1/23/96

# Chapter II

## Symmetries and their consequences:

### Symmetries:

a sphere is "symmetric" w.r.t. rotations

⇒ two things are involved in this statement:

(i) operation: rotations

(ii) invariance (consequence): geometry is invariant in this case

In general, given a system, we can discuss its symmetry properties by first applying some symmetry operations then seeing their consequences.

The symmetry operations could be continuous (i.e., rotations) or discrete (i.e., time-reversal parity transformation).

The concept of symmetry in daily life is actually referring to geometry only, which is a static property of objects.

In physics, we extend it to include dynamics too.

For classical mechanics, one discusses symmetries at the level of equations of motions via the Lagrangian or the Hamiltonian:

e.g. If a system is translational-invariant, i.e., its description doesn't depend on how one chooses the origin, one has the Lagrangian $L$ is invariant under the operation $q_i \to q_i + \delta q_i$.

$$\therefore \quad \frac{\partial L}{\partial q_i} = 0$$

$$\therefore \quad \frac{dp_i}{dt} = \frac{\partial L}{\partial q_i} = 0$$

$\therefore$ the consequence is the conservation law : $p_i =$ const. indept. of time.

Note : the same problem can be formulated in terms of $H$ too :

$H$ is invariant under $q_i \to q_i + \delta q_i$

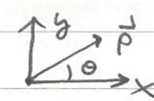
$$\frac{dp_i}{dt} = \frac{\partial H}{\partial q_i} = 0$$

$\therefore$ The consequence of a symmetry is the conservation law!

<u>More example</u> :

(i)

$$L = \frac{1}{2} m \dot{\vec{\rho}}^{\,2} - V(\rho)$$




$$= \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\theta}^2 - V(\rho)$$

$L$ invariant under $\theta \to \theta + \delta\theta$

$$\frac{\partial L}{\partial \dot\theta} = \frac{\partial L}{\partial \dot x}\frac{\partial \dot x}{\partial \dot\theta} + \frac{\partial L}{\partial \dot y}\frac{\partial \dot y}{\partial \dot\theta} = x p_y - y p_x = L_z$$

$$\therefore \quad \frac{d}{dt} L_z = 0 \quad , \quad L_z = \text{constant}.$$

(ii) time-translational invariant

If $t \to t + \delta t$, $H$ is invariant. i.e., no explicit $t$-dependence in $H$.

$$\because \quad dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt$$

$$\therefore \quad \frac{dH}{dt} = \underbrace{\frac{\partial H}{\partial q_i}}_{= -\dot p_i}\frac{dq_i}{dt} + \underbrace{\frac{\partial H}{\partial p_i}}_{= \dot q_i}\frac{dp_i}{dt} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} = 0 \qquad H \text{ is conserved!}$$

1/23/96

For quantum mechanics, symmetry operation $\Rightarrow$ transformation of $|\psi\rangle$.

In general, the symmetry operation maps $|\psi\rangle$ into $|\tilde{\psi}\rangle$:

$$|\tilde{\psi}\rangle = A|\psi\rangle$$

$$|\tilde{\phi}\rangle = A|\phi\rangle .$$

Since the operation does not change the relative probability, we impose the requirement

$$|\langle\tilde{\phi}|\tilde{\psi}\rangle|^2 = |\langle\phi|\psi\rangle|^2 , \quad \cdots \text{①}$$

i.e., the relative "angle" is preserved.

In particular, if $\phi = \psi$, this preserves the total probability.

If $\phi = |n\rangle$, this implies that the probability of finding $|\psi\rangle$ in $|n\rangle$ is not changed!

By appropriately choosing phases for $|\tilde{\psi}\rangle$ & $|\tilde{\phi}\rangle$, there are only two ways for ① ~~to~~ be satisfied: (see Messiah Chapter XV) Quantum Mechanics ↑ or Wigner's Group theory.. Chap. 26

Wigner theorem

(i) $A = U =$ unitary operator

$U^{+}U = \mathbb{I}$ , $\langle\tilde{\phi}|\tilde{\psi}\rangle = \langle\phi|\psi\rangle$

rotations, translations & parity

(ii) $A = \theta =$ antiunitary, $\langle\tilde{\phi}|\tilde{\psi}\rangle = \langle\phi|\psi\rangle^* = \langle\psi|\phi\rangle$, time-reversal

An operator $\theta$ is antiunitary if

$$\langle\tilde{\phi}|\tilde{\psi}\rangle = \langle\phi|\psi\rangle^*$$

$$\theta[c_1|\psi\rangle + c_2|\phi\rangle] = c_1^*\,\theta|\psi\rangle + c_2^*\,\theta|\phi\rangle$$

(antilinear)

The requiremet of "anti linear" is to include "superposition" into $\langle\tilde{\phi}|\tilde{\psi}\rangle = \langle\phi|\psi\rangle^*$.

An operator satisfying $\langle\tilde{\phi}|\tilde{\psi}\rangle = \langle\phi|\psi\rangle^*$ could be a nonlinear operator!

Consider $|\psi\rangle = c_1|\psi_1\rangle$ , $|\tilde{\psi}\rangle = \theta|\psi\rangle$
$|\phi\rangle = c_2|\phi_1\rangle$ , $|\tilde{\phi}\rangle = \theta|\phi\rangle$

For $\langle\tilde{\phi}|\tilde{\psi}\rangle = \langle\phi|\psi\rangle^*$ to be true,

we require : $\langle\tilde{\phi}|\tilde{\psi}\rangle = c_2 c_1^* \langle\phi_1|\psi_1\rangle^* = c_2 c_1^* \langle\tilde{\phi}_1|\tilde{\psi}_1\rangle$

$\therefore$ One requires $\left.\begin{aligned}\theta|\psi\rangle &= c_1^*\,\theta|\psi_1\rangle \\ \theta|\phi\rangle &= c_2^*\,\theta|\phi_1\rangle\end{aligned}\right\} \Rightarrow$ anti linear.

Any antiunitary operator $\theta$ can be written as

$$\theta = \underset{\substack{\uparrow \\ \text{unitary}}}{U}\;\underset{\substack{\uparrow \\ \text{Complex-conjugate operator}}}{K}, \qquad U = \theta K$$

Complex-conjugate operator: take the coefficients of a ket $\Rightarrow$ complex conjugate

<u>Action of K</u>: $Kc|\phi\rangle = c^* K|\phi\rangle$

if we can expand $|\phi\rangle$:

$$|\phi\rangle = \sum_n |n\rangle\langle n|\phi\rangle$$

$$K|\phi\rangle = \sum_n \langle n|\phi\rangle^* \underbrace{K|n\rangle}_{=|n\rangle} = \sum_n \langle n|\phi\rangle^* |n\rangle$$

$\Rightarrow \langle n|K|\phi\rangle = \langle n|\phi\rangle^*$ ($K$ does not change base ket!)

