): + (z-w):\partial_w\psi(\omega)\partial_w\psi(\omega):}{(z-w)^2} + \cdots$$

$$= \frac{1}{2}\,\frac{T(\omega)}{(z-w)^2} + \frac{1}{2}\,\frac{-\pi g\ \underbrace{:\partial_w\psi(\omega)\partial_w\psi(\omega):}_{0\ \text{Fermions}}}{z-w} + \cdots$$

2nd term $= \frac{\pi g}{2}\,\frac{:\partial_z\psi(z)\,\psi(\omega):}{(z-w)^2}$

$:\partial_w\psi(\omega)\,\psi(\omega):$ $= -:\psi(\omega)\partial_w\psi:$

$$= \frac{\pm 1}{2}\,\frac{T(\omega)}{(z-w)^2} + \frac{\pi g}{2}\,\frac{:\partial_w^2\psi(\omega)\,\psi(\omega):}{z-w} + \cdots$$

3rd term $= -\frac{\pi g}{2}\,\frac{:\partial_z\psi(z)\,\partial_w\psi(\omega):}{z-w}$

$$= -\frac{\pi g}{2}\,\cancel{\frac{:\partial_w\psi(\omega)\partial_w\psi(\omega):}{z-w}} + \cdots$$ fermions

4th term $= \pi g\,\frac{:\psi(z)\,\psi(\omega):}{(z-w)^3}$

$$= \pi g\,\underbrace{\frac{:\psi(\omega)\psi(\omega):}{(z-w)^3}}_{0\ \text{Fermions}} + \pi g\,\frac{:\partial_w\psi(\omega)\,\psi(\omega):}{(z-w)^2}$$

$$+\pi g\,\frac{:\partial_w^2\psi(\omega)\,\psi(\omega):}{2(z-w)} + \cdots = \frac{T(\omega)}{(z-w)^2} - \frac{\pi g}{2}\,\frac{:\psi(\omega)\partial_w^2\psi:}{z-w}$$

Adding 4 terms gives:

$$\frac{2T(w)}{(z-w)^2} - \pi g \frac{:\psi(w)\,\partial_w^2\psi(w):}{z-w}$$

$$= \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}$$

$$\therefore\ T(z)T(w) = \frac{\frac{1}{4}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \cdots$$

$$c = 1/2.$$

<u>Physical meaning of the central charge</u>

As we have seen, the algebra obtained by assuming conformal invariance in the classical level is governed by $[\hat{l}_n, \hat{l}_m] = (m-n)\,l_{n+m}$, which becomes $[\hat{L}_n, \hat{L}_m] = (n-m)L_{n+m}$

$$+ \frac{c}{12}(n^3-n)\,\delta_{n+m,0}$$

after quantization.

The central charge, c, is the extra character introduced after quantization.

The phenomena — "classical conservation is broken after quantization" — is often called an anomaly. In this case, it's known as conformal anomaly.

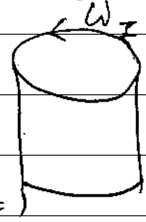
The conformal anomaly implies that conformal symmetry is not exactly maintained at quantum level due to some lengthscale.

Since there is no particular lengthscale in the local Lagrangian, it is an important question to ask what lengthscale causes the existence of c.

Clearly, the lengthscale must be macroscopic and the size L is the obvious choice.

To make it more specific, we consider a conformal invariant system with periodic condition on one direction with length L.

Such a system can be connected to C^2 (infinite 2D) via a conformal transformation



$$z = e^{\frac{2\pi w}{L}}$$

$w = w_R + i w_I$ $\quad w_I \to w_I + L$. z = invariant $T(z)$ is invariant.

Equivalently, $w = \frac{L}{2\pi} \ln z$

Let $T_0(z)$ be the energy-momentum tensor for plane, and

$T_L(\omega)$ be the energy-momentum tensor for systems with periodic boundary condition in ω_I direction.

According to eq. (53), we get

$$T_L(\omega) = \left(\frac{d\omega}{dz}\right)^{-2} \left(T_0(z) - \frac{c}{12}\{\omega; z\} \right) \cdots$$

Since $\frac{d\omega}{dz} = \frac{L}{2\pi z}$, $\frac{d^2\omega}{dz^2} = -\frac{L}{2\pi z^2}$, $\frac{\omega''}{\omega'} = -\frac{1}{z}$

$$\therefore \{\omega; z\} = \left(\frac{\omega''}{\omega'}\right)' - \frac{1}{2}\left(\frac{\omega''}{\omega'}\right)^2$$

$$= \left(\frac{-1}{z}\right)' - \frac{1}{2z^2} = +\frac{1}{2z^2}$$

$$\therefore T_L(\omega) = \left(\frac{2\pi}{L}\right)^2 \left\{ T_0(z) z^2 - \frac{c}{24} \right\} \quad \cdots (64)$$

Eq. (64) implies if $\langle T_0(z) \rangle = 0$ as required by translation invariance, for finite system, this is no longer true, $\langle T_L(\omega) \rangle = -\frac{c\pi^2}{6L^2}$ -- (65)

One sees that finite size induces finite contribution to energy (see later). This is exactly the same as the Casimir effect/energy that is due to vacuum fluctuations.

Specifically, while the zero point energy $\sum_k \frac{1}{2}\hbar\omega_k$

is infinite, it results in finite Casimir energy.

$\frac{\langle E \rangle}{A} = \frac{\hbar c \pi^2}{3 \times 240 d^3}$ for a paralell capacitor with area A and d being the distance between two plates.

By analogy, one realizes C is proportional to the Casimir energy.

Now from eq. (34)-3, one knows $\langle T_{00} \rangle$ is the energy of the system.

Conformal invariance implies $T_{z\bar{z}} = 0$ (eq. 37)

$= \frac{1}{4}(T_{00} + T_{11})$, $\therefore T_{11} = -T_{00}$

$$T_{zz} = \frac{1}{4}(T_{00} - 2i T_{10} - T_{11})$$
$$T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{00} + 2i T_{10} - T_{11})$$
$$\therefore \underbrace{T_{zz}}_{-\frac{T(z)}{2\pi}} + T_{\bar{z}\bar{z}} = T_{00}$$

$$\therefore \langle T_{00} \rangle = -\frac{1}{2\pi} \langle T(z) \rangle \times 2 = \frac{\pi C}{6 L^2}$$

$\delta S = \delta F$ (in Eucleadian field, $\delta(\beta F)$ is more accurate)

$$= -\int d^2 x \, T^{\mu\nu} \partial_\mu \delta X_\nu$$

$\delta X_\nu = \varepsilon X_\nu$ for $\nu = 0$ ($L \to L + \varepsilon L$, $\delta L = \varepsilon L$)

$\therefore \partial_\mu \delta X_\nu = \varepsilon = \frac{\delta L}{L}$ if $\mu = \nu = 0$

$$\therefore \delta F = -\int dw_R \int dw_I \, T_{00} \frac{\delta L}{L} = -\int dw_R \int dw_I \frac{\pi C}{6 L^2} \frac{\delta L}{L}.$$

$\because \int d\omega_I = L$, one defines free energy per unit length along the cylinder by f,

$$\therefore \delta f = \left(f_0 - \frac{\pi c}{6L^2}\right)\delta L$$

f_0: free energy per unit area in $L \to \infty$

After integration, $f = f_0 L - \frac{\pi c}{6L}$ --- (66)

$\therefore$ We see that c does contribute free energy and plays a similar role as the Casimir energy.

In general, c is associated with finite size L. In addition to the above situation (periodic condition), it may also appear in the coefficient of $\ln L$ in βF

$$\beta F = A_2 L^2 + A_1 L + A_0 \ln L + \cdots$$

due to other geometric effect on the Casimir energy (e.g. a wedge of the plane of angle γ, one finds

$$\beta F = -\left(\frac{c\gamma}{24}\left(1 - \left(\frac{\pi}{\gamma}\right)^2\right)\right) \ln L + \cdots$$

see H.W.).

The central charge c can be also explored by measuring the specific heat.

If we interprete ω as a one dimensional system (along ω_R) at temperature T, one has $k_B T = L^{-1}$ i.e. $\beta = L$

then $f = \frac{F}{k_B T}$, $\frac{F_C}{k_B T} = -\frac{\pi c}{6L} = -\frac{\pi c}{6} k_B T$

$\therefore F_C \sim \frac{\pi c}{6} (k_B T)^2$.

which contributes $C \sim \frac{\pi c k_B^2}{3} T$. (linear in T!)

The linear behavior in T is valid when the system is isotropic in space & time (imaginary time). In more technical terms, $\omega \sim v|k|$. i.e. $z = 1$. (dynamical exponent)

$\therefore$ For a one-dimensional system with $z = 1$, the low temperature behavior of the specific heat is linear in T with the slope in proportional to c.

Finally, it's found that for a conformal field theory defined on a curved 2D manifold. The curvature introduces a macroscopic scale so that the trace of T no longer vanishes and is given by $\langle T^{\mu}_{\mu}(x) \rangle = \frac{c}{24\pi} \underset{\substack{\uparrow \\ \text{curvature at } x}}{R(x)}$

This is known as the trace anomaly.

Zamolodchikov's c-theorem

Another meaning of the central charge is that it represents a measure of the number of degrees of freedom in a theory.

