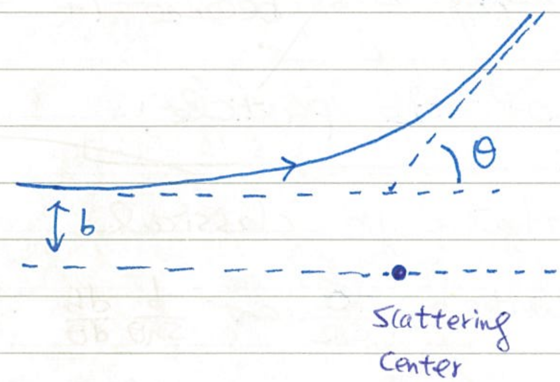
Motivation: One of the best ways to understand the structure of particles and interactions between them is to scatter them off each other: X-ray, photo-emission, . .
In fact, the images that we observe everyday are results of scattering.
Here, let us start from classical scattering theory first.

Classical scattering theory (with a fixed scattering center)

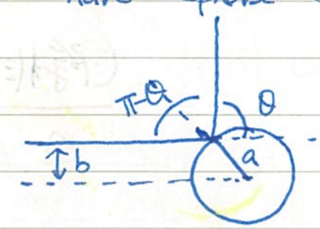


assume: recoil is negligible
azimuthally symmetrical.

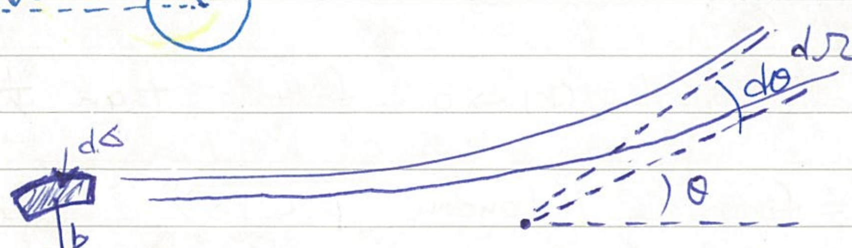
b: describes "accurary of aiming"
⇒ impact parameter

θ: scattering angle.

example: hard-sphere scattering



$b = a\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = a\cos\frac{\theta}{2}$



dN(θ) /per sec
⇑
flux I ↑
dN ↑ of course

This figure shows # of particles with impact parameter ∈ (b, b+db) must correspond to # of particles scattered into dθ

$\sigma(\theta) = \frac{d\sigma}{d\Omega}$ = differential cross-section

$d\sigma = \frac{dN(\theta)}{I}$ = function of θ $\equiv \sigma(\theta)\, d\Omega$

I ← particle/sec area

$(dN(\theta) = 2\pi b\, db \cdot I \neq I d\sigma$

From $I \to \sigma(\theta)$ just changes of variables
from $b\,db\,d\phi$ to $\sin\theta\, d\theta\, d\phi$

$\because$ $d\sigma = b(db)\,d\phi$ $\quad$ ($\therefore$ $I\, 2\pi b\, db = -I\, d\sigma$

$d\Omega = \sin\theta\, d\theta\, d\phi$ $\quad$ $\equiv -I\sigma(\theta)d\Omega$)

using # of particles not changed! $\quad$ $2\pi \sin\theta\, d\theta$

$\therefore \dfrac{d\sigma}{d\Omega} = \dfrac{b}{\sin\theta}\dfrac{db}{d\theta}$ $\quad$ $I \cdot da = I\sigma(\theta)\,d\Omega$

Typically, θ is a decreasing function of b

$\therefore \dfrac{db}{d\theta} < 0$. $\therefore$ One chooses positive value by

setting $\dfrac{d\sigma}{d\Omega} = \dfrac{b}{\sin\theta}\left|\dfrac{db}{d\theta}\right|$

Total cross-section $\quad$ $\sigma = \int \sigma(\theta)\, d\Omega$

$$\sigma \equiv \int \left(\frac{d\sigma}{d\Omega}\right) \cdot d\Omega$$ $\quad$ = trace all area

unit = area, meaning: track down ~~effective~~ areas into which when incident particles move will be scattered!

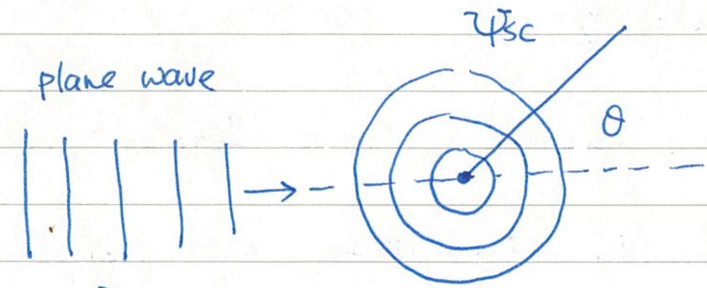
↳ see over

example: hard-sphere scattering : $\dfrac{d\sigma}{d\Omega} = \dfrac{a^2}{4}$ $\quad$ $b = a\cos\theta/2$

$\underline{\sigma = \pi a^2}$

Quantum Scattering

general $\boxed{\text{elastic}}$ theory :

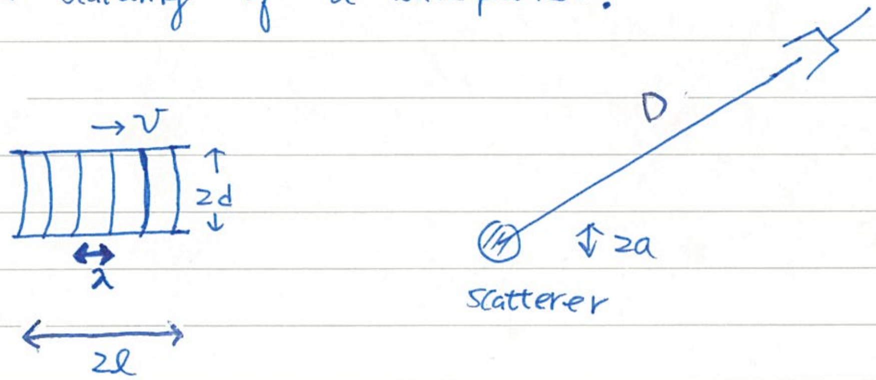


most general : wave packet

$$\psi \to \left(\frac{1}{\sqrt{2\pi}}\right)^3 e^{ik\cdot x} + \psi_{s.c.}$$

↑ original plane wave

Rigorously speaking, in reality, one must deal with the scattering of a wavepacket:



However, if $a \ll d, l$, the collision phenomenon does not depend critically on the particular form of the wave packet.

Furthermore, if $\lambda \ll d$ & $\lambda \ll l$, the incident wave are well approximated by a plane wave ($\because$

$\Delta x \sim l$, $\Delta y \sim d$, $\Delta p_x \sim \frac{\hbar}{l}$, $\Delta p_y \sim \frac{\hbar}{d}$)

$\uparrow$

$\propto \hbar \Delta k \propto \frac{\Delta\lambda}{\lambda}\cdot\frac{1}{\lambda}$

$\frac{\Delta\lambda}{\lambda} \sim \frac{\lambda}{l} \ll 1$

$\Delta q_0 = d$ or l

$\Delta q = \left[(\Delta q_0)^2 + \left(\frac{\Delta p_0 t}{m}\right)^2\right]^{\frac{1}{2}}$

Further consideration of spreading leads to $\sqrt{\bar\lambda D} \ll d, l$.
(to avoid effects due to spreading)

see Messiah's Quantum Mechanics Chapter X for more details.

$\frac{\Delta p_0 t}{m} \simeq \frac{\hbar t}{2m\Delta q_0} = \frac{1}{2\Delta q_0}\frac{\overset{D}{(vt)}\hbar}{mv}$

$\uparrow$

$\Delta p_0 \Delta q_0 = \hbar/2$

$= \frac{D\bar\lambda}{2\Delta q_0}$

$\therefore$ We can simply solve the problem for a single wave vector k: $\psi_k(\vec r)\, e^{-\frac{i}{\hbar}E_k t}$, elastic: $E_k = \frac{\hbar^2 k^2}{2m}$

$\Delta q = \Delta q_0 \left(1 + \frac{1}{4}\left(\frac{D\bar\lambda}{\Delta q_0^2}\right)^2\right)^{\frac{1}{2}}$

$$\frac{-\hbar^2}{2m}\nabla^2\psi_k + V\psi_k = E_k\psi_k$$

The problen then becomes time-independent.

(Note that if one really wants to treat wavepackets, one can form packets from the single k solutions)

Suppose $rV(r) \to 0$ as $r\to\infty$.

$\psi_k(r) \to$ free particles.