☆ Since K does not change the base ket, its form depends on <u>which basis is being used</u>!

$$|n\rangle = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ <u>see over</u>

1/23/96

Verify $UK$ is antiunitory:

anti linear:

$$UK\left[c_1|\phi\rangle + c_2|\psi\rangle\right]$$
$$= U\left[c_1^* K|\phi\rangle + c_2^* K|\psi\rangle\right]$$
$$= c_1^* UK|\phi\rangle + c_2^* UK|\psi\rangle$$

$\langle\hat{\phi}|\hat{\psi}\rangle = \langle\phi|\psi\rangle^*$

$$|\psi\rangle = \sum_n \langle n|\psi\rangle |n\rangle, \quad |\hat{\psi}\rangle = \sum_n \langle n|\psi\rangle^* U|n\rangle$$
$$= \sum_n \langle\psi|n\rangle\, U|n\rangle$$

$$|\hat{\phi}\rangle = \sum_n \langle\phi|n\rangle\, U|n\rangle$$

$$\therefore \langle\hat{\phi}|\hat{\psi}\rangle = \sum_{mn} \langle n|\phi\rangle \langle n|\underbrace{U^+U}_{=\mathbb{1}}|m\rangle \langle\psi|m\rangle$$
$$= \sum_m \langle\psi|m\rangle\langle m|\phi\rangle = \langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$$

## Active and Passive transformations:

Consider the quantity $\langle\phi|\hat{\Omega}|\psi\rangle$ which undergoes to $\langle\phi|A^+\Omega A|\psi\rangle$.

Two points of view:

(i) Active transformation $\psi \to A|\psi\rangle$

$$\langle\phi|\Omega|\psi\rangle \to \langle U\phi|\Omega|U\psi\rangle$$

is due to the change of $|\psi\rangle$, $\hat{\Omega}$ is not changed.

(ii) Passive transformation $|\psi\rangle$ is kept fixed.

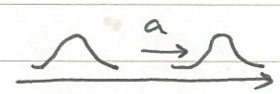
but $\hat{\Omega} \to U^+\hat{\Omega}U$

* Note that here, passive & active are used in different meaning from what we used in discussing translational operators where we were concerned about wether the coordinate axises are fixed or not!

By using the new "meaning" of terms, we have

**Active** $\hat{T}(a)\,|x\rangle = |x+a\rangle$

$\langle x|\,\hat{T}(a)\,|\psi\rangle = \psi(x-a)$




$|\phi\rangle \equiv \hat{T}(a)\,|\psi\rangle$

$\langle\phi|\,\hat{x}\,|\phi\rangle = \int dx\ \psi^*(x-a)\, x\, \psi(x-a)$

$= \langle x\rangle + a$

**Passive** $\hat{x} \to \hat{T}^\dagger(a)\,\hat{x}\,T(a) = e^{\frac{i}{\hbar}a\hat{p}}\,\hat{x}\,e^{-\frac{i}{\hbar}a\hat{p}} = \hat{x}+a$

$|x\rangle \to |x\rangle$

Recall that $\hat{T}(a) = e^{-\frac{i}{\hbar}a\hat{p}}$ for finite $a$.

$\hat{p}$ = generator of translation

because $\hat{p}$ generates the translation transformation.

* **Generator of symmetry** (for continuous symmetry)

Let $\hat{S}$ be a symmetry operator parametrized by $\varepsilon$, i.e., $\hat{S} = \hat{S}(\varepsilon)$. We will assume $\hat{S}$ is additive, i.e., $\hat{S}(a+b) = \hat{S}(a)\hat{S}(b)$.

For $\varepsilon \to 0$, $\hat{S}$ must be deviate from $\mathbb{I}$ by $O(\varepsilon)$.

$\therefore$ One writes $\hat{S}(\varepsilon) = \mathbb{I} - \frac{i\varepsilon}{\hbar}\hat{G}\ \ (\varepsilon\to 0)$

$\because$ $\hat{S}$ is unitary. $\mathbb{I} = \hat{S}^\dagger\hat{S} = (\mathbb{I} + \frac{i\varepsilon}{\hbar}\hat{G}^\dagger)(\mathbb{I} - \frac{i\varepsilon}{\hbar}\hat{G})$

$\Rightarrow \frac{i\varepsilon}{\hbar}(G^\dagger - G) = 0$. $G^\dagger = G$ $\therefore$ $G$ = Hermitian operator (could be observables!)

Given $\hat{G}$, one not only knows $\hat{S}(\varepsilon)$ at $\varepsilon \to 0^+$, but also knows $\hat{S}(\varepsilon)$ for finite $\varepsilon$:

For a finite $\varepsilon$, we divide it by $N$.

$$\varepsilon = \underbrace{\frac{\varepsilon}{N} + \frac{\varepsilon}{N} + \cdots + \frac{\varepsilon}{N}}_{N \text{ times}}$$

$$\therefore \hat{S}(\varepsilon) = \lim_{N\to\infty} \hat{S}\left(\frac{\varepsilon}{N}\right)\cdot \hat{S}\left(\frac{\varepsilon}{N}\right) \cdots \hat{S}\left(\frac{\varepsilon}{N}\right)$$

$$= \lim_{N\to\infty} \left(\mathbb{I} - \frac{i\varepsilon}{\hbar N}\hat{G}\right)^N$$

$$= e^{-\frac{i\varepsilon}{\hbar}\hat{G}}.$$

In other words, $\hat{G}$ generates all possible transformations

$\hat{G}$ is called the generator of "xxx".

↑↑ name of symmetry.

e.g. translation, $\hat{T}(a) = e^{-\frac{i}{\hbar}a\hat{P}}$, $\hat{P}$ = generator of translation

$\hat{P}^\dagger = \hat{P}$ is Hermitian and an observable.

time-translation, $\hat{T}(\Delta t) = e^{-\frac{i}{\hbar}\Delta t \hat{H}}$, $\hat{H}$ = generator of time-translation

$\hat{H}^\dagger = \hat{H}$ is Hermitian and also an observable.

* Symmetry in Quantum Mechanics:

Classical: an operation is a symmetry if it leaves $H$ (or $L$) invariant.

quantum: the same: $\therefore S^\dagger H S = H$ (passive view)

To $O(\varepsilon)$, this implies $[G, H] = 0$

No.