This is due to Zamolodchikov who extends the concept of the central charge to any 2D field theory that needs not to be conformal.

The fact that c represents a measure on # of degrees of freedom can be actually understood by considering a theory of N independent free scalar fields. Since $\langle T_i(z) T_j(z)\rangle$
$= \langle T_i\rangle\langle T_j\rangle = 0$,

∴ the central charge is additive and
the net central charge $= N$.

Hence one sees that c measures # of massless modes in the system.

The above is a rough picture. To address the general property of c, we consider general fields on 2D.

For any field on 2D, one needs to consider T_{zz}, $T_{z\bar{z}}$ (or $T_{\bar{z}\bar{z}}$ with $T_{\bar{z}z}$).

Now, T_{zz} no longer depends only on z.

$\therefore T_{zz} = T(z, \bar{z})$, Similarly, $T_{z\bar{z}} \equiv \Theta(z, \bar{z})$.

From eq. (37)-1, they satisfy (due to translational invariance, $\partial^\mu T_{\mu\nu} = 0$)

$$\partial_{\bar{z}} T + \partial_z \Theta = 0 \quad \text{-- (67)}$$

Now, under rotation, if the system is isotropic (rotational invariance) T_{zz} rotates as $z \cdot z$.

$T_{z\bar{z}}$ rotates as $z \cdot \bar{z}$. Here $z \to z e^{i\theta}$, $\bar{z} \to \bar{z} e^{-i\theta}$ ← θ = rotating angle.

Since $T^{\mu\nu}$ has the dimension -2, $\therefore$ one concludes

$$\langle T(z,\bar{z})\, T(0,0) \rangle = \frac{F(z\bar{z})}{z^4}$$

$$\langle T(z,\bar{z})\, \Theta(0,0) \rangle = \frac{G(z\bar{z})}{z^3 \bar{z}}$$

$$\langle \Theta(z,\bar{z})\, \Theta(0,0) \rangle = \frac{H(z\bar{z})}{z^2 \bar{z}^2} \quad \text{-- (68)}$$

where F, G, & H are functions of $r = \sqrt{z\bar{z}}$.

We shall assume that $F \geq 0$, $H \geq 0$ (known as reflection positivity) are correct for models we consider.

Now, multiplying eq. (67) the left-hand-side by $T(0,0)$ and taking $\langle\cdots\rangle$, using eq. (68) one gets

$$0 = \partial_{\bar z}\left(\frac{F(z\bar z)}{z^4}\right) + \partial_z\left(\frac{G(z\bar z)}{z^3\bar z}\right)$$

$$= \frac{1}{z^4} z F'(z\bar z) + \frac{1}{z^3\bar z}\bar z G'(z\bar z) - \frac{3G(z\bar z)}{z^4\bar z}$$

$$\therefore \quad z\bar z F'(z\bar z) + z\bar z G'(z\bar z) - 3G(z,\bar z) = 0$$

~~Let~~ $t = \ln(z\bar z)$. $\quad z\bar z F'(z\bar z) = \frac{dF}{dt} = \dot F$.

We arrive at $\quad \dot F + \dot G - 3G = 0 \quad \cdots$ (69)

Similarly, if we multiply eq. (67) by $\Theta(0,0)$ and take the average, we get

$$0 = \partial_{\bar z}\left(\frac{G(z\bar z)}{z^3\bar z}\right) + \partial_z\left(\frac{H(z\bar z)}{z^2\bar z^2}\right)$$

$$= \frac{1}{z^3\bar z} z G'(z\bar z) - \frac{G(z\bar z)}{z^3\bar z^2} + \frac{1}{z^2\bar z^2}\bar z H'(z\bar z) - \frac{2H(z\bar z)}{z^3\bar z^2}$$

$$\therefore \quad z\bar z G'(z\bar z) - G(z\bar z) + z\bar z H'(z\bar z) - 2H(z\bar z) = 0$$

∴ We obtain

$$\dot G - G + \dot H - 2H = 0 \quad \cdots (70)$$

If we eliminate G (which is not always positive) by forming (69) $- 3\times$ (70), we get

$$\frac{d}{dt}(F - 2G - 3H) = -6H$$

Since the central charge c corresponds to the situation of conformal invariance where $\Theta = 0$, $F(t) = \text{const} = c/2$, we form the function $C(t) = 2F(t) - 4G(t) - 6H(t)$ and find

$$\frac{dC(t)}{dt} = -12H(t) \quad \cdots \text{(71)}$$

Clearly, we immediately get two conclusions

(i) When $\Theta = 0$, $C(t) = 2F(t)$ & $\frac{dC(t)}{dt} = 0$.

$C(t) = c$, the theory is critical. $C(t)$ is an appropriate generalization of c to non-critical domain.

(ii) $\because H \geq 0$, $\therefore C(t)$ is a monotonically decreasing function of t.

Since during the RG transformation, the scale of $r = \sqrt{z\bar{z}}$ effectively goes up and hence t increases. Eq. (71) then implies C(t) decreases along the RG flow! Hence the RG group flows go "downhill" and are necessarily gradient flows and can't be limit cycles and other exotic flows.

A naive picture of the behavior of C-function given in (71) is that the RG transformation coarse-gram the system so that "information" is lost in the process resulting in the downhill behavior. However, this would imply the C-theorem should also hold in higher dimension. Yet no such theorem has been found.

The hole in the above reasoning is that during the RG transformation, we alway rescale the system back and hence there is no information loss.

More accurate picture is that $C(t)$ is the # of low energy (gapless) degrees of freedom as measured by their contribution to $\langle T(z)\,T(w)\rangle$ (or specific heat). The C-theorem simply reflects the fact that systems tend to form gaps by allowing gapless degrees of freedom to interact and form gaps. During RG transformation interactions are induced between gapless degrees of freedoms and their # is reduced and $C(t)$ is thus decreasing. In higher dimensions, this is no longer true as it's possible to generate new gapless modes — Goldstone modes by breaking continuous symmetries.

Representation of the Virasoro Algebra & Constrained models.

The simplest conformal theories are those with finite # of primary fields called minimal models or rational conformal field theories. In these models, the representation of the Virasoro Algebra is finite dimensional, describing discrete statistical models (Ising, Potts and so on) at their critical points.

The discovery & identification of minimal models with known statistical models are the greatest application of conformal invariance.

We shall start by understanding meanings of the generators $\hat{L}_n$ first.

Meaning of $\hat{L}_n$

If we adopt radial quantization in z-plane for Euclidean field theories, that radial direction can be though as τ in the corresponding cylinder with $z = e^{\sigma_0 + i\sigma_1}$ (page 36)

If we treat σ_0 = Euclidean time τ, $\sigma_1 = x$ = space, the Euclidean quantum field theory the

the cylinder is charactorized by the

Hamiltonian $\hat{H} = \frac{d}{d\tau}$, which

becomes $\hat{H} \to i\frac{d}{dt}$ when $\tau \to -it$

(analytical continuation) in the Minkowski

space.

Now, $\frac{d}{d\tau} = \frac{d}{dz}\frac{dz}{d\sigma_0} + \frac{d}{d\bar{z}}\frac{d\bar{z}}{d\sigma_0}$

$= z\, \partial_z + \bar{z}\, \partial_{\bar{z}}$ which

corresponds generators of dilation: $\vec{r} \to \lambda \vec{r}$
$\to x^\mu \partial_\mu$

in the z plane. $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}$

$\therefore$ In the z plane, the dilation generator

$$D = z\partial_z + \bar{z}\,\partial_{\bar{z}} \quad \text{--- eq. (12)}$$

has its root as Hamiltonian in (σ_0, σ_1) plane.

From the Laurent expansion,

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{\hat{L}_n}{z^{n+2}}, \quad \bar{T}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{\bar{L}_n}{\bar{z}^{n+2}}$$

one gets

$$\hat{L}_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}\, T(z), \quad \bar{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\, \bar{z}^{n+1}\, \bar{T}(\bar{z})$$

In comparison with eq. (45)−2, by setting $\epsilon(z) = z^{n+1}$

We find :

$$[\hat{L}_n, \Phi(w,\bar{w})] = \delta_{\epsilon=z^{n+1},\, \bar{\epsilon}=\bar{z}^{n+1}} \Phi(w,\bar{w}) \quad \text{--- (73)}$$

Hence, $\hat{L}_n$ is the conserved charge associated to the conformal transformation $\delta z = z^{n+1}$ (eq.(41)) and generates the corresponding change in Φ.

$\therefore$ $\hat{L}_n$ is exactly the quantum version of classical ℓ_n

($\ell_n = + z^{n+1} \frac{\partial}{\partial z}$)

For $n=0$, $z \to z + \epsilon z$ which can be rotation ($\epsilon = i\theta$) or scaling ($\epsilon = \lambda$).