Recall that radial part $= \frac{u_{E\ell}(r)}{r}$ which

satisfies $\frac{d^2u_{E\ell}}{dr^2} + \underbrace{\frac{2m}{\hbar^2}\left[E - \left(V(r) + \frac{\ell(\ell+1)\hbar^2}{2mr^2}\right)\right]}_{\substack{\parallel \\ k^2 + U(r)}} u_{E\ell} = 0$

$u_{E\ell}(0) = 0$

Where $U(r) = \frac{-2m}{\hbar^2}V(r) - \frac{\ell(\ell+1)}{r^2}$. (i) Suppose $V(r) \to 0$ slows than $\frac{1}{r^2}$

We can drop ℓ. Try $u_E = f e^{\pm ikr}$ ($R_E = \frac{u_E}{r}$)

$\Rightarrow$ $f'' \pm 2ik f' + U(r) f = 0$ since we expect $|f''(r)| \ll |2ikf'(r)|$

if $f' \sim Uf$, $\frac{1}{r^{\alpha+1}}$, $\frac{1}{r^\alpha}$

$f'' = U'f + Uf'$

$|f''(r)| \ll |2ikf'(r)|$

$\frac{1}{r^{\alpha+1}}$ or $\frac{1}{r^{2\alpha}}$ $\quad U(r)f(r)$

U' $(U)^2$ $\quad \frac{1}{r^\alpha}$

self-const.

$\therefore$ $f(r) = f(r_0)\exp\left[\frac{\pm}{2ik}\int_{r_0}^{r} U(r')dr'\right]$

$f(r) \to$ const only when $V(r)\cdot r \to 0$

$\hookrightarrow$ jump to Schwinger-Lipp. eq.

The above derivation counts on the spherical symmetry. We will have more justification later.

We have seen, for free particles (ii) $V(r) \to 0$ faster than $\frac{1}{r^2}$

$$\psi(r\to\infty) = \sum_{\ell m}\left[A_\ell j_\ell(kr) + B_\ell n_\ell(kr)\right]Y_\ell^m(\Omega)$$

$$\because \quad j_\ell(kr) \to \frac{\sin(kr - \ell\pi/2)}{kr} \qquad \left(e^{-ikr}(-i)^{\ell} = 0\right)$$

$$n_\ell(kr) \to \frac{-\cos(kr-\ell\pi/2)}{kr} \qquad \frac{A_\ell}{B_\ell} = -i \Rightarrow \text{get } e^{ikr}. \text{ outgoing}$$

$$\therefore \quad \psi_k(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{kr}\sum_{\ell m}(-i)^{\ell}(-B_\ell)\,Y_\ell^m(\Omega)$$

$$\equiv e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} f(\theta,\phi) \quad \cdots \quad (1)$$

$\uparrow$

Scattered wave : spherical.

The fact that the scattered wave takes the form

$\frac{e^{ikr}}{r} f(\theta,\phi)$ is **not** surprising and is expected.

since the probability is conserved : what is flowing out radius r should also flow out $r' > r$, otherwise, something will be created between r and r'!

f is known as <u>scattering amplitude</u>.

We shall show that

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2. \quad \cdots \quad (2)$$

so $|f|^2$ has the meaning of differential crosssection.

To show eq. (2), we have to calculate the # of particles that flow into $d\Omega$, i.e., the probability current.

$$\left|\vec{J}_{inc}\right| = \left|\frac{\hbar}{2mi}\left(e^{-i\vec{k}\cdot\vec{r}}\nabla e^{i\vec{k}\cdot\vec{r}} - e^{i\vec{k}\cdot\vec{r}}\nabla e^{-i\vec{k}\cdot\vec{r}}\right)\right| \cdot \frac{1}{V}$$

normalization constant of $e^{i\vec{k}\cdot\vec{r}}$

$$= \underbrace{\frac{\hbar k}{m}}_{\vec{v}} \cdot \underbrace{\frac{1}{V}}_{\rho} \qquad \left(\frac{1}{\sqrt{V}} e^{ik\cdot r}\right) \qquad \text{(particles / sec area)}$$

same normalization as in eq. ③

$$\left|\vec{j}_{sc}\right| = \left|\frac{\hbar}{2mi}\left(\frac{e^{-ikr}}{r}f^*\nabla\frac{e^{ikr}}{r}f - \frac{e^{ikr}}{r}f\nabla\frac{e^{-ikr}}{r}f^*\right)\right| \cdot \frac{1}{V}$$

$\frac{1}{V} \Downarrow \frac{1}{(2\pi)^3}$

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \underbrace{\hat{\theta}\frac{\partial}{r\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}}$$

$O(\frac{1}{r^2})$ when acting on $\left(\frac{e^{ikr}}{r}f\right)$

f is defined under the normalization form in eq. ③

$$\frac{\partial}{\partial r}\frac{e^{ikr}}{r}f = ik\frac{e^{ikr}}{r}f + O\left(\frac{1}{r^2}\right)$$

$$\therefore \quad \vec{j}_{sc} = \frac{\hat{r}}{r^2}\left|f(\theta,\phi)\right|^2\frac{\hbar k}{m}\left(\frac{1}{V}\right) + O\left(\frac{1}{r^3}\right)$$

$\frac{1}{V} \to \frac{1}{(2\pi)^3}$

$$\text{particles / per sec through } d\Omega = \vec{j}_{sc}\cdot\hat{r}\, r^2 d\Omega$$

$$= |f|^2\frac{\hbar k}{m}d\Omega \cdot \frac{1}{V}$$

$$\therefore \quad \frac{|f|^2\frac{\hbar k}{m}d\Omega}{\frac{\hbar k}{m}} = \frac{d\sigma}{d\Omega}\cdot d\Omega.$$

$$\frac{d\sigma}{d\Omega} = \left|f(\theta,\phi)\right|^2$$

Note that rigorously speaking, there are interference terms between $e^{i\vec{k}\cdot\vec{r}}$ & $\frac{e^{ikr}}{r}f(\theta,\phi)$ when calculating the current. However, since in real experiments. it

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Many scattering centers

(often encounted in Condensed Matter physics)

$$\psi_s(r) = \sum_i \psi_s(r-r_i) \quad \text{or } N_s(r-r_i) \text{ (classically)} \quad (\text{assuming no spin})$$

$$I \propto \langle |\psi_s(r)|^2 \rangle$$

↑ average over positions of r_i

$$= \sum_{ij} \langle \psi_s(r-r_i)\, \psi_s(r-r_j) \rangle$$

$$= \int dr_1 \int dr_2 \, \langle \sum_i \delta(r_1-r_i)\, \psi_s(r-r_1) \sum_j \delta(r_2-r_j)\, \psi_s(r-r_2) \rangle$$

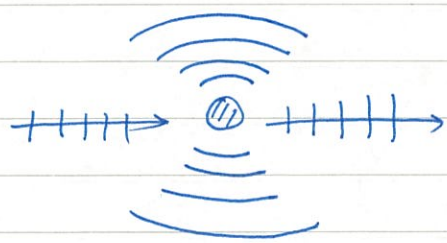
$$= \int dr_1 \int dr_2 \, \underbrace{\langle n(r_1)\, n(r_2) \rangle}_{\text{correlation function}} \, \psi_s(r-r_1)\, \psi_s(r-r_2)$$

If spins are involved

$$\Rightarrow \langle \vec{S}(r_1) \cdot \vec{S}(r_2) \rangle$$

∴ Scattering type expts. measure correlations.

occurs in the following way



∴ We may drop the interference terms.

The Lippmann-Schwinger equation

Let us write $H = H_0 + V$, $H_0 = \frac{p^2}{2m}$

$$V = V(\vec{r})$$

We would like to develope the perturbation theory around

$$H_0 |\phi\rangle = E|\phi\rangle$$

For elastic scatterings, we need to solve

$$(H_0 + V)|\psi\rangle = E|\psi\rangle \quad \text{for the same } E.$$

Since $(E - H_0)|\psi\rangle = V|\psi\rangle$ & $(E-H_0)|\phi\rangle = 0$,

We may try to write $|\psi\rangle = |\phi\rangle + \frac{V}{E-H_0}|\psi\rangle$.

This is not quite right as we have shown in 5/18/96, 2: the correct way to invert $E - H_0$ is

$$|\psi^{\pm}\rangle = |\phi\rangle + \frac{1}{E - H_0 \pm i\varepsilon} V|\psi^{\pm}\rangle$$

3/4/97. 1.

As we have also shown, in real space, this equation leads to

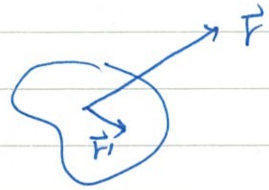
integral converges only when $|r'V| \to 0$ as $r' \to \infty$

$$\psi^{\pm}(\vec r) = \phi_{\vec p}(\vec r) - \frac{2m}{\hbar^2} \int d^3\vec r' \, \frac{e^{\pm ik|\vec r - \vec r'|}}{4\pi|\vec r - \vec r'|} V(\vec r') \psi^{\pm}(\vec r') \qquad k = |\vec p|$$

$\because |r \psi^{\pm}(\vec r')| \to 1$ as $r' \to \infty$

For a finite range potential, the region that $V(\vec r')$ contributes is limited in space.