Date

1/23/96 

$\therefore \quad \hat{G}_H = e^{\frac{i}{\hbar}Ht}\,\hat{G}\,e^{-\frac{i}{\hbar}Ht} = \hat{G}$

$$\frac{d\hat{G}_H}{dt} = \frac{-i}{\hbar}[\hat{G}_H, H] = \frac{-i}{\hbar}[\hat{G}, H] = 0$$

$\hat{G}_H$ = a constant !

$\therefore \langle G \rangle$ = a constant, $\hat{G}$ is conserved !

Example: If $\hat{H}$ is invariant under $x \to x + \delta x$,

$\Rightarrow \hat{p}$ is conserved ! $\quad (\dot{p} = 0)$

This agrees with our intuition : " $\hat{H}$ is invariant under $x \to x + \delta x$" implies there is no external interaction (i.e., $\hat{H}$ describes an isolated system !)

$\therefore \vec{p}$ is conserved !

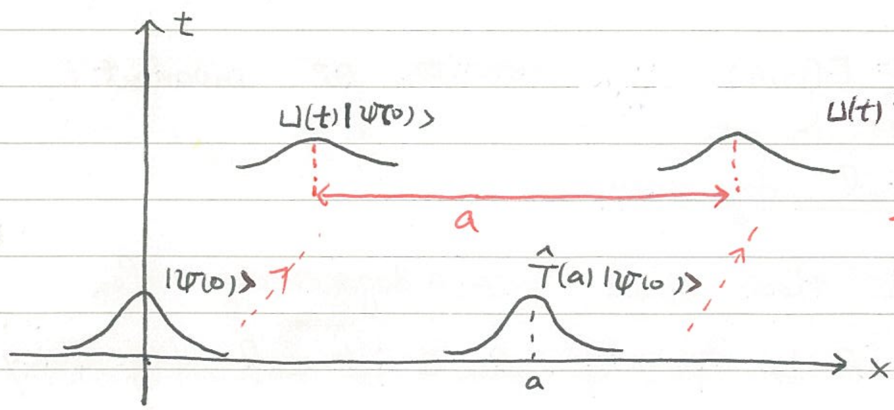
$\Rightarrow$ see over

<u>pictorial understanding</u> :

$[\hat{p}, \hat{H}] = 0 \qquad U = e^{\frac{-i\hat{H}}{\hbar}t}$

$\Rightarrow [\hat{p}, \hat{U}(t)] = 0 \qquad$ i.e. $\quad [\hat{T}(a), \hat{U}(t)] = 0$

$\therefore \quad \hat{T}(a)\,\hat{U}(t)\,|\psi(0)\rangle = \hat{U}(t)\,\hat{T}(a)\,|\psi(0)\rangle$




to compare $U(t)|\psi(0)\rangle$ & $U(t)\hat{T}(a)|\psi(0)\rangle$ one shift $U(t)|\psi(0)\rangle$ to $\hat{T}(a)U(t)|\psi(0)\rangle$ $= U(t)\,\hat{T}(a)\,|\psi(0)\rangle$

$\therefore$ translated state $(\hat{T}(a)|\psi(0)\rangle)$ evolves into the same state as evolved untranslated state $(\hat{U}(t)|\psi(0)\rangle)$ being translated again !

the state obtained by

They will not be the same if $|\psi(0)\rangle$ and $\hat{T}(a)|\psi(0)\rangle$ have different momenta or under different "acceleration"!

<u>System of many particles</u>:

$$\hat{T}(a)\,|x_1, x_2, \ldots, x_N\rangle = |x_1+a, x_2+a, \ldots, x_N+a\rangle$$

i.e. $$\hat{T}(a) = \hat{T}^{(1)}(a) \otimes \hat{T}^{(2)}(a) \otimes \cdots \otimes \hat{T}^{(N)}(a)$$

e.g. $$\hat{T}(a)\,|x_1 x_2, S\rangle = \hat{T}(a)\frac{1}{\sqrt{2}}\left(|x_1 x_2\rangle + |x_2 x_1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|x_1+a, x_2+a\rangle + |x_2+a, x_1+a\rangle\right)$$

$$= |x_1+a, x_2+a, S\rangle$$

$$\therefore\ \hat{T}(a) = e^{-\frac{ia}{\hbar}\hat{p}^{(1)}} \otimes e^{-\frac{ia}{\hbar}\hat{p}^{(2)}} \otimes \cdots \otimes e^{-\frac{ia}{\hbar}\hat{p}^{(N)}}$$

$$= e^{-\frac{i}{\hbar}a\,[\underbrace{\hat{p}^{(1)} \oplus \hat{p}^{(2)} \oplus \cdots \oplus \hat{p}^{(N)}}_{\hat{P}}]}$$

$\hat{p}^{(i)}$ is actually $\mathbb{I}^{(1)} \otimes \mathbb{I}^{(2)} \otimes \cdots \otimes \hat{p}^{(i)} \otimes \mathbb{I}^{(i+1)} \otimes \cdots \otimes \mathbb{I}^{(N)}$.
$= \hat{p}_i^{(1)\otimes(2)\otimes\cdots\otimes(N)}$.

i.e., $$\hat{P} = \frac{i}{\hbar}\left(\frac{d}{dx_1} + \frac{d}{dx_2} + \cdots + \frac{d}{dx_N}\right)$$

Symmetry: $$\hat{T}^{+}(a)\, H(x_1, \ldots, x_N)\, \hat{T}(a) = H(x_1, \ldots, x_N)$$

example: $$\hat{H} = \frac{1}{2m_1}\hat{P}_1^{\,2} + \frac{1}{2m_2}\hat{P}_2^{\,2} + V(x_1 - x_2)$$

$$\hat{T}^{+}(a)\, H(x_1, x_2)\, \hat{T}(a) = \hat{H}(x_1+a, x_2+a) = \frac{1}{2m_1}\hat{P}_1^{\,2} + \frac{1}{2m_2}\hat{P}_2^{\,2} + V(x_1+a-x_2-a)$$

$$= \hat{H}(x_1, x_2)$$

$\vec{P}$ is conserved!

The translational invariance of an isolated system reflects the uniformity or homogeneity of space, i.e., the dynamics of the isolated system depends only on relative coordinates of the constituents not on where the system is!

Because of this property, one expt. performed in U.S. must produce the same result as the one performed in Taiwan.