$\therefore$ L_0 & $\bar{L}_0$ are generators for dilation & rotations

The quantum version to D is $L_0 + \bar{L}_0$

$\therefore$ $L_0 + \bar{L}_0$ has the meaning of energy / Hamiltonian due to its root with $\hat{H}$ in the cylinder geometry. --- (74)

Similarly, $-i(x\,\partial_y - y\,\partial_x) = z\,\partial_z - \bar{z}\,\partial_{\bar{z}}$

$\therefore$ $L_0 - \bar{L}_0$ generates rotations --- (75)

$n = -1$ $\delta z = $ const $\therefore$ L_{-1} & $\bar{L}_{-1}$ generate translation in the z-plane --- (76)−1

Finally, the special conformal transformation corresponds to $z \to z + \epsilon z^2$, i.e. $\epsilon(z) = z^2$. L_1, $\bar{L}_1$ are corresponding generators. --- (76)−2

Hermitian conjugate, reflection positivity & unitary

In radial quantization, one tries to interprete statistical field theories as quantum field theories in Minkowski space analytically continued to Euclidean space (called Euclidean field theory), so that radial ordering corresponds time-ordering in the corresponding quantum field theory.

The formal connection is made by setting

$$t = -i\tau$$

$$\& \qquad z = e^{i\sigma_0 + i\sigma_1} \quad \text{with } \sigma_0 = \tau \quad \cdots (77)$$

However, such connection spoils the usual definition of Hermition conjugate:

$$\phi(x,t) = e^{iHt}\,\phi(x,0)\,e^{-iHt}$$

$$\Rightarrow \quad \phi^{\dagger}(x,t) = e^{iHt}\,\phi^{\dagger}(x,0)\,e^{-iHt} \quad \text{if } H^{\dagger} = H \quad \cdots (78)$$

In the Euclidean field theory,

$$\phi_E(x,\tau) = e^{\tau H}\,\phi(x,0)\,e^{-\tau H}$$

the usual Hermitian conjugate $\phi^{\dagger}(x,\tau) = e^{-\tau H}\,\phi^{\dagger}(x,0)\,e^{\tau H}$ does not correspond to the analytic continuation of $\phi^{\dagger}(x,t)\big|_{t\to -i\tau} = e^{\tau H}\,\phi^{\dagger}(x,0)\,e^{-\tau H}$

Instead, the Hermitian conjugate that corresponds to $\phi^{+}(x,t)\big|_{t\to -i\tau}$ should be defined as $\Theta\Phi_E(x,\tau) = \Phi_E^{+}(x,-\tau)$ --- (79)

According to $z = e^{\tau+i\sigma_1}$ (eq. (77)), this amounts to change $z \to \frac{1}{\bar{z}}$, $\bar{z} \to \frac{1}{z}$

$$\therefore\ \Theta\Phi_E(z,\bar{z}) = \Phi_E^{+}\left(\frac{1}{\bar{z}}, \frac{1}{z}\right) \quad \text{--- (80)}$$

We shall omit the subscript E later when complex #s $z, \bar{z}$ are used.

reflection positivity

After defining the Hermitian Conjugation for a given Euclidean field theory, one needs to check the positivity requirement.

In quantum mechanics (quantum field theory), if one denotes $|\phi\rangle = \hat{\phi}|0\rangle$

$\langle\phi| = \langle 0|\hat{\phi}^{+}$ is the ket to $|\phi\rangle$

$\because\ \langle\phi|\phi\rangle \geq 0 \quad \therefore\ \langle\hat{\phi}^{+}\hat{\phi}\rangle \geq 0$

Therefore, in order that the Euclidean field theory has a corresponding quantum field theory, one requires

$$\langle \Theta\, \Phi(x,z)\, \Phi(x,z) \rangle \geq 0 \quad \text{--- } (\mathcal{R}1)-1$$

or generally $\langle \Theta \hat{O}(x,z)\ \hat{O}(x,z) \rangle \geq$ for any operator $\hat{O}$

Eq. ($\mathcal{R}1$) is termed reflection-positivity as it means $\langle \Phi^{+}(x,-z)\, \Phi(x,z) \rangle \geq 0$

In 2D, $\langle \Phi^{+}(\frac{1}{\bar{z}}, \frac{1}{z})\, \Phi(z,\bar{z}) \rangle \geq 0$ --- ($\mathcal{R}1$) -2

In general, given a Euclidean field theory of $\Phi(x_1, x_2)$, it's not necessarily that there is a corresponding quantum field theory. Eq. ($\mathcal{R}1$) is the essential requirement in order that this Euclidean field theory can have a corresponding quantum field theory!

transfer matrix & unitary

Given a Euclidean field theory on $(x_1, x_2, \ldots, x_d)$, if the axis x_d can be thought as the analytic continuation of a quantum field theory on (x_1, x_2, x_d) the translation operator along $x_d \equiv e^{\varepsilon_d \hat{P}_d}$

$\hat{P}_d = \frac{d}{dx_d}$ will correspond to Hamiltonian H.

the corresponding
in quantum field theory.

Hence such correspondence is possible only when

$$P_d^\dagger = P_d \quad \text{(Hermitian)}$$

& P_d has a lower bound. --- (f2)

The Hermitian implies that the corresponding evolution in the quantum field theory is unitary:

$$U = e^{i\hat{P}_d}, \quad U^\dagger U = \mathbb{I}$$

so that probability is conserved.

The connection between $\Phi(x_1, \ldots x_{d-1}, x_d+a)$ and $\Phi(x_1, \ldots, x_{d-1}, x_d)$ is through the transfer matrix $\hat{T}$ by

--- (f3)

$$\Phi(x_1, \ldots, x_{d-1}, x_d+a) = \hat{T}(a)\, \Phi(x_1 \ldots, x_{d-1}, x_d)$$

which is well-defined for Euclidean field theory on a lattice.

$\because\ \hat{T}(a) = e^{-aP_d} = \hat{T}^\dagger$ for two-point correlation functions on a lattice, one has

$$\langle \underset{\substack{\uparrow \\ x_d = +ka}}{\Theta\Phi(r_1)}\ \underset{\substack{\uparrow \\ x_d = 0}}{\Phi(r_2)} \rangle = \lim_{N\to\infty} \frac{\mathrm{Tr}\ T^{N-k}\ \overset{\text{"}\Phi^\dagger(\ldots,-ka)\text{"}}{\Phi^\dagger(r_2)}\ (T(a))^k\ \Phi(r_2)}{\mathrm{Tr}(T^N) \leftarrow e^{-S}} \quad \text{--- (f4)}$$

N = total # of steps

$$\left(\Phi^\dagger_{(x,\tau)} = e^{-\tau H}\, \Phi^\dagger(x,0)\, e^{+\tau H}, \quad \Phi^\dagger(x, -ka) = e^{-kaH}\, \Phi^\dagger(x,0)\, e^{-kaH}\right)$$

$$= \lim_{N\to\infty} \sum_{n\geq 0} \frac{\langle 0 | T^{N-K} \phi^{\dagger}(r_2) | n \rangle \langle n | T^{K} | n \rangle \langle n | \phi(r_2) | 0 \rangle}{\langle 0 | T^{N} | 0 \rangle}$$

~~where~~ $P_d | n \rangle = E_n | n \rangle$

$$\therefore \langle \Theta \phi(r_1) \phi(r_2) \rangle = \sum_{n\geq 0} \frac{|\langle 0 | \phi(r_2) | n \rangle|^2 e^{-E_n K} e^{-(N-K)E_0}}{e^{-NE_0}}$$

$$= \sum_{n\geq 0} |\langle \phi | \phi(r_2) | n \rangle|^2 e^{-(E_n - E_0)K} \quad \cdots (ff)$$

Eq. (ff) is derived under the assumption $P_d^{\dagger} = P_d$. $\cdots$ (ff)

Clearly, $\langle \Theta \phi^{\dagger}(r_1) \phi(r_2) \rangle \geq 0$ and is well-defined when $E_n \geq E_0$ i.e. P_d has a lower bound.

It's clearly that if $\langle \Theta \phi(r_1) \phi(r_2) \rangle \geq 0$ is violated, $P_d^{\dagger} = P_d$ is no-longer guaranteed!

Hence, $\langle \Theta \phi^{\dagger}(r_1) \phi(r_2) \rangle \geq 0$ the positivity condition can lead to a quantum field theory correspondence with $\hat{H}^{\dagger} = \hat{H}$ so that unitarity is maintained!

Boltzmann weight

In general, if the Euclidean field theory results from the statistical Mechanics with the Boltzman weight $e^{-\beta H}$, the dimension x_d corresponding to β (i.e. the so-called direction with Matsubara frequency) automatically gives a Hermitian $P_d = H$ with lower bound.

Hence, in this case, reflection positivity or eq. (86) is automatically satisfied.

For general statistical systems, however, this is no longer true because the statistical weight $P(\{x_i\})$ is a positive probability which needs not to be resulted from $e^{-\beta H}$. The reflection positity / eq. (86) can be violated and thus the theory is non-unitary.

An example is the system with quenched impurities when the average over impurities renders P not in the form $e^{-\beta H}$ any more!

Hermitian conjugate & reflection positivity in complex variables

For 2D, complex variables z & $\bar{z}$ are natural variables.

In this case, one also needs to take $z \to \bar{z}$ in the relevant coefficients when taking the Hermitian conjugate.

Specifically, since components with indices z & $\bar{z}$ are linear combinations of original components with indices x & y, factors involved with only z or $\bar{z}$ has to be removed!