$\therefore$ We can approximate $r >> r'$ in the integral:



$$|\vec r - \vec r'| = r\sqrt{1 - \frac{2r'}{r}\hat r \cdot \hat r' + \left(\frac{r'}{r}\right)^2}$$

$$\cong r\left(1 - \frac{2r'}{2r}\hat r \cdot \hat r' + o\left(\frac{r'}{r}\right)^2\right)$$

$$= r - \hat r \cdot \vec r'$$

If we define $\vec k' \equiv k\hat r$, which is the wave vector for the outgoing particle, we have

$$e^{\pm ik|\vec r - \vec r'|} = e^{\pm ikr} \, e^{\mp i\vec k' \cdot \vec r'}$$

$\phi_p + \int (\,)\phi_p + \int\int (\,)(\,)\phi_p + \cdots$

$$\therefore \psi^{+}(\vec r) \to \phi_{\vec p}(\vec r) - \frac{1}{4\pi}\frac{2m}{\hbar^2}\frac{e^{ikr}}{r} \int d^3\vec r' \, e^{-i\vec k' \cdot \vec r'} V(\vec r') \psi^{+}(\vec r')$$

$$= \frac{1}{(2\pi)^{3/2}}\left[e^{i\vec k \cdot \vec r} + \frac{e^{ikr}}{r} f(\vec k', \vec k)\right] \quad ---③$$

(非 perturbative view)

if box normalization is used. $\frac{1}{(2\pi)^{3/2}} \to \frac{1}{\sqrt{\Omega}}$

$$f(\vec k', \vec k) = \frac{-1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3 \int d^3\vec r' \, \frac{e^{-i\vec k' \cdot \vec r'}}{(2\pi)^{3/2}} V(\vec r') \langle \vec r' | \psi^{+} \rangle$$

$$= \frac{-1}{4\pi}(2\pi)^3 \frac{2m}{\hbar^2} \langle \vec k' | V | \psi^{+} \rangle = f(\theta, \phi)$$

$\Rightarrow$ back to interprete f

Now, ③ is correct even $V(\vec r) \neq V(r)$!

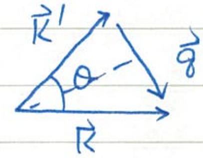
3/4/97 * Born Approximation

One can iterate eq. ③ and get the 1st order Born Approximation

$$f^{(1)}(\vec k', \vec k) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3r'\, e^{i(\vec k-\vec k')\cdot\vec r'}\, V(\vec r')$$

$$\therefore \quad \frac{d\sigma^{(1)}}{d\Omega} = |f^{(1)}(\vec k',\vec k)|^2 = \left|\frac{m}{2\pi\hbar^2}\int d^3r'\, e^{+i\vec q\cdot\vec r'}\, V(\vec r')\right|^2$$

$$\hbar\vec q = \vec p_i - \vec p_f, \quad \vec q = \vec k - \vec k'$$



For central potentials $V(\vec r) = V(r)$,

$|\vec k - \vec k'| = 2k\sin\frac{\theta}{2} = q$

$f^{(1)}(\theta,\phi)$

$$= -\frac{1}{2}\frac{2m}{\hbar^2}\frac{1}{iq}\int_0^\infty \frac{r^2}{r}V(r)\left(e^{iqr}-e^{-iqr}\right)dr$$

↑

$$\int e^{iqr\cos\theta_r}\sin\theta_r\, d\theta_r = \frac{e^{iqr}-e^{-iqr}}{iqr}$$

$$= -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty rV(r)\sin qr\, dr \underset{\uparrow\, q=2k\sin\frac{\theta}{2}}{=} f^{(1)}(\theta)$$

example: $V(r) = \dfrac{V_0 e^{-\alpha r}}{\alpha r}$

$$f^{(1)}(\theta) = -\frac{2mV_0}{\alpha\hbar^2}\frac{1}{q^2+\alpha^2} \quad \left(\because \int_0^\infty rV(r)\sin qr\, dr = \mathrm{Im}\int_0^\infty rV(r)e^{iqr}\, dr\right)$$

$$\therefore \quad \frac{d\sigma}{d\Omega} \simeq \left(\frac{2mV_0}{\hbar^2\alpha}\right)^2 \frac{1}{[2k^2(1-\cos\theta)+\alpha^2]^2} \qquad q^2 = 2k^2(1-\cos\theta)$$

take $\alpha \to 0$, while keeping $\frac{V_0}{\alpha}$ fixed $= ZZ'e^2$

We get : for Coulomb interaction

$$\frac{d\sigma}{d\Omega} \simeq \frac{(2m)^2 (ZZ'e^2)^2}{\hbar^4} \frac{1}{16 k^4 \sin^4\frac{\theta}{2}}$$

(Rutherford scattering cross section that was obtained in classical electrodynamics !)

Remarks :

(i) Validity of $\underbrace{}_{\text{Born Approx.}}$ requires $|r V(r)| \to 0$ as $r \to \infty$

$$\frac{|\psi_{sc}(0)|}{|e^{i\vec{k}\cdot\vec{r}}|} = \left| \frac{2m}{4\pi\hbar^2} \int \underset{\uparrow \atop \left.\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\right|_{r=0}}{\frac{e^{ikr'}}{r'}} V(\vec{r}') e^{-i\vec{k}\cdot\vec{r}'} d\vec{r}' \right| << 1$$

$\therefore$ $\underline{|r' V(\vec{r}')| \to 0 \text{ as } r' \to \infty}$! (same as what we said in the central potential case.)

$\therefore$ Rigorously speaking, it can not be applied to Coulomb interaction. But it turns out that rigorous treatment also leads to the same $\frac{d\sigma}{d\Omega}$

see Sakurai's book § 7.13 for more details.

(ii) T-matrix (transition operator)

It is often convenient to define

$$V|\psi^+\rangle = T|\phi\rangle$$

So, the Lippmann – Schwinger equation becomes

$$V|\psi^{\pm}\rangle = V|\phi\rangle + V\frac{1}{E-H_0+i\varepsilon}T|\phi\rangle$$

$\uparrow$

$T|\phi\rangle$

$$\therefore \quad T = V + V\frac{1}{E-H_0+i\varepsilon}T \quad = V + V\frac{1}{E-H_0+i\varepsilon}V + \cdots$$

$$\therefore \quad f(\vec{k}',\vec{k}) = \frac{-1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\int d^3\vec{r}'\,\frac{e^{-i\vec{k}'\cdot\vec{r}'}}{(2\pi)^{3/2}}V(\vec{r}')\langle\vec{r}'|\psi^{\pm}\rangle$$

$$= -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\int d^3\vec{r}'\,\langle\vec{k}'|\vec{r}'\rangle\langle\vec{r}'|V|\psi^{\pm}\rangle$$

$$= \frac{-1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\int d^3\vec{r}'\,\langle\vec{k}'|\vec{r}'\rangle\underbrace{\langle\vec{r}'|T|\phi\rangle}_{\langle\vec{r}'|T|\vec{k}\rangle}$$

$$= -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\langle\vec{k}'|T|\vec{k}\rangle$$

i.e. $f(\vec{k}',\vec{k})$ is the matrix element of $\frac{-1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3 T$

Higher order expansion was discussed before so we do not repeat it here.

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Relation of T matrix to S matrix

S-matrix is defined as

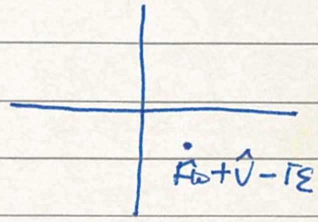
$$\lim_{\substack{t'\to\infty \\ t\to-\infty}} \langle f|\,\hat{U}(t',t)\,|i\rangle \equiv \langle\phi_f|S|\phi_i\rangle \qquad |i\rangle = e^{-iE_i t}|\phi_i\rangle, \quad |f\rangle = e^{-iE_f t}|\phi_f\rangle$$

Let us assume $H = H_0 + V$ $\quad V$ = time-independent.