Time-translation invariance:

Expt 1. $t_1$ $|\psi_0\rangle$, $|\psi(t_1+\varepsilon)\rangle = e^{\frac{-i\varepsilon}{\hbar}\hat{H}(t_1)}|\psi_0\rangle$

$(\varepsilon \ll 1)$

$\left( |\psi(\Delta t)\rangle = |\psi(0)\rangle + \Delta t \frac{d}{dt}|\psi(0)\rangle = |\psi(0)\rangle - \frac{i\Delta t}{\hbar}\hat{H}(0)|\psi(0)\rangle \right)$

$= \left(1 - \frac{i\varepsilon}{\hbar}\hat{H}(t_1)\right)|\psi_0\rangle$

Expt. 2. $t_2$ $|\psi_0\rangle$, $|\psi(t_2+\varepsilon)\rangle = \left(1 - \frac{i\varepsilon}{\hbar}\hat{H}(t_2)\right)|\psi_0\rangle$

invariance $\Rightarrow$ $|\psi(t_1+\varepsilon)\rangle = |\psi(t_2+\varepsilon)\rangle$

$\hat{H}(t_1) = \hat{H}(t_2)$

i.e. $\hat{H}$ doesn't have explicit $t$ dependence!

$\frac{\partial \hat{H}}{\partial t} = 0$

$\therefore \langle \dot{\hat{H}} \rangle = \frac{i}{\hbar}\langle [H, H] \rangle + \langle \frac{\partial H}{\partial t} \rangle$

$= 0$

$\langle H \rangle$ is a constant

$\Rightarrow$ Conservation of energy: all known interactions obey this invariance!

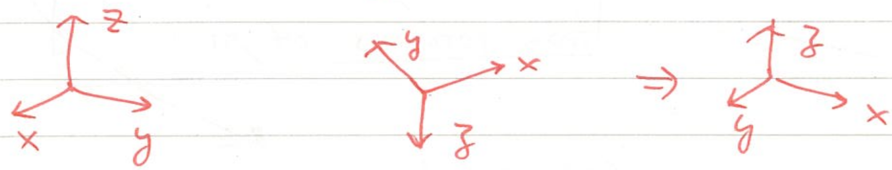
1/28/96

Discrete symmetries : operations that can be obtained by applying successively infinitesimal symmetry operations are continuous symmetry operations.

Those that can't be obtained this way ⇒ discrete. (have no generators!)

(i) Parity (Space inversion) : $\Pi$ $\boxed{\det = -1}$

right-handed → left handed



Active : $\Pi|x\rangle = |-x\rangle$ $\quad (\Pi|\vec{r}\rangle = |-\vec{r}\rangle)$ ⇒ all over

Passive : $\Pi^{+}\vec{r}\,\Pi = -\vec{r}$

$\because \langle x'|\Pi^{+}\hat{x}\,\Pi|x\rangle = \langle -x'|\hat{x}|-x\rangle = -x\,\delta(x'-x)$

$= -\langle x'|\hat{x}|x'\rangle \Rightarrow \Pi^{+}\hat{x}\,\Pi = -\hat{x}$

$\Pi$ in principle, a phase factor $e^{i\delta}$ is allowed. But conventially, it's set to be 1.

Wave function under parity transformation :

$$\Pi|\psi\rangle = \Pi\int dx\ |x\rangle\langle x|\psi\rangle$$

$$= \int dx\ |-x\rangle\langle x|\psi\rangle$$

$$= \int dx\ |x\rangle\langle -x|\psi\rangle$$

$$\therefore \langle x|\Pi|\psi\rangle = \langle -x|\psi\rangle = \psi(-x)$$

This implies $\langle x|\Pi|p\rangle = \langle -x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\,e^{-\frac{i x}{\hbar}p}$

$= \langle x|-p\rangle \quad \therefore \quad \Pi|p\rangle = |-p\rangle, \Rightarrow \Pi^{+}\hat{p}\,\Pi = -\hat{p}$

* $\Pi$ : unitary $\quad \langle\phi|\Pi^{+}\Pi|\psi\rangle = \int dx\ \langle\phi|\Pi^{+}|x\rangle\langle x|\Pi|\psi\rangle$

$= \int dx\,[\langle x|\Pi|\phi\rangle]^{*}\langle x|\Pi|\psi\rangle$

$= \int dx \; \langle \phi | -x \rangle \langle -x | \psi \rangle = \int dy \; \langle \phi | y \rangle \langle y | \psi \rangle$

$= \langle \phi | \psi \rangle$ for any $|\psi\rangle$ & $|\phi\rangle$

$\therefore \; \pi^{+} \pi = \mathbb{I}$

* $\pi^{+} \vec{r} \pi = -\vec{r} \Rightarrow \pi \vec{r} = -\vec{r} \pi$, $\therefore \{\pi, \vec{r}\} = 0$. $\vec{r}, \vec{p}$ : polar vectors

similarly, $\{\pi, \vec{p}\} = 0$

$\therefore \; \pi^{+} \vec{r} \times \vec{p} \, \pi = \vec{r} \times \vec{p}$, i.e., $\pi^{+} \vec{L} \pi = \vec{L}$

$\pi^{+} \vec{S} \pi = \pi^{+} \vec{S} \pi$, $\pi^{+} \vec{J} \pi = \vec{J}$, $\vec{J} = \vec{L} + \vec{S}$ (axial vectors or pseudo vectors)

* $\hat{\pi} \hat{\pi} |x\rangle = \hat{\pi} |-x\rangle = |x\rangle$

$\therefore \; \hat{\pi}^{2} = \mathbb{I}$

$\pi^{-1} = \pi = \pi^{+}$, $\pi$ is Hermitian

* eigenvalues of $\pi = \pm 1$

$+1$, $\pi |\phi\rangle = |\phi\rangle$, $\Rightarrow$ even function $(\phi(-x) = \phi(x))$ : even parity

$-1$, $\pi |\phi\rangle = -|\phi\rangle$, $\Rightarrow$ odd function $(\phi(-x) = -\phi(x))$ : odd 〃

* <u>relation to mirror image</u> :

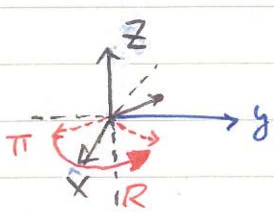
e.g. reflection (mirror reflection) w.r.t. XY plane

$z \to -z$

$= \underbrace{\text{rotation } \pi \text{ about } z}_{R} + \text{parity}$

$R \, \pi |x, y, z\rangle = R |-x, -y, -z\rangle$

$= |x, y, -z\rangle$



* $H(x,p)$ is parity invariant if

$$\pi^{+} H(x,p)\, \pi = H(-x,-p) = H(x,p)$$

$\therefore [\pi, H] = 0$ e.g. $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$ ✓ $H = \frac{I}{2}L^2$ ✓ $S \cdot p$ X

(i) $\Rightarrow \langle \frac{d\pi}{dt} \rangle = 0$ $\therefore$ if initially, if $|\psi(0)\rangle$ is a parity eigenstate $\Rightarrow |\psi(t)\rangle$ is a parity eigenstate with the same parity if $[\pi, H] = 0$.