$\therefore$ In eq. (82), if $\phi_E^+ = \phi$ (observable), and ϕ is a quasi-primary field, one has

$$\phi_E^+\left(\frac{1}{\bar{z}}, \frac{1}{z}\right) = \phi\left(\frac{1}{\bar{z}}, \frac{1}{z}\right) \frac{1}{\bar{z}^{2h}} \frac{1}{z^{2\bar{h}}} \quad \cdots (86)$$

Similar to the derivation of eq. (86), for Hermitian operator (in the sense of quantum mechanic $O^+ = O$),

$$\langle O^+(r_1) O(r_2) \rangle = \langle O(r_1) O(r_2) \rangle$$

$$= \langle e^{X_d^1 H} O(r_1^\perp, 0) \underbrace{e^{-X_d^1 H} e^{X_d^2 H}}_{e^{(X_d^2 - X_d^1) H}} O(r_2^\perp, 0) e^{-X_d^2 H} \rangle$$

$$\geq 0$$

$\therefore$ $\langle T_{\mu\nu}(x,y) T_{\mu\nu}(0,0) \rangle \geq 0$. When transforming to $z, \bar{z}$ variables, one thus require

$F(z,\bar{z}) \geq 0$, $H(z,\bar{z}) \geq 0$ in eq. (68)

In addition to positivity of H & F, the requirement of Hermitian of $T(z)$ leads to

$$T^+(z) = \sum_m \frac{L_m^+}{\bar{z}^{m+2}} = T\left(\frac{1}{\bar{z}}\right)\frac{1}{\bar{z}^4}$$

$$= \frac{1}{\bar{z}^4}\sum_m \frac{1}{(1/\bar{z})^{m+2}} L_m$$

$$= \sum_m \frac{1}{\bar{z}^{-m+2}} L_m = \sum_m \frac{1}{\bar{z}^{m+2}} L_{-m}$$

$$\therefore \quad L_n^+ = L_{-n} \quad \cdots \quad (87)$$

Similarly, $\bar{L}_n^+ = \bar{L}_{-n}$

Verma Modulos

The role played by the Virasoro algebra

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12}(n^2-1)n\,\delta_{n+m,0}$$

is exactly the same as the role played by the algebra of angular momentum J_x, J_y, J_z : $[J_\alpha, J_\beta] = i\hbar\,\epsilon_{\alpha\beta\gamma} J_\gamma$.

Constructing representations (matrix representation) of $\vec{J}$ enables one to find how the wavefunction changes under rotations with allowed j & m.

Here constructing representations of L_n enables one to find how primary fields ϕ changes under conformal transformation with allowed c & scaling dimension according to eq. (49)

In the case of rotations: representations of $\vec{J}$ can be used to reduce the work of calculation of energy to each (j,m) section.

For the conformal theory, representations of $\hat{L}_n$ can be used to calculate all correlations according to eq. (50) and in particular, it helps to pin down the undetermined functions such as f in eq. (33) (which was derived by using only global conformal transformation)!

This is the powerful of local conformal transformations and it only occurs at 2D!

Similarly to the rotational group where $[J_0, J_\pm] = \pm J_\pm$, $[J_+, J_-] = 2J_0$ and one can only diagonalize one of $J_0, J_\pm$, we can only diagonalize one of L_n due to their non-commutability!

Clearly, on one hand, according to eq. (40), one has

$$T(z)\phi(0,0) = \frac{h}{z^2}\phi(0,0) + \frac{1}{z}\partial\phi(0,0)$$

for primary fields.

On the other hand,

$$T(z)\phi(0) = \sum_{n=-\infty}^{\infty} \frac{1}{z^{n+2}} \hat{L}_n \phi(0)$$

$\therefore$ For a primary field ϕ, it satisfies

$$L_0 \phi(0) = h\phi(0) \qquad \text{--- (88)-1}$$

$$L_n \phi(0) = 0 \quad \text{if } n \geq 1 \qquad \text{--- (88)-2}$$

(n=1, 2 are sufficient)

$$L_{-1}\phi = \partial_z \phi \qquad \text{--- (88)-3}$$

Therefore, it's convenience to diagonalize L_0 ($\&$ $\bar{L}_0$)

Now, $[L_0, L_n] = -nL_n$
$[L_0, L_{-n}] = nL_{-n}$ $\quad$ & $L_{-n} = L_n^{+}$

$\therefore$ L_n (L_{-n}) is a raising (lowering) operator!

Similar to the construction of $|jm\rangle$ for $\vec{J}$ where one starts the highest $m=j$, $|jj\rangle$ and applies J_- successively to get $|jm\rangle$, we start from the highest weight state (or a primary state) for a given primary operator defined by

$$\phi(0)|0\rangle \equiv |h\rangle$$

$$\therefore \quad L_0|h\rangle = h|h\rangle \qquad \text{--- (89)}$$

$$L_n|h\rangle = 0 \qquad n \geq 1$$

Similarly, to include $\bar{L}_0$, $|h\rangle \to |h, \bar{h}\rangle$

$$\bar{L}_0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle$$

$$\bar{L}_n|h,\bar{h}\rangle = 0 \quad \text{if } n \geq 1$$

Note that $|h\rangle$ in fact has lowest scaling dimension, by using $[L_0, L_{-n}] = nL_{-n}$, one finds that the scaling dimension of $(L_{-n}|h\rangle)$ is $h+n$: $L_0(L_{-n}|h\rangle) = (h+n)(L_{-n}|h\rangle)$ -- (90)

$\because L_n^{+} = L_{-n}$, $\therefore$ eq. (89) implies

$$\langle h|L_0 = h\langle h|$$

and $\langle h|L_n = 0$ for $n \leq -1$ --- (91)

Descendant states

Starting from $|h\rangle$, one can apply L_{-n} successively to get states (operators) with higher scaling dimensions. These are called descendant states with the form

$$L_{-n_1}L_{-n_2}\cdots L_{-n_k}|h\rangle \quad n_i > 0 \qquad \text{--- (92)}$$

and the eigenvalue of $L_0 = (n_1 + \cdots + n_k) + h$.

All together, the space formed by these states is called a Verma module:

	L_0 eigenvalue	states
level 0	h	$\vert h\rangle$
level 1	$h+1$	$L_{-1}\vert h\rangle$
2	$h+2$	$L_{-1}^2\vert h\rangle$, $L_{-2}\vert h\rangle$
3	$h+3$	$L_{-1}^3\vert h\rangle$, $L_{-1}L_{-2}\vert h\rangle$, $L_{-3}\vert h\rangle$
⋮		⋮

No.
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As an example, we consider the Verma module built on the identity operator $\mathbb{I}$.

Clearly, from eq. (88)-3, $L_{-1}\mathbb{I} = \partial_z \mathbb{I} = 0$

$$\therefore L_{-1}|h\rangle = L_{-1}\mathbb{I}|0\rangle = 0$$

i.e. $L_{-1}|0\rangle = 0$ agreeing with the fact that $|0\rangle$ is global conformal invariant ($L_{\pm1}, L_0$ are generators)

To get the action of T on $\mathbb{I}$, we equate $T(z)\,\phi(0,0) = \sum_{n=\infty}^{\infty} \frac{1}{z^{n+2}} \hat{L}_n \phi(0)$

$\therefore$ only $n=-2$ contributes, we get

$$L_{-2}\mathbb{I} = T(z), \quad L_0\mathbb{I} = 0, \quad L_n\mathbb{I} = 0 \quad n \geq 1 \qquad \text{--- (93)}$$

$\therefore T(z)$ is not a primary field and is the descendant field to $\mathbb{I}$. (secondary fields).

Note that $\because L_1 T = L_1 L_{-2}\mathbb{I} = (L_{-2}L_1 + 3L_{-1})\mathbb{I} \underset{L_1\mathbb{I}=0}{=} 0$ --- (94)

$$L_0 T = L_0 L_{-2}\mathbb{I} = (L_{-2}L_0 + 2L_{-2})\mathbb{I} = 2L_{-2}\mathbb{I} = 2T(z)$$

$$L_2 T = L_2 L_{-2}\mathbb{I} = \left(L_{-2}L_2 + 4L_0 + \frac{c}{2}\right)\mathbb{I} = \frac{c}{2}\mathbb{I} \neq 0$$

$$L_n T = L_n L_{-2}\mathbb{I} = (L_{-2}L_n + (n+2)L_{n-2})\mathbb{I} = 0 \quad \text{for } n \geq 3$$

$\therefore T$ scales with dimension 2 but is not a primary field. ($L_n T \neq 0$, $n \geq 1$).

Since the vacuum is invariant under global conformal transformations, $L_{\pm 1}|0\rangle = 0$, $L_0|0\rangle = 0$.

Using eq (73) $[L_1, T] = \delta_{\epsilon = z^2} T$ for $\epsilon = z^2$ (special conformal transformation^{SCT}), eq. (84)

implies $[L_1, T]|0\rangle = 0$

$\therefore \langle \delta T \rangle = 0$ under SCT

Similarly, $[L_{-1}, T]|0\rangle = 0$, $[L_0, T]|0\rangle = 0$.

$\therefore \langle \delta T \rangle = 0$ under global conformal transformation

$\therefore$ According to the definition, T is a quasi-primary field!

general requirement on (c, h)

we have constructed the Verma module ~~for a given primary field. The space generated~~ by $|h\rangle$ & its descendants is closed under the action of the Virasoro generators.

So far, it may seem that values of (c, h) are not relevant.

We shall see, however, by requirements of irreducibility & unitary, possible values of c and h will be greatly reduced.