This is the situation for our scattering problem in which we may set when $t<0$, $|i\rangle = |\psi_{in}\rangle$ and for $t>0$, the particle comes in contact with the scattering center and get scattered!

$$\therefore\ U(t,0) \Rightarrow U(t,0)\,\theta(t) = e^{-i(\hat{H}_0+\hat{V})t}\underbrace{\theta(t)}_{=0 \text{ for } t<0}$$

$$= \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \underbrace{\frac{i}{\omega-\hat{H}_0-\hat{V}+i\varepsilon}}_{\hat{U}(\omega)} \qquad \text{with } \varepsilon\to 0^+$$



$$-i\,\hat{U}(\omega) = \frac{1}{\omega-\hat{H}_0-\hat{V}+i\varepsilon}$$

$$= \frac{1}{\omega-\hat{H}_0+i\varepsilon} + \frac{1}{\omega-\hat{H}_0+i\varepsilon}\,\hat{V}\,\frac{1}{\omega-\hat{H}_0+i\varepsilon}$$

$$+ \frac{1}{\omega-\hat{H}_0+i\varepsilon}\,\hat{V}\,\frac{1}{\omega-\hat{H}_0+i\varepsilon}\,\hat{V}\,\frac{1}{\omega-\hat{H}_0+i\varepsilon} + \cdots \qquad (1)$$

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$$= \frac{1}{\omega - \hat{H}_0 + i\varepsilon} + \frac{1}{\omega - \hat{H}_0 + i\varepsilon} \hat{T}(\omega) \frac{1}{\omega - \hat{H}_0 + i\varepsilon} \quad \text{---} (2)$$

where we have made use of the identity:

$$(\hat{A} - \hat{B}) \left[\frac{1}{A} + \frac{1}{A} B \frac{1}{A} + \frac{1}{A} B \frac{1}{A} B \frac{1}{A} + \cdots \right]$$

$$= \left(1 - B\frac{1}{A} + B\frac{1}{A} B\frac{1}{A} + \cdots \right)$$

$$- B\frac{1}{A} - B\frac{1}{A} B\frac{1}{A} \cdots = \mathbb{I}$$

$$\therefore \frac{1}{A-B} = \frac{1}{A} + \frac{1}{A} B \frac{1}{A} + \frac{1}{A} B \frac{1}{A} B \frac{1}{A} + \cdots$$

Note that $\hat{T} = \hat{V} + \hat{V} \frac{1}{E - \hat{H}_0 + i\varepsilon} \hat{T}$

$$= \hat{V} + \hat{V} \frac{1}{E - \hat{H}_0 + i\varepsilon} \hat{V} + \cdots$$

Now, we can take out e^{-iEt} factor from $|i\rangle$ & $|f\rangle$ by setting $|i\rangle = e^{-iE_i t} |\phi_i\rangle$, $|f\rangle = e^{-iE_f t} |\phi_f\rangle$

$$\therefore \hat{U}(t', t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{U}(\omega) e^{-i\omega(t'-t)}$$

$$\langle f | S | i \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i(E_f t' - E_i t)} e^{-i\omega(t'-t)} \langle \phi_f | \hat{U}(\omega) | \phi_i \rangle$$

$$\overset{(2)}{=} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i(E_f - \omega)t'} e^{-i(E_i - \omega)t} \frac{i}{\omega - E_i + i\varepsilon} \langle \phi_f | \phi_i \rangle$$

$$+ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, i \, \frac{e^{i(E_f - \omega)t'}}{\omega - E_f + i\varepsilon} \langle \phi_f | \hat{T}(\omega) | \phi_i \rangle \frac{e^{-i(E_i - \omega)t}}{\omega - E_i + i\varepsilon}$$

$$\text{---} (3)$$

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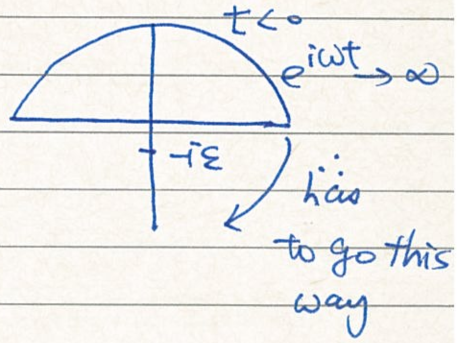
To proceed, the following identity is useful :



$$\lim_{t\to-\infty} \int_{-\infty}^{\infty} d\omega \, f(\omega) \frac{e^{i\omega t}}{\omega + i\varepsilon}$$

$$= \lim_{t\to-\infty} \lim_{\varepsilon\to0^+} -2\pi i \, f(-i\varepsilon) \, e^{\varepsilon t}$$

$$= -2\pi i f(0) \qquad \therefore \quad \lim_{t\to\infty} \frac{e^{i\omega t}}{\omega+i\varepsilon} = -2\pi i \, \delta(\omega)$$

$$\therefore \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i(E_f-\omega)t'} \, i \, \boxed{\frac{e^{-i(E_i-\omega)t}}{\omega - E_i + i\varepsilon}} \langle \phi_f | \phi_i \rangle$$

$$\xrightarrow[t\to-\infty]{} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i(E_f-\omega)t'} \, i \, \left(-2\pi i \, \delta(\omega - E_i)\right) \langle \phi_f | \phi_i \rangle$$

$$= \frac{2\pi}{2\pi} e^{i(E_f-E_i)t'} \langle \phi_f | \phi_i \rangle = \underline{\delta_{fi}}$$

The second term in ③

$$\to \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, i \, \frac{e^{i(E_f-\omega)t'}}{\omega - E_f + i\varepsilon} \left[-2\pi i \, \delta(\omega - E_i)\right] \langle \phi_f | \hat{T}(\omega) | \phi_i \rangle$$

$$= \frac{e^{i(E_f-E_i)t'}}{E_i - E_f + i\varepsilon} \langle \phi_f | \hat{T}(E_i) | \phi_i \rangle$$

$$\xrightarrow[(-t')\to-\infty]{=} \frac{e^{+i(E_i-E_f)(-t')}}{E_i - E_f + i\varepsilon} \langle \phi_f | \hat{T}(E_i) | \phi_i \rangle$$

$$\Rightarrow \quad -2\pi i \, \delta(E_i - E_f) \langle \phi_f | \hat{T}(E_i) | \phi_i \rangle$$

$$\therefore \quad \hat{S} = \hat{1} - 2\pi i \, \delta(E_i - E_f) \hat{T}$$

$$\Leftrightarrow \quad S_\ell = 1 + 2ki \, \underbrace{a_\ell}_{-\frac{\pi}{k} T_\ell(E)}$$ (page 6)

3/31/97

Revisit partial wave analysis

* plane waves vs spherical waves for free particles

$(H, \vec{p})$ $\qquad\qquad$ (H, L^2, L_z) $\qquad\qquad$ ⇒ see over

$|\vec{k}\rangle$ $\qquad\qquad$ $|E, l, m\rangle$ $\quad , \quad E = \frac{\hbar^2 k^2}{2m}$

Recall that

$$\hat{k} \parallel \hat{z} , \quad e^{i\vec{k}\cdot\vec{r}} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l (2l+1)\, j_l(kr)\, P_l(\cos\theta) \quad \cdots (3)$$

This relation is actually the transformation relation between $|k\rangle$ & $|E, l, m\rangle$:

$$|\vec{k}\rangle = \sum_{l,m} \int dE \; |Elm\rangle \langle Elm|\vec{k}\rangle$$

The transfomation is specified by the function

$$\langle \vec{k}|Elm\rangle ,$$

where $|Elm\rangle$ are normalized : $\langle E'l'm'|Elm\rangle = \delta(E-E')\delta_{ll'}\delta_{mm'}$

Obviously, $\langle \vec{k}|Elm\rangle$ is the solution to $H = \frac{\hat{p}^2}{2m}\, |\psi\rangle = E|\psi\rangle$ in the Fourier space. So it must satisfy

$$\frac{\hbar^2 k^2}{2m} \langle \vec{k}|Elm\rangle = E \langle \vec{k}|Elm\rangle \quad \cdots (4)$$

$$\langle \vec{k}|\hat{L}^2|Elm\rangle = \hbar^2 l(l+1) \langle \vec{k}|Elm\rangle \quad \cdots (5)$$

$$\langle \vec{k}|L_z|Elm\rangle = \hbar m \langle \vec{k}|Elm\rangle \quad \cdots (6)$$

Since the angular momentum has been exactly the same form both in the real & Fourier spaces, so $\langle \vec{k}|Elm\rangle \propto Y_l^m(\hat{k})$

e.g. $L_z = xp_y - yp_x = x\frac{\hbar}{i}\frac{d}{dy} - y\frac{\hbar}{i}\frac{d}{dx} = i\hbar\left[y\frac{d}{dx} - x\frac{d}{dy}\right]$

$$= i\hbar\left[p_y\frac{d}{dp_x} - p_x\frac{d}{dp_y}\right]$$

$p_x \Leftrightarrow x$, $p_y \Leftrightarrow y$.