(ii) If the energy eigenkets are nondegenerate, they must have definite parity. ($[\pi, H] = 0$)

pf: Suppose $\hat{H}|n\rangle = E_n|n\rangle$ $\Rightarrow$ see over for another easier proof.

Consider $\frac{1}{2}(1 \pm \pi)|n\rangle$

$$\because \hat{H}\left[\frac{1}{2}(1 \pm \pi)|n\rangle\right] = E_n\left[\frac{1}{2}(1 \pm \pi)|n\rangle\right]$$

$\therefore \frac{1}{2}(1 \pm \pi)|n\rangle$ must $= \alpha|n\rangle$ ($\because$ nondegenerate)

$\therefore |n\rangle$ must be a parity eigenket.

e.g. 1D harmonic oscillator, $\psi_n(-x) = (-1)^n \psi_n(x)$

* $H$ parity invariant,

$$\hat{\pi}\hat{U}(t) = \hat{U}(t)\hat{\pi}$$

Two routes produce the same state

$$\begin{tikzcd}
\pi|\psi(0)\rangle \arrow[r, leftrightarrow] \arrow[d, "U(t)"] & |\psi(0)\rangle \arrow[d, "U(t)"] \\
U(t)\pi|\psi(0)\rangle & \hat{U}(t)|\psi(0)\rangle = |\psi(t)\rangle \\
\pi|\psi(t)\rangle \arrow[u, equal] \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

No.

Date

3/4/96



## * Parity non conservation in weak interaction

mirror image = rotation + parity

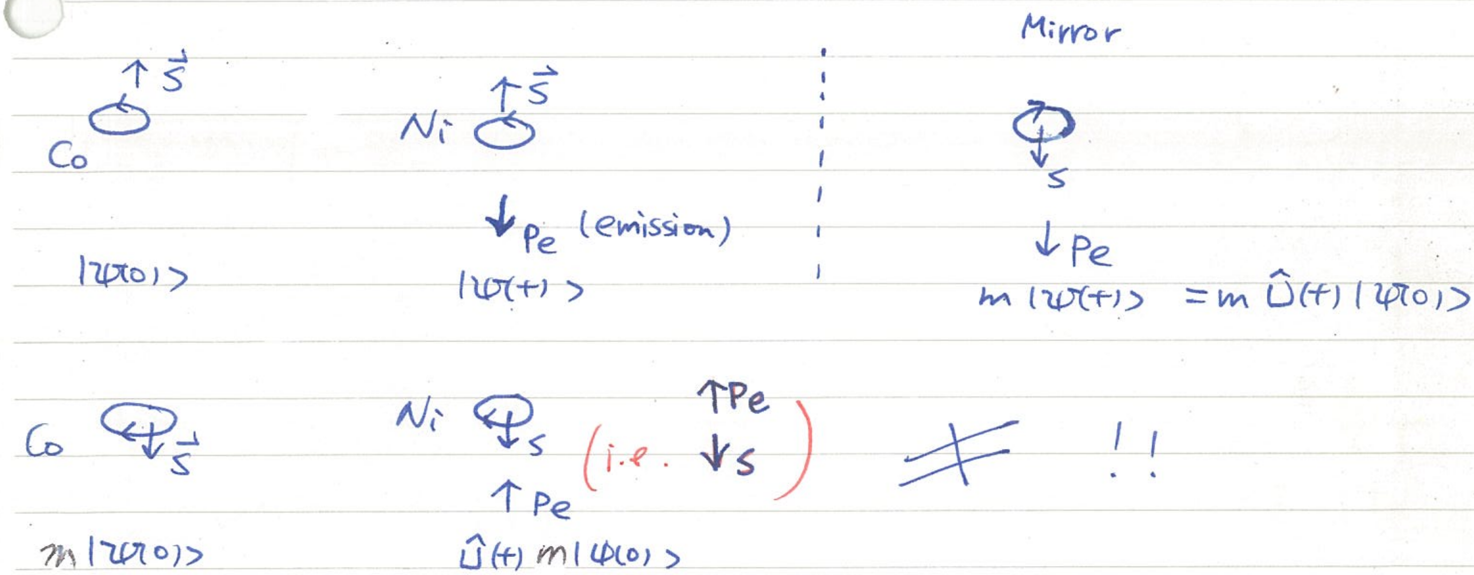
For weak interaction, rotational invariance is Known to hold !

∴ Check mirror-image invariane = Check parity invariance.

$$^{27}Co^{60} \rightarrow {}^{28}Ni^{60} + e^- + \bar{\nu} \quad (\beta\text{-decay})$$

(Lee & Yang suggested, Wu performed).

$(n \rightarrow p + \bar{e} + \bar{\nu})$




Co's dipole moments are aligned in a very strong magnetic field in a low temperature.

Note that if the momentum of electrons are emitted symmetric v.s. ⟳ plane.

⇒ no parity violation!

3/4/96 $\vec{B}$ ? 

The fact that Wu's expt. shows is that there is a preferred direction to emit electrons for a given $\vec{S}$ of Co.

i.e. the angular dependence of emission depends on $\underbrace{\langle \vec{p} \rangle \cdot \vec{S}}_{\substack{\shortparallel \\ S|\langle \vec{p} \rangle|\cos\theta}}$ . giving a maximum at $\theta = \pi$

This indicates that H depends on $\vec{S} \cdot \langle \vec{p} \rangle$ which is not parity invariant.