Before we discuss irreducibility & unitary, we shall consider the requirement of positive Hilbert space first:

$$\because [L_2, \underset{\|}{L_{-2}}^{L_2^+}] = 4L_0 + \frac{c}{2}$$

$$L_0|0\rangle = 0 \quad (\text{eq. } (P3))$$

$$\therefore \langle 0|[L_2, L_{-2}]|0\rangle = \frac{c}{2}$$

$$\langle 0|L_2 L_2^+|0\rangle - \underbrace{\langle 0|L_2^+ L_2|0\rangle}_{\substack{0 \\ \text{eq. } (P3)}} = c/2$$

$$\therefore \frac{c}{2} = |L_2^+|0\rangle|^2 \geq 0$$

Similarly, $\langle h|L_{-n}^+ L_{-n}|h\rangle = |L_{-n}|h\rangle|^2 \geq 0$

$$= \langle h|[L_n, L_{-n}] + \cancel{L_{-n}L_n|h\rangle}^{\,0}$$

$$= 2n\langle h|L_0|h\rangle + \frac{c}{12}(n^3-n)\langle h|h\rangle$$

$$= \left(2nh + \frac{c}{12}(n^3-n)\right)\langle h|h\rangle \geq 0$$

$\therefore c \geq 0$, $h \geq 0$ (taking $n=1$) ...... (95)

Null states and unitary minimal models

Just as the representation for $SU(2)$, the representation is finite dimensional because $(J_-)^{2j}|j,j\rangle = 0$.

The representation of Virasoro algebra is generally infinite dimension unless there exists so-call null states at nth level.

From the point of view of linearity, given c & h, there is no reason why these states at the same level are linearly independent.

In addition, it is also possible that not all c & h allow the theory to be unitary.

Hence imposing the unitary condition allows one to constraint possible values of c and h.

As we have learned, the unitary condition requires $\langle\psi|\psi\rangle \geq 0$ (semi-positivity).

When $\langle\psi|\psi\rangle = 0$, $|\psi\rangle$ is a null state.

It's clear null states set the boundary of the constraint set by the unitary condition.

The necessary and sufficient conditions for ~~unitarity~~ can be found by considering the matrix of inner products between all basis states.

For instance, at level 2, one considers

$$M^{(2)} = \begin{pmatrix} \langle h|L_2 L_{-2}|h\rangle & \langle h|L_1^2 L_{-2}|h\rangle \\ \langle h|L_2 L_{-1}^2|h\rangle & \langle h|L_1^2 L_{-1}^2|h\rangle \end{pmatrix} \quad \cdots (96)$$

At level 1, one considers $\langle h|L_1 L_{-1}|h\rangle$.

0 " " $\langle h|h\rangle = 1$

Clearly, states from different levels are ~~orthogonal~~: e.g.

$$\langle h|L_1 L_{-2}|h\rangle = \langle h|L_{-2}L_1 + 3L_{-1}|h\rangle \overset{L_1|h\rangle=0}{=} 0$$

$$\underbrace{\langle h|}_{=0} L_{-1}|h\rangle = 0$$

∴ we don't need to consider inner products of crossed levels.

The unitary condition imposes

$$\det M^{(N)} \geq 0 \quad \text{for all } N \cdots (97)$$

In the case of $N=1$, $L_{-1}|h\rangle = 0$

~~∴ $|h\rangle = |0\rangle$ (since vacuum is invariant under~~ global conformal transformation. ($L_{\pm1}, L_0|0\rangle = 0$)

Formally, $[L_1, L_{-1}] = 2L_0 + \frac{c}{12}(\cancel{1^3-1})^{0}$

$\therefore \langle h|L_1L_{-1} + L_{-1}\underbrace{L_1|h\rangle}_{0} = 2\langle h|L_0|h\rangle = 2h$

$\therefore \langle h|L_1L_{-1}|h\rangle = 2h = 0 \qquad h = 0$

For $N=2$. $\langle h|L_2L_{-2}|h\rangle = \langle h|\cancel{L_{-2}L_2}^{0} + 4L_0 + \frac{c}{12}(2^3-2)|h\rangle$

$= 4h + \frac{c}{2}$

$\langle h|L_1^2L_{-2}|h\rangle = \langle h|L_1(\cancel{L_{-2}L_1}^{0} + 3L_{-1})|h\rangle$

$= 3\langle h|L_1L_{-1}|h\rangle = 6h = \langle h|L_2L_{-1}^2|h\rangle$

$\langle h|L_1^2L_{-1}^2|h\rangle = \langle h|L_1(L_{-1}L_1 + 2L_0)L_{-1}|h\rangle$

$[L_1, L_{-1}] = 2L_0$

$\langle h|L_1L_0L_{-1}|h\rangle = \langle h|L_1(L_{-1}L_0 + L_{-1})|h\rangle = (h+1)\langle h|L_1L_{-1}|h\rangle = 2h(h+1)$

$\therefore \langle h|L_1L_{-1}L_1L_{-1}|h\rangle = \langle h|L_1L_{-1}(\cancel{L_{-1}L_1}^{0} + 2L_0)|h\rangle$

$= 2h\langle h|L_1L_{-1}|h\rangle = 2h\langle h|\cancel{L_{-1}L_1}^{0} + 2L_0|h\rangle$

$= (2h)^2$

$\therefore \langle h|L_1^2L_{-1}^2|h\rangle = 4h(h+1) + 4h^2 = 4h(2h+1)$

$\therefore M^{(2)} = \begin{pmatrix} 4h + c/2 & 6h \\ 6h & 4h(2h+1) \end{pmatrix} \Rightarrow \det M^{(2)} = 16h^3 - 10h^2 + 2h^2c + hc$

$= 32(h - h_{1,1}(c))(h - h_{1,2}(c))(h - h_{2,1}(c))$

Where $h_{p,q} \equiv \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}$

$m = \frac{-1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}}$ - - - (98)

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There exists a general formula, due to Kac, for $\det M^{(N)}$ as already given in eq. (98) with

$$\det M^{(N)} = \alpha_N \prod_{pq \le N} (h - h_{pq})^{P(N-pq)} \quad \text{--- (99)}$$

(sum over positive integers p, q with $pq \le N$)

$P(N-pq)$ = # of partitions of the integer $N-pq$

and α_N is a positive constant independent of h or c

By analyzing eq. (98), one can determined allowed values of c & h for the theory to be unitary.

(i) There are two allowed Categories:

(i) $c > 1$, $h \geq 0$ unitary representation

~~Since any $c>1$, $h\geq 0$ are allowed, this is the infinite dimensional representation.~~

Roughly, it follows from eq. (98) by noting that

for $1 < c < 25$, m = complex

with $4m(m+1) = \frac{6}{1-c} < 0$

$\therefore h_{pq}$ is either complex or negative ($p \neq q$)

$\therefore \det M^{(N)}$ has no real roots, is always positive

For $c \geq 25$, $-1 < m < 0$, $4m(m+1) < 0$

$$[(m+1)p - mq]^2 = [(1-|m|)p + |m|q]^2 \geq 1$$

$\therefore h_{pq} < 0$ $\therefore \det M^{(N)}$ is always positive.

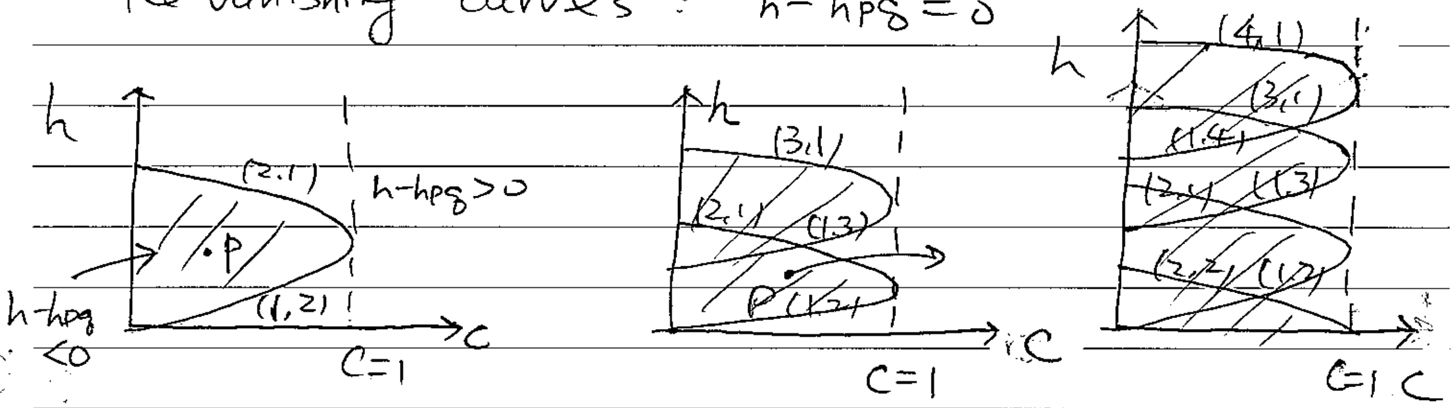
(ii) $C<1$. Unitary representations

only discrete points are allowed:

$$C = 1 - \frac{6}{m(m+1)} \qquad m = 3, 4, 5, 6, \ldots$$

$$h_{pq}(m) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \qquad (1 \le p \le m-1,\ 1 \le q \le p) \qquad (100)$$

This can be understood roughly by plotting out the vanishing curves: $h - h_{pq} = 0$



$N=2$ $\qquad$ $N=3$

Given any point p in the region $c<1$ $h>0$, if p does not reside on one of the vanishing curve, it's easy to see, for some particular level N_0, there is a continuous path linking p to $c>1$ $h>0$ region with one crossing point across one vanishing curve.