Eq. ④ implies that $\langle\vec{k}|E\ell m\rangle \propto \delta(E - \frac{\hbar^2k^2}{2m})$

∴ We conclude

$$\langle\vec{k}|E\ell m\rangle = C\,\delta(E - \frac{\hbar^2k^2}{2m})\,Y_\ell^m(\hat{k})$$

C is fixed by the normalization $\langle E'\ell'm'|E\ell m\rangle$
$= \delta(E-E')\,\delta_{\ell\ell'}\,\delta_{mm'}$:

$$\langle E'\ell'm'|E\ell m\rangle = \int d^3\vec{k}\,\langle E'\ell'm'|\vec{k}\rangle\langle\vec{k}|E\ell m\rangle$$

$$= |C|^2\int k^2dk\int d\Omega_k\,Y_\ell^m(\Omega_k)\,Y_{\ell'}^{m'*}(\Omega_k)$$

$$\times\,\delta(E' - \frac{\hbar^2k^2}{2m})\,\delta(E - \frac{\hbar^2k^2}{2m})$$

Let $E'' = \frac{\hbar^2k^2}{2m}$ $\quad\int k^2dk = \frac{1}{2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\sqrt{E''}\,dE''$

$$\therefore\ \langle E'\ell'm'|E\ell m\rangle = |C|^2\,\delta_{\ell\ell'}\,\delta_{mm'}\,\frac{1}{2}\left(\frac{2m}{\hbar^2}\right)^{3/2}$$

$$\times\int dE''\,\sqrt{E''}\,\delta(E-E'')\,\delta(E'-E'')$$

$$= |C|^2\,\frac{mk}{\hbar^2}\,\delta(E-E')\,\delta_{\ell\ell'}\,\delta_{mm'}.$$

∴ We can choose $C = \frac{\hbar}{\sqrt{mk}}$, thus $\underline{\langle\vec{k}|E\ell m\rangle = \frac{\hbar}{\sqrt{mk}}\,\delta(E - \frac{\hbar^2k^2}{2m})\,Y_\ell^m(\hat{k})}$

└-- ⑦

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Combinations of eqs. (3), (7) & addition theorem

$$P_l(\cos\theta) = \frac{4\pi}{2l+1}\sum_m Y_l^m(\hat{r})\, Y_l^{m*}(\hat{k})$$ give

$$\langle \vec{r}\,|\,E l m\rangle = \frac{i^l}{\hbar}\cancel{\sqrt{\frac{2mk}{\pi}}}\ \sqrt{mk}\cdot 4\pi\ \ j_l(kr)\, Y_l^m(\hat{r}).$$

* The advantage of using the partial wave representation shows up when we measure $\vec{r}$ from the scattering center :

incident particles coming out from the accelerator

a uniform beam

For the particles in the cylinder of radius ρ:

$$L \simeq \hbar l \simeq \rho(\hbar k)$$

$\therefore$ If the range of the potential (r_0) is finite, only $l \lesssim kr_0$ will contribute ! In other words, one needs only to consider a few partial waves. For example, for a 100 MeV neutron incident upon a fixed neucleus (size $\sim 10^{-15}$m), $l \lesssim 2$.

Only about 2 partial waves need to be considered, this property greatly simplies the whole analysis on the scattering problem! (The above arguement is classical, for Q.M. see page 8.)

* <u>Including interaction</u>

The usage of spherical waves also simplifies the analysis when the potential is central.

do not need this assumption

In this case, $\because T = V + V \frac{1}{E - H_0 + i\varepsilon} V + \cdots$ (Scalar)

$\therefore [T, L^2] = 0, \quad [T, L_z] = 0.$

As a result, <u>l & m are conserved during scattering</u>!

Technically, it means that <u>T matrix is diagonal in $\{|Elm\rangle\}$ basis</u>. also ~~valid in general~~

(from Wigner - Eckart theorem)

$$\langle E l' m' | T | E l m \rangle = T_l(E)\, \delta_{ll'} \delta_{mm'}$$

T is a scalar

depends only on l. (central potential)

For elastic scatterings, $\frac{\hbar^2 k'^2}{2m} = \frac{\hbar^2 k^2}{2m}$, therefore

$$f(\vec{k}', \vec{k}) = \frac{-1}{4\pi} \frac{2m}{\hbar^2} (2\pi)^3 \langle \vec{k}' | T | \vec{k} \rangle$$

$$= \frac{-1}{4\pi} \frac{2m}{\hbar^2} (2\pi)^3 \sum_{lm} \sum_{l'm'} \int dE \int dE' \langle \vec{k}' | E' l' m' \rangle$$

$$\langle E' l' m' | T | E l m \rangle \langle E l m | \vec{k} \rangle$$

$\sum_{lm} Y_l^m(\Omega) Y_l^{m*}(\Omega') = \sum_l \frac{2l+1}{4\pi} P_l(\cos\theta)$

$$= \frac{-1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3 \frac{\hbar^2}{mk} \sum_{lm} T_l(E)\Big|_{E=\frac{\hbar^2k^2}{2m}} Y_l^m(\hat{k}') Y_l^{m*}(\hat{k})$$

$$= -\frac{4\pi^2}{k}\sum_{lm} T_l(E)\Big|_{E=\frac{\hbar^2k^2}{2m}} Y_l^m(\hat{k}') Y_l^{m*}(\hat{k})$$

$$= \sum_{l=0} \underbrace{\left(-\frac{\pi}{k}T_l(E)\right)}_{\parallel\; a_l(E)} (2l+1) P_l(\cos\theta) \qquad \cos\theta = \hat{k}'\cdot\hat{k}$$

$\cdots$ (8)

Now, $\because \psi(\vec{r}) \to \frac{1}{(2\pi)^{3/2}}\left[e^{i\vec{k}\cdot\vec{r}} + f(\vec{k}',\vec{k})\frac{e^{ikr}}{r}\right]$ ... (9)

Combining eqs. (3) & (8), eq. (9) becomes

$$\psi(\vec{r}) \longrightarrow \frac{1}{(2\pi)^{3/2}}\left[\sum_l (2l+1)P_l(\cos\theta)\frac{e^{ikr} - e^{-i(kr-l\pi)}}{2ikr}\right.$$

$$\left. + \sum_l (2l+1) P_l(\cos\theta)\, a_l(E)\frac{e^{ikr}}{r}\right]$$

$$\left(j_l(kr) \to \frac{e^{i(kr-l\pi/2)} - e^{-i(kr-l\pi/2)}}{2ikr}, \quad i^l = e^{i\pi l/2}\right)$$

$$\to \frac{1}{(2\pi)^{3/2}} \sum_{l=0}^{\infty}(2l+1)\frac{P_l}{2ik}\left[(1+2ika_l)\frac{e^{ikr}}{r} - \frac{e^{-i(kr-l\pi)}}{r}\right] \cdots (10)$$

Eq. (10) makes scattering clear: scatterer changes $1 \to 1+2ika_l$

i.e., it changes the outgoing waves from $\frac{e^{ikr}}{r}$ to

$(1+2ia_l k)\frac{e^{ikr}}{r}$! In coming waves are not changed.

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<u>Unitarity</u> : First, we know that l is conserved.

$\therefore$ One can consider each l each time.

In other words, each term in ⑩ represents the situation: lth wave comes in and comes out.

The conservation of probability :

scattering matrix — elastic (inelastic, not so!)

$$\int j \cdot d\vec{s} = 0 \Rightarrow |S_l| \equiv |1 + 2k a_l i| = 1$$

(Note : $S_l \cong \dfrac{\text{outgoing wave amplitude}}{\text{incoming} \quad ''}$ $= 1 - 2\pi i T_l(E)$)

$\therefore$ $1 + 2k a_l i = e^{2i\delta_l}$ $\quad \delta_l \equiv$ phase shift.

$$a_l = \frac{e^{2i\delta_l} - 1}{2ik} = \frac{e^{i\delta_l}\sin\delta_l}{k}$$

i.e. $R_l(r) \to \frac{1}{r}\sin(kr - \frac{l\pi}{2} + \delta_l)$

$$\therefore f(\theta) = \sum_{l=0}^{\infty} \underbrace{\frac{(2l+1)}{k} e^{i\delta_l}\sin\delta_l}_{(2l+1)\,a_E(l)} P_l(\cos\theta) \quad \cdots ⑪$$

total cross section $\equiv \sigma_t = \int |f|^2 d\Omega$

$$= \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l \quad = 4\pi\sum_{l=0}^{\infty}(2l+1)|a_l|^2 \quad \cdots ⑫$$

$\uparrow$

$$\int P_l(\cos\theta) P_{l'}(\cos\theta)\, d\cos\theta = \frac{2}{2l+1}\delta_{ll'}$$

"unitary limit"

$\sigma_l^{max} = \frac{4\pi}{k^2}(2l+1)$

$\because$ unitary $\Rightarrow \delta_l =$ real

$\therefore \sin^2\delta_l \le 1$

$\therefore$ σ_t depends only on phase shifts.