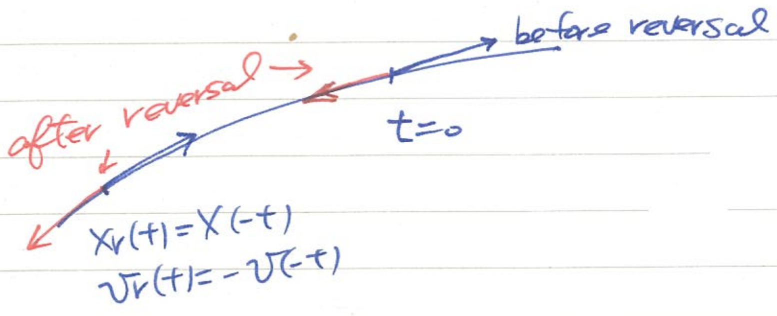
(ii) Time-reversal symmetry

classically, $t \to -t$ given a time $t$. $x_r(t) = x(-t)$

$$x_r(t) \equiv x(-t)$$

$$\dot{x}_r(t) = \frac{d\,x(-t)}{dt} = -\frac{dx(-t)}{d(-t)} = -\dot{x}(-t)$$

i.e. $v_r(t) = -v(-t)$




$x_r(t)$ still satisfies Newton's law:

$$m\frac{d^2x_r(t)}{dt^2} = F(\underline{x_r}) \quad \text{(not } v\text{!)}$$

$$\because \quad m\frac{d^2x_r(t)}{dt^2} = m\frac{d^2x(-t)}{d(-t)^2} = F(x(-t))$$

$$= F(x_r(t))$$

$\Rightarrow$ this is time-reversal invariance in Newtonian physics:
reversing a particle's velocity will trace back its old orbit. for $\underline{t<0}$

3/4/96

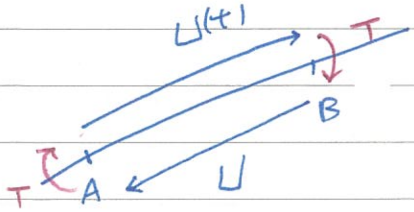
$p \to -p, \ldots$

(T) acts on states

Note that the time-reversal operation ~~will inevitably involve time-translation operation (U)~~, when combined with U

$\Rightarrow$ Change $t \to -t$

Take the above as an example, what should be stated is : $TU(t)\,TU(t) = 1$




Obviously,

$U(-t)U(t) = 1$

$T^2 = 1$

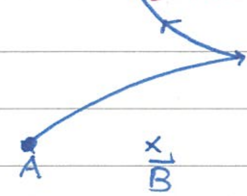
T-invariant $\Downarrow$ $\quad t \to -t \to$ same as ①'

$\therefore$ This implies : $TU(t) = U(-t)\,T \quad \ldots\ldots$ ①

i.e. $T\,U(t)\,T^{-1} = U(-t)$

(Note that the time-reversal symmetry is not obeyed if there is a magnetic field (external)

time-reversal path.




$T_H = U(-t)\,T\,U^{-1}(t)$
$= U(t)\,T\,U(-t)$
$= U^2(t)\,T$

This is because we did not switch $\vec{B} \to -\vec{B}$. If we also reverse $\vec{B}$ ($\because \vec{B}$ is produced by electrons or other charge carriers), then the time-reversal symmetry is obeyed again.)

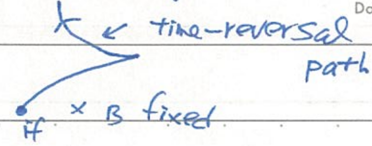
① implies. $\quad T e^{-\frac{i}{\hbar}Ht} = e^{\frac{i}{\hbar}Ht}\,T$.

In the limit of small $t$, we get

$$-iHT = TiH$$

two possibilities. (i) $T$ ~~unitary~~ linear $\Rightarrow TH = -TH$ i.e. $\{T, H\} = 0$

3/4/96 Note that if there is a magnetic field, $\vec{B} \to -\vec{B}$ 




Quantum Mechanics

$t=0$. $|\Psi\rangle = |\alpha\rangle$

$t=\delta t$ $|\Psi, \delta t\rangle = \left(1 - \frac{iH}{\hbar}\delta t\right)|\alpha\rangle$

$t=0$. $|\tilde{\Psi}\rangle = T|\alpha\rangle$ ↙ $t=0$

$t=\delta t$, $|\tilde{\Psi}, \delta t\rangle = \left(1 - \frac{iH}{\hbar}\delta t\right) T|\alpha\rangle$

Time-reversal invariant : $|\tilde{\Psi}, \delta t\rangle = T\,|\Psi, -\delta t\rangle$

trace back the state at $-\delta t$ with $p \to -p$, ...

$$\Rightarrow \left(1 - \frac{iH}{\hbar}\delta t\right) T|\alpha\rangle = T\left(1 + \frac{iH}{\hbar}\delta t\right)|\alpha\rangle \quad \cdots \text{①}'$$

$$\therefore -iHT = TiH$$

Two possibilities

(i) $T$ is ~~unitary~~ linear $\Rightarrow$ $TH = -TH$

$\therefore \{T, H\} = 0$ (H is chiral)

This implies that if $|\psi_E\rangle$ is an energey eigenket

$H|\psi_E\rangle = E|\psi_E\rangle$ , $T|\psi_E\rangle$ has negative energy $-E$ !

Since usually, there is no upper bound for E, this then implies there is no lower bound ! This is not acceptable !

(ii) $T$ antilinear $\therefore$ $TH = HT$, i.e. $[T, H] = 0$ ($\because TU(t) = U(-t)T$)

$T|\psi_E\rangle$ & $|\psi_E\rangle$ have the same energy

$\Rightarrow$ Kramer degeneracy (half-integer spin system)

$\because$ Any antiunitary operator can be rewritten as

$$T = \tilde{U}K$$

$\tilde{U}$ ↑ unitary, $K$ ← complex conjugate

$\therefore$ $H\tilde{U}K = \tilde{U}KH = \tilde{U}H^*K$

~~$\therefore H^* = \tilde{U}^\dagger H \tilde{U}$ —— ①~~

~~$(\tilde{U} = \Pi$ $\Pi^\dagger H \Pi = H^*)$~~

can find $\tilde{U}$ such that.

i.e. if the system is time-reversal invariant $\Leftrightarrow$ $H^* = \tilde{U}^\dagger H \tilde{U}$

$H^* = T^{-1}HT$

What is $\tilde{U}$ ?, i.e., what is time-reversal operator ?