Since $h - h_{pq} > 0$ on $c>1$, $h>0$, one concludes $\det M^{N_0} < 0$ at p. Hence all shaded areas in the figures are non-unitary.

Since unitarity requires det $M^N \geq 0$ for all N, these shaded areas are not unitary.
As N increases, more points are included in the region $c \leq 1$, $h > 0$ eventually any points that are not on any vanishing curve are included inside some vanishing curve, they are thus non-unitary.

As a results, only points on vanishing curves can support unitarity. It turns out that on vanishing curves, the Verma modules contain negative-norm states as well except for discrete points given by eq. 100

Hence we see that the constraint of unitarity is so strong that for $c \leq 1$, $h > 0$, there are only finite # of $h_{p,q}$ for a given c.
Therefore, given a central charge $c \leq 1$, # of primary fields is finite and the representation of the Virasoro algebra is finite.

null states

One of the consequences from the above discussion is that unitary models for $c \leq 1$ $h > 0$ must contain null states since (c, h) ~~all~~ fall on vanishing curves!

The null states impose partial relations among $L_{-n_1} L_{-n_2} \cdots L_{-n_k}$ that result in addition differential equations to be satisfied by correlation functions.

These equations allow one to find the exact form of correlation functions.

We shall come back to this point by considering the 2D Ising model as an example.

Note that when we define null states as $\langle \chi | \chi \rangle = 0$, it is constructed by linear combination of $L_{-n_1} L_{-n_2} \cdots L_{-n_k} |h\rangle$ and require $\sum a_i L_{-n_1} L_{-n_2} \cdots L_{-n_k} |h\rangle = |\chi\rangle$, the inner product $\langle \chi | \chi \rangle = 0$ as a result of $\det M^{(a)} = 0$

From operator point of view, one has.

$$\sum a_i^* \langle \phi^\dagger L_{n_K} L_{n_{K-1}} \cdots L_{n_1} L_{-n_1} \cdots L_{-n_K} \phi \rangle = 0 . \quad \text{--- (101)}$$

It doesn't mean $\sum_i a_i L_{-n_1} \cdots L_{-n_K} \phi = 0$

as usual implied, it has to be put in the average w.r.t. the ground state $|0\rangle$

Therefore, if $|\chi\rangle$ = null state, $|\chi\rangle \neq 0$

In general, given a primary field, ϕ, with $|h\rangle = \phi|0\rangle$ being the highest weight state, any state $|\chi\rangle \neq |h\rangle$ (descendant) that is annihilated by L_n for all $n > 0$

$$L_n |\chi\rangle = 0 \quad \text{--- (102)}$$

is called a null state.

Such a state generates its own Verma module included in the Verma module of $|h\rangle$.

Furthermore, $|\chi\rangle$ is orthogonal to the whole Verma module of $|h\rangle$:

$$\langle \chi | L_{-k_1} L_{-k_2} \cdots L_{-k_n} | h \rangle$$
$$= \langle h | L_{k_n} \cdots L_{k_2} \underbrace{L_{k_1} |\chi\rangle}_{0}{}^* = 0 \quad \text{--- (103)}$$

In particular, $\langle \chi | \chi \rangle = 0$ --- (104)

Furthermore, all descendants of $|\chi\rangle$ are also orthogonal to the whole Verma module:

for $\langle h| L_{k_n} \cdots L_{k_1} L_{-r_1} \cdots L_{-r_m} |\chi\rangle$
$\uparrow$
level N

to be relevant $\sum_i k_i = N + \sum_i r_i$ (# of L_{-n} = # of L_n)

Any imbalance between # of L_n & L_{-n} is automatically zero. $\because \sum_i k_i > \sum_i r_i$, commuting L_{k_i} by commuting L_n to $|h\rangle$ to right, one yields terms as in eq. (102) $\therefore \langle h| L_{k_n} \cdots L_{k_1} L_{-r_1} \cdots L_{-r_m} |\chi\rangle = 0$ — (105)

Therefore, the Verma module of $|\chi\rangle \perp$ the whole Verma module. The representation based on ϕ is reducible! To get irreducible representation, one has to quotient the Verma module of $|\chi\rangle$ out!

Finally, all the descendants to $|\chi\rangle$ also satisfy $\langle \psi | \psi \rangle = 0$ — (106)

Eqs (104) & (106) agree the previous definition of a null states at the beginning

The existence of such a state is assured by $\det M^{(N)} = 0$

From $\det M = 0$, one decides the relative ratio $a_1, a_2, \ldots$ in eq. (101):

$$M \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \end{pmatrix} = 0$$

As an example, consider $N=2$.

One has $M^{(2)} = \begin{pmatrix} 4h + c/2 & 6h \\ 6h & 4h(2h+1) \end{pmatrix}$

$\because \det M^{(2)} = 32h^3 - 20h^2 + 4h^2 c + 2hc = 0$

$\therefore c = \dfrac{10h - 16h^2}{2h+1}$ Substituting back to $M^{(2)}$, one needs to find

$$\begin{pmatrix} 4h + \frac{5h - 8h^2}{2h+1} & 6h \\ 6h & 4h(2h+1) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \quad \text{--- (107)}$$

$\therefore 6h a_1 + 4h(2h+1) a_2 = 0$

$$a_1 : a_2 = 1 : -\frac{3}{2(2h+1)} \quad \text{--- (108)}$$

From eq. (96), one concludes

$$|\chi\rangle = a_1 L_{-2} |h\rangle + a_2 L_{-1}^2 |h\rangle$$
$$= \left(L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2\right) |h\rangle \quad \text{--- (109)}$$

At the operator level, the Verma module of $|\chi\rangle$ is generated by the field

$$\chi(z) = \left(L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2\right)\phi(z) \quad \cdots (110)$$

"$\chi(z)$ is a null state" means

$$\langle \chi \cdot \hat{O} \rangle = 0 \quad \text{for any operator}.$$

$$\therefore \left\langle \left(L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2\right)\phi(z,\bar{z})\,\hat{O} \right\rangle = 0 \quad \cdots (111)$$

Eq. (111) can be converted into a differential equation by first noting that for a primary field

$$T(z)\phi(\omega,\bar{\omega}) = \sum_n \frac{1}{(z-\omega)^{n+2}} L_n \phi(\omega,\bar{\omega})$$

$$= \frac{1}{(z-\omega)^2} L_0\phi + \frac{1}{(z-\omega)} L_{-1}\phi + L_{-2}\phi$$

$L_n\phi = 0$, $n \geq 1$

$$+ \cdots + (z-\omega)^{n-2} L_{-n}\phi + \cdots$$

$$\therefore L_{-n}\phi(\omega,\bar{\omega}) = \oint \frac{dz}{2\pi i} \frac{1}{(z-\omega)^{n-1}} T(z)\phi(\omega,\bar{\omega})$$

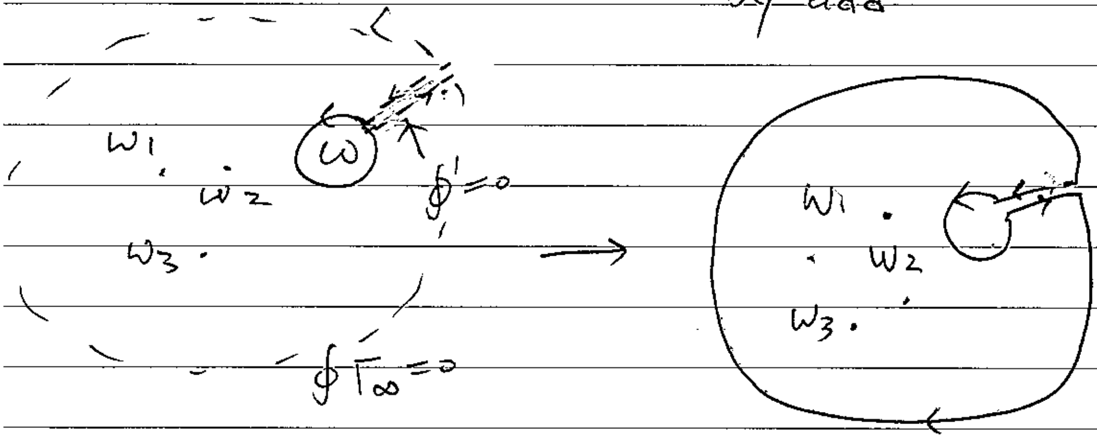
Using the Ward identity in eq. (12), one gets

$$\langle L_{-n}\phi(\omega,\bar{\omega})\phi_1(\omega_1,\bar{\omega}_1)\cdots\phi_K(\omega_K,\bar{\omega}_K)\rangle$$

$$= \oint \frac{dz}{2\pi i} \frac{1}{(z-\omega)^{n-1}} \left(\sum_{j=1}^{K} \frac{h_j}{(z-\omega_j)^2} + \frac{1}{z-\omega_j}\frac{d}{d\omega_j}\right) \langle \phi(\omega,\bar{\omega})\phi_1\cdots\phi_K\rangle$$