From ⑪ & ⑫ $\Rightarrow$ optical theorem $\quad \sigma_t = \frac{4\pi}{k}\,\mathrm{Im}(f(0))$

<u>Determination of phase shifts</u>

In terms of δ_l, eq. ⑩ becomes

$$\psi(\vec{r}) \to \frac{1}{(2\pi)^{3/2}}\sum_{l=0}^{\infty}(2l+1)\, i^l P_l \left[\frac{e^{2i\delta_l}e^{i(kr - \frac{l\pi}{2})} - e^{-i(kr - l\pi/2)}}{2ikr}\right] \cdots ⑬$$

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$$\therefore \tan\delta_l = \frac{KR\, j_l'(KR) - \beta_l\, j_l(KR)}{KR\, n_l'(KR) - \beta_l\, n_l(KR)}$$

provide a cut off of l!

for $l \gg KR = l_{max}$

$j_l(KR) \sim \frac{1}{(2l+1)!!}(KR)^l$

$n_l(KR) \sim (2l+1)!!\,(KR)^{-l-1}$

$\delta_l \sim -\frac{l-\beta_l}{l+1+\beta_l}\frac{(KR)^{2l+1}}{(2l+1)!!(2l-1)!!} \ll 1$

$\therefore$ To determine δ_l, we need only to solve the 1D Schrödinger equation (usually numerically) up to $r=R$ to get $\beta_l(R)$ {by continuity. $\beta_l(R^-)=\beta_l(R^+)$:

$$\therefore \sigma_t \simeq \frac{4\pi}{k^2}\sum_{l=0}^{l_{max}}(2l+1)\sin^2\delta_l$$

$$\frac{d^2}{dr^2}[rA_l] + \left[K^2 - \frac{2m}{\hbar^2}V - \frac{l(l+1)}{r^2}\right](rA_l) = 0$$

$$rA_l\Big|_{r=0} = 0.$$

example: Hard sphere

$$A_l(R) = 0 \qquad e^{i\delta_l}[\cos\delta_l\, j_l(KR) - \sin\delta_l\, n_l(KR)] = 0$$

$$\therefore \tan\delta_l = \frac{j_l(KR)}{n_l(KR)}$$

$$\because \quad j_0(x) = \sin x/x, \qquad n_0(x) = -\cos x/x$$

$$\therefore \tan\delta_0 = -\tan(KR). \qquad \delta_0 = -KR$$

i.e. $$A_0(r) = e^{i\delta_0}[\cos\delta_0\, j_0(Kr) - \sin\delta_0\, n_0(Kr)]$$

$$= e^{i\delta_0}\frac{\sin(Kr+\delta_0)}{Kr} = e^{-iKR}\frac{\sin K(r-R)}{Kr}$$

$\therefore$ The hard sphere has just pushed out the wave function, forcing it to start the oscillation at $r=R$!

Note that $\because$ $j_l(x) \to x^l/(2l+1)!!$ $\quad n_l(x) \to -x^{-(l+1)}(2l-1)!!$ $\quad x\to 0$

$\therefore$ $\delta_l \sim -(KR)^{2l+1}$ at low energies. $\therefore$ Only small l get phase shifted, i.e. get scattered!

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Recall that

$$j_\ell(kr) \to \frac{\sin(kr-\frac{\ell\pi}{2})}{kr}$$
$$n_\ell(kr) \to \frac{-\cos(kr-\ell\pi/2)}{kr}$$
as $r\to\infty$

$$\therefore\ j_\ell(kr)+in_\ell(kr) \to \frac{1}{ikr}\, e^{i(kr-\ell\pi/2)}$$
$$j_\ell(kr)-in_\ell(kr) \to \frac{-1}{ikr}\, e^{-i(kr-\ell\pi/2)}$$

$$[\cdots] \text{ in } (13) = e^{2i\delta_\ell}\,\frac{j_\ell(kr)+in_\ell(kr)}{2} + \frac{j_\ell(kr)-in_\ell(kr)}{2}$$
$$= e^{i\delta_\ell}\left[\cos\delta_\ell\, j_\ell(kr) - \sin\delta_\ell\, n_\ell(kr)\right]$$

$$\therefore\ \psi(\vec r) \to \frac{1}{(2\pi)^{3/2}}\sum_{\ell=0}^{\infty} i^\ell(2\ell+1)\,A_\ell(r)\,P_\ell(\cos\theta)$$

$$\to \frac{e^{i\delta_\ell}}{kr}\sin\left(kr-\frac{\ell\pi}{2}+\delta_\ell\right)$$

$$A_\ell(r) \equiv e^{i\delta_\ell}\left[\cos\delta_\ell\, j_\ell(kr) - \sin\delta_\ell\, n_\ell(kr)\right] \cdots (14)$$

Which satisfies
$$\left[\frac{-\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2\ell(\ell+1)}{2mr^2}\right](rA_\ell) = E\,(rA_\ell). \qquad E=\frac{\hbar^2k^2}{2m}.$$

This is nothing but the statement : $r\to\infty$ $\psi\to$ solutions of free particles ! (We have actually looked at this problem via this point of view when we discussed central potentials.)

For a finite range potential $(r<R)$, eq. (14) is valid only for $r\geq R$. To extract δ_ℓ, one forms a dimensionless parameter

$$\beta_\ell \equiv \frac{r}{A_\ell}\frac{dA_\ell}{dr}\bigg|_{r=R} = kR\,\frac{j_\ell'(kR)\cos\delta_\ell - n_\ell'(kR)\sin\delta_\ell}{j_\ell(kR)\cos\delta_\ell - n_\ell(kR)\sin\delta_\ell}$$

NO.
DATE: 1.

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Zero-energy scattering & scattering length

$\sigma = \sum_\ell \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell$
$\rightarrow 4\pi \left(\frac{\delta_{\ell=0}}{k}\right)^2$
$= 4\pi a^2$
in $E \to 0$

Consider the case $V=0$ for $r \geq R$

and $E \to 0$

then $\frac{-\hbar^2}{2m}\frac{d^2}{dr^2}(\underbrace{rR_\ell}_{u_\ell}) = 0$ for $\ell = 0$, $r > R$

($E \to 0$ only $\delta_{\ell=0}$ survives see below!)

$\therefore u_0 = \text{Const.}\,(r-a)$

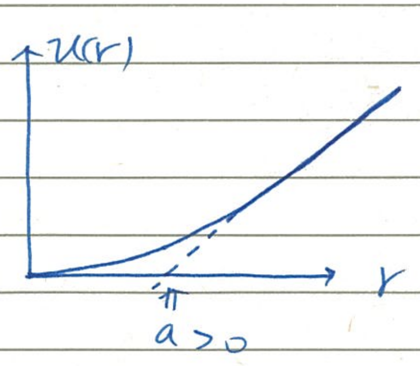
$a \equiv$ scattering length

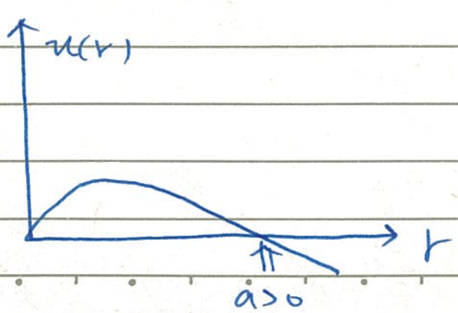
This can be also viewed as the limit of

$$\lim_{k\to 0} \underbrace{\sin(kr+\delta_0)}_{u_0} = \lim_{k\to 0} \sin k(r + \underset{\Uparrow \atop -a}{\delta_0/k}) \quad \cdots \quad (1)$$

$\delta_\ell \sim (kR)^{2\ell+1}$

a can be both positive or negative:



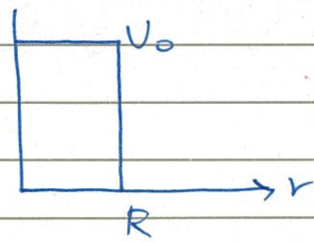


a can differ from R by orders of magnitude.

For hard spheres, $a = R$ ($\because u = \sin k(r-R) \quad r \geq R$)

There is no general relation between attractive/repulsive and sign of a:

repulsive:



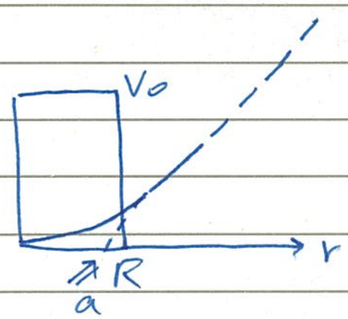
$E \to 0$ $\uparrow$ ($u(r)$ is $\sinh \kappa r$ for $0 \leq r \leq R$ $\quad \frac{\hbar^2\kappa^2}{2m} = V_0$)

$(u(0) = 0)$

$\therefore u' = \cosh \kappa r$

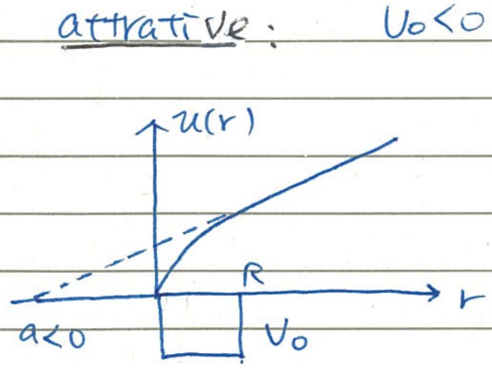
$u'(R) = \cosh \kappa R > 0$

$\Rightarrow$



$a > 0$

attrative: $V_0 < 0$



$|V_0|$ small enough

$u(r) \sim \sin kr$. $\quad k = \sqrt{\frac{2m|V_0|}{\hbar^2}}$ is small $\xrightarrow{\sin}$ so that $u'(R) > 0$!