Let $|\psi'\rangle = T|\psi\rangle$, we require

$$\langle\psi'|\vec{r}|\psi'\rangle = \langle\psi|\vec{r}|\psi\rangle$$

$$\langle \psi' | \vec{p} | \psi' \rangle = - \langle \psi | \vec{p} | \psi \rangle$$

$$\langle \psi' | \vec{L} | \psi' \rangle = - \langle \psi | \vec{L} | \psi \rangle$$

Claim: when no spin involved. $\Uparrow$ i.e. $T = K^*$

$\psi' = T\psi = \psi^*$ can satisfy the above requirements (i.e. $U = \mathbb{1}$) ($\because H^* = H$)

i.e. $\langle \vec{r} | T | \psi \rangle = \psi^*(\vec{r})$

$$\langle \vec{r} \rangle' = \int (\psi^*)^* \vec{r}\, \psi^*\, d^3\vec{r} = \int \psi^* \vec{r}\, \psi\, d^3\vec{r} = \langle \vec{r} \rangle$$

$$\langle \vec{p} \rangle' = \int \psi\, (-i\hbar\vec{\nabla})\, \psi^*\, d^3\vec{r} = \int \psi^* (i\hbar\vec{\nabla})\, \psi\, d^3\vec{r} = -\langle \vec{p} \rangle$$

integration by part

$$\langle \vec{L} \rangle' = \int \psi\, \vec{r} \times (-i\hbar\vec{\nabla})\, \psi^*\, d^3\vec{r}$$

$$= \int \psi^*\, i\hbar\, \vec{r} \times \vec{\nabla}\, \psi\, d^3\vec{r} = -\langle \vec{L} \rangle$$

$$\nabla \times \vec{r} = 0, \quad \vec{r} \times \psi \vec{\nabla} \psi^* = \underbrace{\vec{r} \times \nabla(\psi\psi^*)}_{\nabla \times (\psi^*\psi\vec{r}) - \psi^*\psi\, \nabla \times \vec{r} \to 0} - \psi^* \vec{r} \times \nabla\psi$$

What about $\psi'(\vec{r}, t)$?

Note that $\psi'(\vec{r}, t) \neq T\psi$ instead $\psi' = U(t)T\psi$ !

$$\psi'(\vec{r}, t) = \langle \vec{r} | U(t)\, T | \psi(0) \rangle$$

$$= \langle \vec{r} | T\, U(-t) | \psi(0) \rangle = \langle \vec{r} | T | \psi(-t) \rangle$$

$= \psi^*(\vec{r}, -t)$. which will be checked in Schrödinger eq.

example. plane wave: $\psi(\vec{r}, t) = e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{r} - Et)}$

$$\psi'(\vec{r}, t) = \psi^*(\vec{r}, -t) = e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{r} - \frac{i}{\hbar}Et}$$

i.e. changing $\vec{p} \to -\vec{p}$, $t \to -t$ is not enough, taking

3/4/96 

Schrödinger eq.

$$i\hbar \frac{d}{dt}\psi(r,t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi(r,t)$$

if $V^* = V$

$$\Rightarrow \quad -i\hbar \frac{d}{dt}\psi^*(r,t) = \left[\frac{-\hbar^2}{2m}\nabla^2 + V(r)\right]\psi^*(r,t)$$

$$\therefore i\hbar \frac{d}{dt}\psi^*(r,-t) = \left[\frac{-\hbar^2}{2m}\nabla^2 + V(r)\right]\psi^*(r,-t)$$

$$\langle p\rangle(t) = \int d\vec{r}\,[\psi^*(\vec{r},-t)]^* \frac{\hbar}{i}\nabla\,\psi^*(\vec{r},-t) \underset{\text{integrated by part}}{=} \int \psi^*(\vec{r},-t)\left(-\frac{\hbar}{i}\nabla\right)\psi(\vec{r},-t) = -\langle p\rangle(-t)$$

i.e. if $\psi(r,t)$ is a solution

$\Rightarrow$ $\psi^*(r,-t)$ is also a solution!

$$\psi'(r,t') = \psi^*(r,-t) = \psi^*(r,t') \quad !$$

$\Leftarrow$ check for plane wave!

In the presence of a magnetic field $\vec{B}$,

$$H^*(\vec{r},\vec{B}) = H(\vec{r},-\vec{B}) \neq H(\vec{r},\vec{B})$$

$\therefore$ only if we also change $\vec{B} \to -\vec{B}$ at the same time,

$\psi' = \psi^*(r,t')$ is also a solution.

Otherwise, this is not correct!

Note that there is no conservation law associated with time-reversal invariance even though $[T,H]=0$.

This is because $T$ is antiunitary, $\frac{dT}{dt}$ does not obey a similar eq. of motion: $\frac{dA_H}{dt} = \frac{1}{i\hbar}[A_H, H]$.

Recall how this eq was derived: $A_H = e^{\frac{i}{\hbar}Ht} A\, e^{-\frac{i}{\hbar}Ht}$

We use $AiH = iAH$ which is not correct for $T$!

$A\,\frac{i}{\hbar}H = \frac{i}{\hbar}AH$

* ~~PCT~~ CPT theorem

↑

Charge conjugation : symmetry between particles & antiparticles

charges $\begin{matrix} + \\ - \end{matrix}$ $\begin{matrix} - \\ + \end{matrix}$

$K^0$, $\bar{K}^0$ decay violates CP

$\Rightarrow$ T is not invariant in this system!

(Assume a local quantum field theory + quantization respects the spin statistics + action principle involving an hermitian Lorentz-invariant lagrangian.)

Any Lorentz invariant local quantum field theory with a Hermitian Hamiltonian $\Rightarrow$ CPT

* Broken symmetry (Note that CP violation is not an example of what is said below!)

As we said, when the system has certain symmetry, the generator of that symmetry, G, commutes with H.

As a result, we can use g to further label the eigenket of H! This causes degeneracy!

This, by no means, implies that when there is degeneracy, the eigenkets of H must also be eigenkets to G!

physical realizable

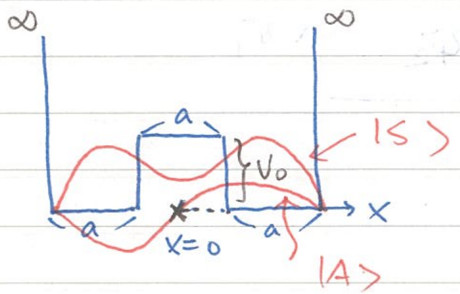
The reasoning is simple : $H|\psi\rangle = E|\psi\rangle$

$$[H, G] = 0 \Rightarrow H(G|\psi\rangle) = E(G|\psi\rangle)$$

$\because$ degenerate, $\therefore$ $G|\psi\rangle$ need not to be proportional to $|\psi\rangle$!

When this happens, the symmetry of $H$ is not realized in $|\Psi\rangle$ (observed state). We say that the symmetry is spontaneously broken.