$\cdots (112)$

The contour integration in (112) can be deformed ~~by add~~



$\therefore$ RHS of eq. (112) $= -\sum_i$ Residue at w_i

~~Now~~

for $n \neq 1$ $\quad \frac{1}{(z-\omega)^{n-1}} = \cdots - (n-1)\frac{1}{(\omega_j-\omega)^n}(z-\omega_j) + \cdots$

$$\therefore \oint \frac{dz}{2\pi i}\frac{1}{(z-\omega)^{n-1}}\sum_{j=1}^{K}\frac{h_j}{(z-\omega_j)^2}\langle \Phi(\omega,\bar\omega)\phi_1\cdots\phi_K\rangle$$

$$= (n-1)\sum_{j=1}^{K}\frac{h_j}{(\omega_j-\omega)^n}$$

Similarly, $\oint \frac{dz}{2\pi i}\frac{1}{(z-\omega)^{n-1}}\sum_{j=1}^{K}\frac{1}{z-\omega_j}\frac{\partial}{\partial\omega_j}\langle \Phi(\omega,\bar\omega)\phi_1\cdots\phi_K\rangle$

$$= \sum_j \frac{1}{(\omega_j-\omega)^{n-1}}\frac{\partial}{\partial\omega_j}\langle \Phi(\omega,\bar\omega)\phi_1\cdots\phi_K\rangle$$

$$\therefore \langle L_{-n}\Phi(\omega,\bar\omega)\phi_1(\omega_1,\bar\omega_1)\cdots\phi_K(\omega_K,\bar\omega_K)\rangle$$

$$= -\sum_{j=1}^{K}\left[\frac{(1-n)h_j}{(\omega_j-\omega)^n} + \frac{1}{(\omega_j-\omega)^{n-1}}\frac{\partial}{\partial\omega_j}\right]\langle\Phi(\omega,\bar\omega)\phi(\omega_1,\bar\omega_1)\cdots\phi_K(\omega_K,\bar\omega_K)\rangle$$

$(n \geq 2)$

For $n=1$, only $\frac{1}{z-\omega_j}\frac{\partial}{\partial\omega_j}$ yields residue. One has

$$\langle L_{-1}\Phi(\omega,\bar\omega)\phi_1\cdots\phi_K\rangle = \sum_{j=1}^{K}\frac{\partial}{\partial\omega_j}\langle\Phi(\omega,\bar\omega)\phi_1\cdots\phi_K\rangle \quad \cdots (113)$$

Using eq. (113), eq. (111) becomes

$$\left\{ \sum_{i=1}^{K} \left[\frac{1}{z-z_i} \frac{d}{dz_i} + \frac{h_i}{(z-z_i)^2} \right] - \frac{3}{2(2h+1)} \frac{d^2}{dz^2} \right\} \langle \phi(z,\bar{z}) \hat{O} \rangle = 0$$

$$\text{--- (114)}$$

Conformal Grids

As seen in the above, the $c<1$ unitary conformal theories provide a set of possible 2D critical systems that can be matched up with that of known statistical mechanics systems.

For this purpose, it's useful to write down all allowed scaling dimensions $h_{pq}(m)$.

As noted in eq. (102), $1 \le p \le m$, $1 \le q \le p$, is half of the square $1 \le p < m$, $1 \le q < m$.

Since $h_{pq} = h_{m-p, m+1-q}$, it's useful to extend it to the region $1 \le p < m$, $1 \le q < m$.

One gets the conformal grids : $(m \times (m-1))$

$\uparrow q$

	$p=1$	$p=2$
$q=3$	1/2	0
$q=2$	1/16	1/16
$q=1$	0	1/2

$\longrightarrow p$

$m=3$, $c=1/2$

3/2	7/16	0
3/5	3/80	1/10
1/10	3/80	3/5
0	7/16	3/2

$m=4$, $c=7/10$

3	7/5	2/5	0
13/8	21/40	1/40	1/8
2/3	1/15	1/15	2/3
1/8	1/40	21/40	13/8
0	2/5	7/5	3

$m=5$, $c=4/5$

...

One sees that for a given h_{pq} value, it only appears in one of the grids.

Identification of the critical Ising model as $m=3$

We shall identify the Ising model at its critical point as $m=3$, $c=\frac{1}{2}$.

The simplest possibility is to consider left-right symmetric field

$$\phi_{pq}(z,\bar{z}) = \phi_{pq}(z)\,\bar{\phi}_{pq}(\bar{z}) \quad \cdots (115)$$

When $h=\bar{h}$, $\langle \phi_{pq}(z,\bar{z})\,\phi_{pq}(0,0)\rangle$ depends only on $r=\sqrt{z\bar{z}}$. We shall assume $c=\bar{c}$.

This is exactly what one encounters in the Ising model. To identify c & h_{pq} for the Ising model at the critical point. We first examine the critical exponents for the Ising model at $d=2$.

First, one has $\quad \xi \sim t^{-\nu} \quad$ $t=\frac{T-T_c}{T_c}$, ξ = correlation length

$$f \sim \xi^{-d} \sim t^{\nu d}$$

$$C \sim \frac{\partial^2 f}{\partial t^2} \sim t^{\nu d-2} \quad \text{(specific heat)}$$

$$= t^{-\alpha}$$

$$\therefore \alpha = 2-\nu d$$

$f = \beta \varepsilon \quad \therefore \quad \frac{\partial f}{\partial \beta} = \varepsilon \qquad \frac{\partial f}{\partial \beta} = \frac{\partial f}{\partial T}\frac{\partial T}{\partial \beta} \sim k_B T^2 \frac{\partial f}{\partial T}$

$\uparrow$ energy density $\qquad \sim \frac{\partial f}{\partial t}$ to leading term

$$\therefore \varepsilon \sim \frac{\partial f}{\partial t} \sim t^{\nu d - 1} \sim \xi^{-\frac{\nu d - 1}{\nu}} = \xi^{-(d - 1/\nu)}$$

η is defined by

$$\langle \sigma(r) \sigma(0) \rangle \sim \frac{1}{r^{d-2+\eta}}$$

For 2D Ising model, it's found $\eta = 1/4$

$$\langle \sigma(r), \sigma(0) \rangle \sim \frac{1}{r^{1/4}} = \frac{1}{z^{\frac{1}{8}} \bar{z}^{\frac{1}{8}}} = \frac{1}{z^{2h_\sigma} \bar{z}^{2\bar{h}_\sigma}}$$

$\therefore h_\sigma = \bar{h}_\sigma = 1/16$ --- (115)

$\because C \sim \log t \quad \therefore \alpha = 0 = 2 - \nu d = 2 - 2\nu$

$\nu = 1$

$$\langle \varepsilon(r) \varepsilon(0) \rangle \sim \frac{1}{r^{2(d-1/\nu)}} = \frac{1}{r^2} = \frac{1}{z \bar{z}}$$

$\uparrow$

scales as $\xi^{-(d-1/\nu)}$

$$= \frac{1}{z^{2h_\varepsilon} \bar{z}^{2\bar{h}_\varepsilon}}$$

$\therefore h_\varepsilon = 1/2 = \bar{h}_\varepsilon$ --- (116)

Comparing with the conformal grid, it is clear that we can identify the critical Ising model with $m = 3$, $c = 1/2 = \bar{c}$

$$\phi_{2,1} \Rightarrow (\overset{h}{\tfrac{1}{2}}, \overset{\bar{h}}{\tfrac{1}{2}}) = \varepsilon = \text{energy density}$$

$$\phi_{1,2} \Rightarrow (\tfrac{1}{16}, \tfrac{1}{16}) = \sigma = \text{spin density}$$

--- (117)

Through the above identification, one can use eq. (114) to find the correlation function of

$$\langle \sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle .$$

For this purpose, we set $\begin{cases}\phi = \sigma(z_4) \text{ in eq. (114)} \\ h = 1/16\end{cases}$

and obtain $(z = z_4)$

$$\left(\frac{4}{3}\frac{\partial^2}{\partial z_4^2} - \sum_{j=1}^{3}\left[\frac{1/16}{(z_4-z_j)^2} + \frac{1}{z_4-z_j}\frac{\partial}{\partial z_j}\right]\right)\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle = 0$$

$\leftarrow$ (118)

Now, applying eq. (33) to this case with $\sigma(z\bar{z}) = \sigma(z)\bar{\sigma}(\bar{z})$

$$h_i = h_j = 1/16 \qquad h = \sum_{i=1}^{4} h_i = \frac{1}{4}$$

$$\therefore \prod_{i<j}^{4} z_{ij}^{h/3 - h_i - h_j} = \prod_{i<j}^{4} z_{ij}^{\frac{1}{12}-\frac{1}{8}}$$

$$\prod_{i<j}^{4} \bar{z}_{ij}^{\bar{h}/3-\bar{h}_i-\bar{h}_j} = \prod_{i<j}^{4} \bar{z}_{ij}^{\frac{1}{12}-\frac{1}{8}}, \quad \& \; x = \eta = \frac{z_{12}z_{34}}{z_{13}z_{24}}$$