NO.

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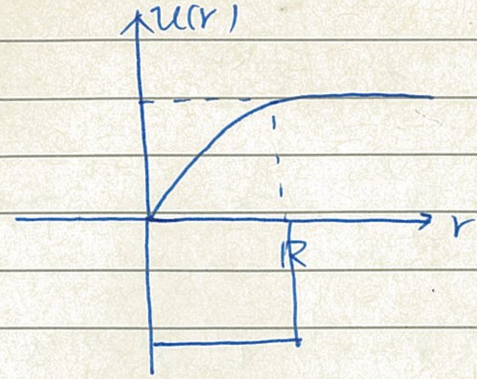
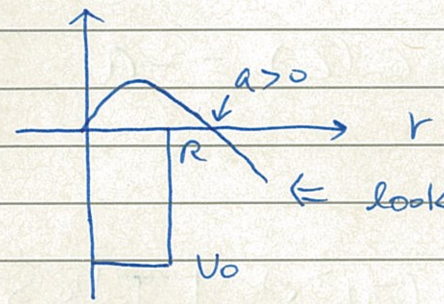
Intermediat U_0 $\qquad a = \infty$



e.g. $KR = \frac{\pi}{2}$ $\qquad |V_0| = \frac{\hbar^2}{2m}\left(\frac{1}{R}\right)^2 \cdot \frac{\pi^2}{4}$

large U_0



⇐ looks like exponential decay if $a \gg R$

bound state may form if $a \gg R$ ⇒ see over

<u>Cross section</u> : $\because a_\ell = \frac{e^{i\delta_\ell}\sin\delta_\ell}{K} = \frac{\sin\delta_\ell}{K(\cos\delta_\ell - i\sin\delta_\ell)}$

$$= \frac{1}{K\cot\delta_\ell - iK}$$

$$\therefore \lim_{K\to 0} a_\ell(K) = \lim_{K\to 0} \frac{1}{K\cot\delta_\ell} \quad \cdots ②$$

but we have seen $\delta_\ell \sim K^{2\ell+1}$ $\quad \therefore a_\ell(K) \sim \frac{\sin\delta_\ell}{K} \sim 0$

for $\ell \neq 0$, $K \to 0$

$\therefore$ only $a_0(K)$ survives (s-wave, $\delta_0(K)$)

$$\therefore \sigma_{tot}(K\to 0) = 4\pi \left(\lim_{K\to 0} a_0(K)\right)^2$$

quick way

$\frac{\sin\delta_\ell}{K} \sim \frac{\delta_\ell}{K} = (-a)$

$\delta_\ell \to 0 \quad e^{i\delta_\ell} \to 1$

NO.

DATE:

$\underline{a \gg R}$ $\underline{r > R}$ $u \sim e^{-\kappa r}$

$$\therefore \frac{u'}{u} = -\kappa = \frac{1}{r-a}\Big|_{r=R} \quad \text{(page 3)}$$

$$\therefore \kappa \simeq \frac{1}{a} \text{ if } R \ll a$$

$$\therefore E = -\frac{\hbar^2 \kappa^2}{2m} = \frac{-\hbar^2}{2m}\left(\frac{1}{a}\right)^2$$

$$f(k', k) \longrightarrow a_0(E) = -a$$

eq (8)

$$= \frac{-1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3 \langle k' | T | k \rangle$$

$\parallel$

V_0 Born-Approx

$\delta(r)$

$$= \frac{-1}{4\pi}\frac{2m}{\hbar^2} V_0$$

$$U = \frac{2\pi a \hbar^2}{m}\delta(r)$$

$\uparrow$

pseudo potential

$$\therefore V_0 = \frac{2\pi a \hbar^2}{m}$$

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Now, from ①

$$\frac{u_0'}{u_0} = k \cot k\left(r + \frac{\delta_0}{k}\right)$$

$$\xrightarrow[\substack{k\to 0\\ r=0}]{} k\cot\delta_0 \text{ should } \to \left.\frac{1}{r-a}\right|_{r=0}$$

← see over

substituted in Born-Approx should be 2π

$$\therefore \lim_{k\to 0} \frac{1}{k\cot\delta_0} = -a = \lim_{k\to 0} a_0(k)$$

→ pseudo potential $V(r) = \frac{4\pi a\hbar^2}{m}\delta(\vec{r})$ (Fermi, 1936)

$$\therefore \sigma_t(k\to 0) = \boxed{4}\pi a^2 \; !$$

⇑ 4!

i.e. the intercept of the zero energy wave function $u(r)$ ~~determines $\sigma_t(k\to 0)$~~!

Friedel Sum rule

Consider putting the scattering event in side a box L^3, the scattering potential is at the Center. A good example is the electron inside the metal, scattered by impurities.

In the absence of impurities,

$\because R_{k\ell}(r) \sim \frac{1}{r}\sin(kr - \frac{1}{2}\ell\pi)$ has to vanish at $r=L$

$$\therefore \quad k_{n\ell} = \frac{n\pi}{L} + \frac{\ell\pi}{2L} \quad \therefore \quad \frac{dk}{dn} = \frac{\pi}{L}, \quad \frac{dn}{dk} = \frac{L}{\pi}$$

= density of states for fixed ℓ

In the presence of impurities

$$k_{nl} = \frac{n\pi}{L} + \frac{l\pi}{2L} - \frac{\delta_l(k_{nl})}{L}$$

$$\therefore\ 1 = \frac{\pi}{L}\frac{dn}{dk} - \frac{1}{L}\frac{d\delta_l}{dk}$$

$$\therefore\ \frac{dn}{dk} = \frac{L}{\pi} + \frac{d\delta_l}{\pi\, dk}$$

Obviously, the extra states are due to impurities.

$$\frac{d\delta n}{dk} = \frac{1}{\pi}\frac{d\delta_l}{dk}$$

$$\therefore\ \frac{dN}{dk} \equiv \text{total change in the density of states}$$

$$= \frac{d}{dk}\sum_{\substack{l, m_l \\ m_s}} \frac{\delta_l}{\pi}$$

In the case of metal, electrons are filled up to k_F

$$\Rightarrow \int_0^{k_F} \frac{dN}{dk}\, dk = \text{total change of electron \#}$$

$$= \sum_{m_s, m_l, l} \frac{\delta_l(k_F)}{\pi}$$

To maintain charge neutrality, total change of electron # = charge on the impurity $= Z$

$$\therefore\ Z = \sum_{\substack{m_s, m_l \\ l}} \frac{\delta_l(k_F)}{\pi}\ . \Leftarrow \text{Friedel Sum Rule}$$

neutral even though there will be + charge (due to exposed ions) on the outer surface!

6/1/'98

Analytic properties of $S_{\ell=0}$ & Resonance scattering

$$\frac{\text{The ratio of outgoing}}{\text{incoming wave}} = S_\ell = 1 - 2\pi i\, T_\ell(E)$$

In other words, for $\ell = 0$,

$$R \sim S_{\ell=0}(k)\frac{e^{ikr}}{r} - \frac{e^{-ikr}}{r}$$

where $E = \frac{\hbar^2 k^2}{2m}$ $\left(\text{or } E - U(r\to\infty) = \frac{\hbar^2 k^2}{2m}\right)$

Now, suppose $E < 0$, i.e. $k \to i\chi$ $\quad \frac{\hbar^2 \chi^2}{2m} = -E$

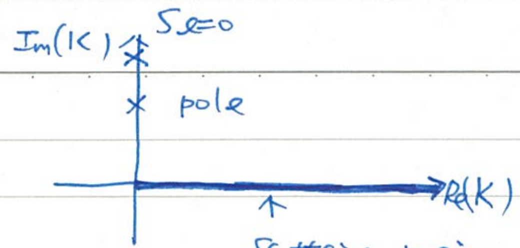
and it corresponds to a bound state, one expects:

$\Big\langle$ no incident wave
only $\frac{e^{-\chi r}}{r}$ is present.

In other words, $\frac{\text{outgoing}}{\text{incoming}} = \infty$ $\quad \therefore \ S_{\ell=0}(i\chi) = \infty$

Therefore, we expect that if we consider $\overset{k \text{ in}}{S_{\ell=0}(k)}$ as a complex number, $S_{\ell=0}(k)$ has a pole at $i\chi$!