<u>A good example</u>: symmetrical double-well potential




$H(-x) = H(x)$, i.e., $[H, \pi] = 0$

$\pi|S\rangle = |S\rangle$

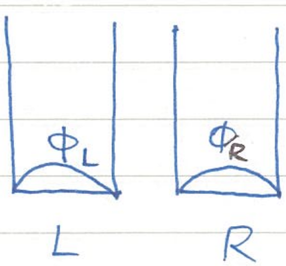
$\pi|A\rangle = -|A\rangle$

$\therefore$ $|S\rangle$, $|A\rangle$ are simultaneous eigenkets of $H$ & $\pi$.

In fact, $|S\rangle$ has lower energy!

$E_A > E_S$ i.e. they are not degenerate!

When $V_0 \to \infty$, $|S\rangle$ & $|A\rangle$ becomes $\phi_L \pm \phi_R$




$\phi \propto \sin\frac{\pi}{a}x$

Furthermore, $|S\rangle$ & $|A\rangle$ become degenerate!

("violate the non-degenerate theorem ⇒ see over

Obviously, the physical realizable states are $|L\rangle$ or $|R\rangle$ because when one puts a particle in the wells, it will not escape from the well it resides since $V_0 \to \infty$ and the tunneling rate $\to 0$!

$\therefore$ $|L\rangle$ & $|R\rangle$ do not realize the symmetry of $H$!

This is an example of broken symmetry



Quite often, when the symmetry is broken, the Hilbert space is broken into several disconnected regimes which can not tunnel into other regimes!

Another often-mentioned example is the ferromagnets:

$\uparrow\uparrow\uparrow\ |\vec{M}\rangle$ flip to $\longrightarrow$ $\searrow\searrow\searrow\ |\vec{M}'\rangle$ (In terms of Ising model)

causes $\infty$ energy!

$H$ = rotational invariant.

in fact is not

$\Rightarrow$ ~~symmetry broken~~. $\Longrightarrow$ not quite right

$\because$ each $|\vec{M}\rangle$ is still the eigenket to the generator $\vec{J}$ (angular momentum)

In nature, we observe a lot of symmetry broken examples. The prevailing belief is that they are not resulted from $\hat{H}$ but because the physical realizable state breaks the symmetry!!

But how do we tell which is the right one?

In the example of ferromagnets, when $H$ is rotational invariant, one will find that single ferromagnet has a definite $\vec{M}$, but an assemble of them can have various directions of $\vec{M}$, each of which is equally probable. If, $H$ is not rotational invariant, one finds that some of $\vec{M}$ will be more easily realized!

3/6/97

Remarks on gauge invariance

Similarly transformation : (unitary) Consider a transformation $A$ ← unitary $A^{-1} = A^{+}$

$$H|\Psi\rangle = E|\Psi\rangle$$

$$AH|\Psi\rangle = EA|\Psi\rangle$$

$$\underbrace{AHA^{-1}}_{H'}(\underbrace{A|\Psi\rangle}_{|\Psi'\rangle}) = E\,A|\Psi\rangle$$

$\therefore \langle\Psi'|H'|\Psi'\rangle = E\langle\Psi'|\Psi'\rangle$, i.e. $\langle H\rangle$ invariant! --- ①

Note that the most general transformation is

$H \to H'$ and $H'$ is not necessarily $AHA^{-1}$!

$\therefore$ Given $H'$, we can not write $H'$ as $AHA^{-1}$ in general!

However, as we mentioned, most transformations we encountered are ! (such as translations, rotations, ...)

i.e. $H(r+a) = T_a^{+} H T_a$ ...

↑ $H'$ is usually what we have in mind!

The similar transformations have the nice property that one can do transformation on the individual part of the operator under concerned !

$(\because A\hat{O}\hat{P}A^{-1} = A\hat{O}A^{-1}A\hat{P}A^{-1} = \hat{O}'\hat{P}')$

$H' = H(x+a)$ ] moves origin O

$\psi' = T_a^{\dagger}\psi = \psi(x+a)$ ] $\psi \xrightarrow{a} \psi'$

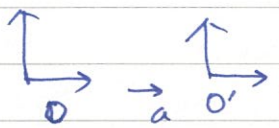
3/6/97

The invariant property, however, is trivial. It has nothing to do with the invariance we mentioned in the symmetry consideration!

Let me clarify this by the translational operator:

Recall passive or active.

external $\to H$




system $\to |\psi\rangle$

$\Leftarrow$ here both passive and active simultaneously

The invariance in ① is simply the statement:

"the intrinsic property (not the external coordinate)" of a system is independent of the coordinate. So if one moves the origin, one should get the same result.

moving the origin

$H(x) \to H(x+a) = T_a^{\dagger} H T_a$

$\psi(x) \to \psi(x+a) = e^{\frac{i}{\hbar}a\hat{p}}\psi(x) = T_a^{\dagger}\psi(x)$

$T_a = e^{-\frac{i}{\hbar}a\hat{p}}$

* Note that if one takes the point of view of fixing O, then both the external & system have moved! (e.g. $V(x) \to V(x+a)$ too!)

$\langle H \rangle$ invariant

In discussing the invariance for certain symmetry, we were referring the invariance of moving (transforming) the state only! And we know, technically, this is correct, only if $H' = H$ i.e. $AHA^{-1} = H$.

For translations, we ask if $H(x+a) = H(x)$.

Not all transformations are unitary so that ① is obeyed!

· <u>Gauge invariance</u> is the trivial invariance we mentioned in ①.

All it says is that gauge transformation is a similar transformation:

$$H(A+\nabla\Lambda, \overbrace{\phi - \tfrac{1}{c}\tfrac{\partial\Lambda}{\partial t}}) - i\hbar\frac{\partial}{\partial t} = \frac{1}{2m}\left(\vec{p} - \frac{q\vec{A}}{c} - \frac{q}{c}\nabla\Lambda\right)^2 + q\left(\phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}\right) - i\hbar\frac{\partial}{\partial t}$$

$$= e^{\frac{iq}{c\hbar}\Lambda}\left(H(A,\phi) - i\hbar\frac{\partial}{\partial t}\right)e^{-\frac{iq}{\hbar c}\Lambda}$$

So $\psi \to \psi\, e^{\frac{iq}{\hbar c}\Lambda}$

∴ accompanying with the gauge transformation, in order that the physical quantities $\hat{O}$ have to be invariant, they must also transform similarly:

$$\hat{O}\left(A+\nabla\Lambda, \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}\right) = e^{\frac{iq}{\hbar c}\Lambda}\,\hat{O}(A,\phi)\,e^{-\frac{iq}{\hbar c}\Lambda}$$

This in particular implies that all derivatives have to be covariant derivatives: $\frac{\hbar}{i}\nabla - \frac{q}{c}A$ & $i\hbar\frac{\partial}{\partial t} - q\phi$