We can absorb some factor of x into $f(x,\bar{x})$:

$$\prod_{i<j}^{4} z_{ij}^{h/3-h_i-h_j} = \frac{(z_{12}z_{23}z_{34}z_{14}z_{13}z_{24})^{\frac{1}{12}}}{(z_{12}z_{23}z_{34}z_{14}z_{13}z_{24})^{1/8}}$$

$$= \underbrace{\left(\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{14}}\right)^{\frac{1}{8}}}_{\text{global scaling } \lambda^{-\frac{1}{4}} \text{ the same as } \langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle} \; \underbrace{\frac{(z_{13}z_{24})^{\frac{1}{12}}}{(z_{13}z_{24})^{1/8\times 2}}}_{} (z_{12}z_{23}z_{34}z_{14})^{\frac{1}{12}}$$

$$\parallel$$

$$\frac{1}{(z_{13}z_{24})^{\frac{1}{12}}\,(z_{13}z_{24})^{1/12}}$$

$$= \left(\frac{z_{13}\, z_{24}}{z_{12}\, z_{23}\, z_{34}\, z_{14}}\right)^{\frac{1}{8}} \underbrace{\left(\frac{z_{12}\, z_{34}}{z_{13}\, z_{24}}\right)}_{x}{}^{\frac{1}{12}} \underbrace{\left(\frac{z_{23}\, z_{14}}{z_{13}\, z_{24}}\right)}_{1-x}{}^{\frac{1}{12}}$$

$\therefore$ One can write (with $z_{14} \to z_{41}$, $\bar{z}_{14} \to \bar{z}_{41}$ without changing sign)

$$\langle \sigma(z_1)\, \sigma(z_2)\, \sigma(z_3)\, \sigma(z_4) \rangle = \left(\frac{z_{13}\, z_{24}}{z_{12}\, z_{23}\, z_{34}\, z_{41}}\right)^{1/8} F(x) \quad \text{--- (119)}$$

with $x = \dfrac{z_{12}\, z_{34}}{z_{13}\, z_{24}}$

Substituting eq. (119) into eq. (118), one finds

$$\left(x(1-x) \frac{d^2}{dx^2} + \left(\frac{1}{2} - x\right) \frac{d}{dx} + \frac{1}{16} \right) F(x) = 0 \quad \text{--- (120)}$$

Eq. (120) has two independent solutions

$$f_{1,2}(x) = \left(1 \pm \sqrt{1-x}\right)^{1/2} \quad \text{--- (121)}$$

Which are multi-valued functions.

Since $\langle \sigma(z_1, \bar{z}_1)\, \sigma(z_2, \bar{z}_2)\, \sigma(z_3, \bar{z}_3)\, \sigma(z_4, \bar{z}_4) \rangle$ is a single-valued function, one must form

$$f_1(x) \underbrace{f_1(\bar{x})}_{=\,\overline{f_1(x)}} + f_2(x) f_2(\bar{x})$$

$\therefore \langle \sigma(z_1, \bar{z}_1)\, \sigma(z_2, \bar{z}_2)\, \sigma(z_3, \bar{z}_3)\, \sigma(z_4, \bar{z}_4) \rangle$

$$= a \left| \frac{z_{13}\, z_{24}}{z_{12}\, z_{23}\, z_{34}\, z_{41}} \right|^{\frac{1}{4}} \left(\left|1 + \sqrt{1-x}\right| + \left|1 - \sqrt{1-x}\right| \right) \quad \text{--- (122)}$$

some const.

where a is some constant. If we can fix a, $\langle \phi(z_1\bar{z}_1)\phi(z_2\bar{z}_2)\phi(z_3\bar{z}_3)\phi(z_4\bar{z}_4)\rangle$ is completely solved.

To find a, we explore $\langle \phi\phi\phi\phi \rangle$ from another point of view, the operator product expansion (OPE).

The expansion of radial ordering of operators as their arguements approach to the same can be expanded in term of operators with less # product is the operator product expansion.

An example is $T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w)$

$$+ \frac{1}{z-w}\partial T(w) + \cdots$$

In general, $A(x)B(y) \sim \sum_i C_i(x-y)\,O_i(y)$ --- (123)

where O_i's are a complete set of local operators and C_i's are (singular) numerical coefficients. By dimensional counting, if A, B, O are scaling fields,

$$C_i \sim \frac{1}{|x-y|^{\Delta_A + \Delta_B - \Delta_{O_i}}}$$

In complex notations at 2D, one writes

$$\phi_i(z,\bar{z})\,\phi_j(\omega,\bar{\omega}) \sim \sum_k C_{ijk}\,(z-\omega)^{h_k-h_i-h_j} \times (\bar{z}-\bar{\omega})^{\bar{h}_k-\bar{h}_i-\bar{h}_j}\,\phi_k(\omega,\bar{\omega})$$
$(|\omega - z| \to 0)$ (124)

so that $\langle \phi_i\,\phi_j\,\phi_k\rangle$ coincides with eq. (32)

if ϕ_i's are quasi-primary fields and $\langle \phi_i\,\phi_j\rangle$ are

normalized as $\langle \phi_i(z,\bar{z})\,\phi_j(\omega,\bar{\omega})\rangle$

$$= \delta_{ij}\,\frac{1}{(z-\omega)^{2h_i}}\,\frac{1}{(\bar{z}-\bar{\omega})^{2\bar{h}_i}}$$

Now, under spin reversal $\sigma \to -\sigma$, ε & $\sigma\sigma$ are

not changed. Therefore, one can write

$$\sigma(z_1\bar{z}_1)\,\sigma(z_2\bar{z}_2) \sim \frac{1}{z_{12}^{1/8}\,\bar{z}_{12}^{1/8}} + C_{\sigma\sigma\varepsilon}\, z_{12}^{h_\varepsilon - 2h_\sigma} \times \bar{z}_{12}^{\bar{h}_\varepsilon - 2\bar{h}_\sigma}\,\varepsilon(z_2,\bar{z}_2) + \cdots$$

$$\therefore\ \sigma(z_1\bar{z}_1)\,\sigma(z_2\bar{z}_2) = \frac{1}{|z_{12}|^{1/4}} + C_{\sigma\sigma\varepsilon}\,|z_{12}|^{3/4}\,\varepsilon(z_2,\bar{z}_2) + \cdots$$

When $z_{12} \to 0$ $\qquad h_\varepsilon = 1/2,\ h_\sigma = 1/16$

$\therefore$ By taking $z_{12} \to 0$ & $z_{34} \to 0$, we have

$$\langle \sigma(z_1\bar{z}_1)\,\sigma(z_2\bar{z}_2)\,\sigma(z_3\bar{z}_3)\,\sigma(z_4\bar{z}_4)\rangle$$
$$= \frac{1}{|z_{12}|^{1/4}\,|z_{34}|^{1/4}} + C^2_{\sigma\sigma\varepsilon}\,|z_{12}|^{3/4}\,|z_{34}|^{3/4}\,\langle \varepsilon(z_2\bar{z}_2)\,\varepsilon(z_4\bar{z}_4)\rangle + \cdots$$

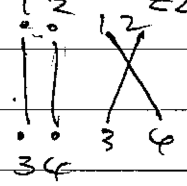
$$= \frac{1}{|z_{12}|^{1/4}|z_{34}|^{1/4}} + C^2_{\sigma\sigma\epsilon}\frac{|z_{12}|^{3/4}|z_{34}|^{3/4}}{|z_{24}|^2} + \cdots \quad \text{(124)}$$

On the other hand, when $z_{12}\to 0$, $z_{34}\to 0$, one has

(i) $x = \frac{z_{12}z_{34}}{z_{13}z_{24}} \to 0$.

(ii) $1+\sqrt{1-x} = 1+1-\frac{x}{2}$, $1-\sqrt{1-x} = \frac{x}{2}$

(iii) $\left|\frac{z_{13}z_{24}}{z_{23}z_{41}}\right| \sim 1$, $z_{13}\sim z_{24}$, $x \sim \frac{z_{12}z_{23}}{z_{24}^2}$



$\therefore$ Eq. (122) becomes

$$\langle \sigma(z_1\bar{z}_1)\sigma(z_2\bar{z}_2)\sigma(z_3\bar{z}_3)\sigma(z_4\bar{z}_4)\rangle$$

$$= a\left|\frac{1}{z_{12}z_{34}}\right|^{\frac14}\left(\left|2-\frac12\frac{z_{12}z_{34}}{z_{24}^2}\right| + \left|\frac12\frac{z_{12}z_{34}}{z_{24}^2}\right|\right)$$

$$\sim a\frac{1}{|z_{12}|^{\frac14}|z_{34}|^{1/4}}\left(2+\frac14\frac{|z_{12}||z_{34}|}{|z_{24}|^2}+\cdots\right)$$

Comparing with eq. (124), we got $a = 1/2$

$$C_{\sigma\sigma\epsilon} = \frac12$$

Hence $\langle \sigma(z_1\bar{z}_1)\sigma(z_2\bar{z}_2)\sigma(z_3\bar{z}_3)\sigma(z_4\bar{z}_4)\rangle$

$$= \frac12\left|\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{41}}\right|^{\frac14}\left(|1+\sqrt{1-x}|+|1-\sqrt{1-x}|\right)$$

In addition to $m=3$, $m=4$ can be identified with the tricritical Ising model (in $4-\varepsilon$ dimensions, this is described by the potential $V(\phi) = r/2\,\phi^2 + u\,\phi^4 + v\,\phi^6$ & tricritical point is at $r=u=0$), while $m=5$ is the 3-state Potts model.

We shall not go into further these identifications and end our introduction to conformal field theories here.