It can be shown that such poles are simple poles (for $\ell=0$ & $\ell \neq 0$!) Ref. Scattering theory, Aleksei G. Sitenko, Springer-Verlag, 1991



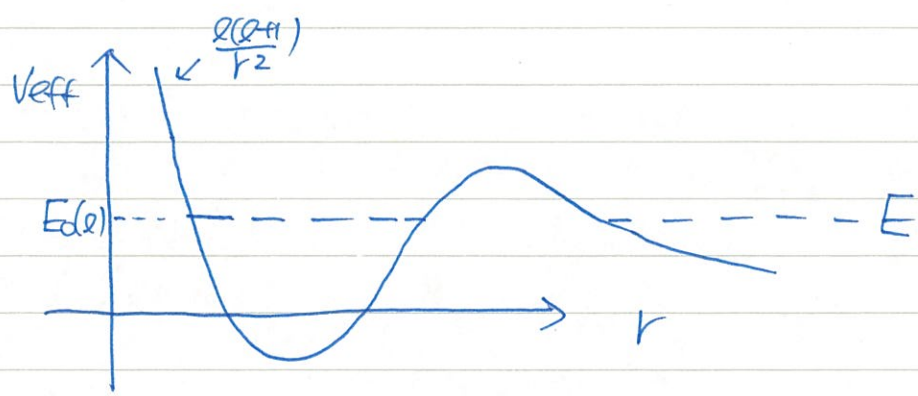
Scattering region ; $S_0 = e^{2i\delta_0(K)}$ $|S_0| = 1$

Note that $\because \lim_{K\to 0} \frac{1}{K\cot\delta_0} = -a$

$\therefore \delta_0(K\to 0) = 0, \pm\pi, \ldots \quad \therefore S_0(K=0) = 1.$

Resonance Scattering

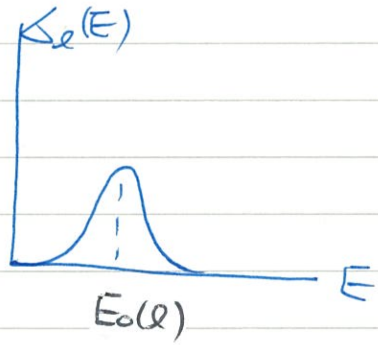
In some cases, when $E>0$, even though the system does not have a bound state, it may happen that the effective $V_{eff} = V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}$ has a potential well that may trap the particle in a "quasi-bound" state. (e.g. Feshbach resonance)



Such a state has a finite lifetime due to quantum mechanical tunneling.

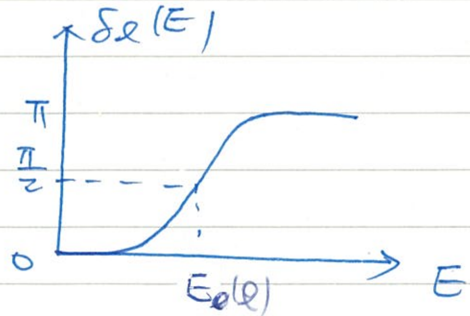
Since $\sigma = \sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell$ for "ℓ-wave", one expects that as we vary E through $E_0(\ell)$, σ_ℓ will go through a max. right at $E_0(\ell)$!

If only one quasi-bound state is present, & for $E \ll E_0(\ell)$, and $E \gg E_0(\ell)$, there is no scattering. We have



The phase shifts at two limits $E \ll E_0(\ell)$ & $E \gg E_0(\ell)$ are 0 & π $\quad (S_\ell = e^{2i\delta_\ell})$

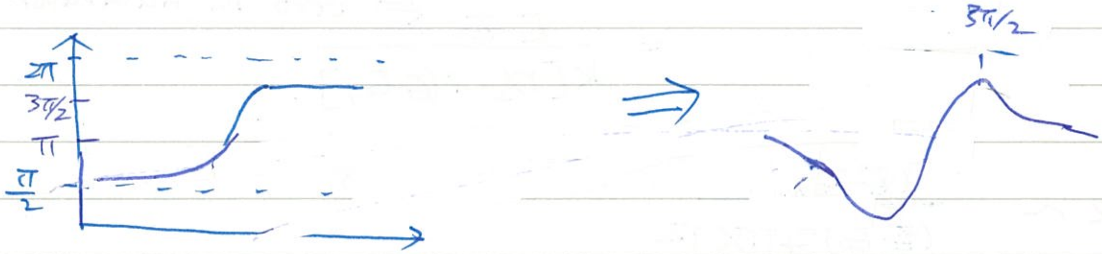
$\therefore$



$$\delta_\ell(E) \simeq \tan^{-1}\left(\frac{\Gamma/2}{E_0 - E}\right)$$

$\therefore$ At $E_0(\ell)$, $\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)$ (unitary limit) reaches max !!

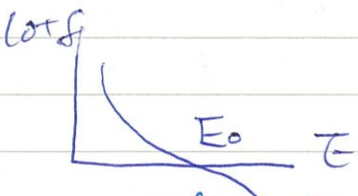
There are also other ways to go through the resonant energy $E_0(\ell)$: (due to background ; other quasi-bound states)



We will concentrate on the 1st case. (others see over)

In this case, $\because \ \delta_\ell(E_0(\ell)) = \pi/2 \quad \therefore \cot\delta_\ell \Big|_{E=E_0} = 0$

$$\therefore \cot\delta_\ell(E) \cong + \frac{d\cot\delta_\ell}{dE}\Big|_{E=E_0} (E-E_0) \Rightarrow \delta_\ell \simeq \tan^{-1}\left(\frac{\Gamma/2}{E_0-E}\right)$$



$\because \dfrac{d\cot\delta_\ell}{dE}\Big|_{E=E_0} < 0 \quad \therefore$ usually, one sets it to $\dfrac{d\cot\delta_\ell}{dE_0} = -\dfrac{2}{\Gamma}$.

$$\therefore a_\ell = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k} = \frac{1}{k\cot\delta_\ell - ik} = -\frac{\Gamma/2}{k[(E-E_0)+i\Gamma/2]}$$

$$\therefore \sigma_l = 4\pi(2l+1)\, a_l^2 = \frac{4\pi}{k^2} \frac{(2l+1)(\Gamma/2)^2}{(E-E_0)^2 + \Gamma^2/4}$$

This is known as the Breit-Wigner formula.

$$\Delta E = \Gamma/2, \quad \therefore \quad \Delta t \sim \frac{\hbar}{2}\frac{1}{\Delta E} = \frac{\hbar}{2}\frac{2}{\Gamma}$$

lifetime of quai-bound state!

Note that $\because \; \delta_l \simeq \tan^{-1} \dfrac{\Gamma/2}{E_0 - E}$

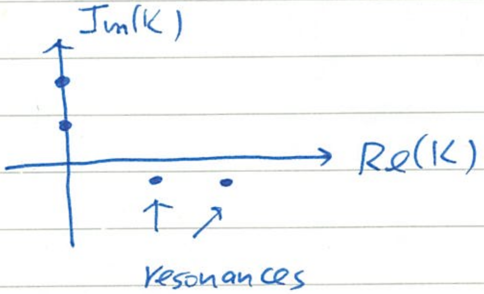
$$\therefore \; S_l(k) = e^{2i\delta_l} = \frac{e^{i\delta_l}}{e^{-i\delta_l}}$$

$$= \frac{1 + i\tan\delta_l}{1 - i\tan\delta_l} = \frac{E - E_0 - i\Gamma/2}{E - E_0 + i\Gamma/2}$$

$\therefore$ resonance also corresponds to a pole

at $E_0 - i\Gamma/2$. $\left(e^{-iEt/\hbar} \simeq e^{-iE_0 t/\hbar} \underset{\Uparrow}{e^{-\frac{\Gamma t}{2\hbar}}} \right)$

$$\Delta t \simeq \frac{\hbar}{\Gamma}$$

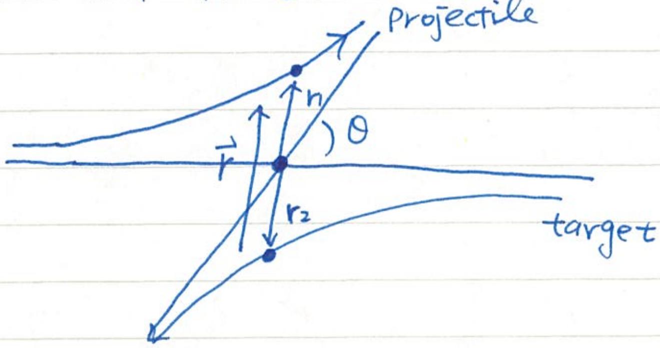


Two-particle scattering

The above formulation applies to the case when the scattering center is fixed.

When we consider scattering for two particles, there are two useful frames that we can look at this problem.

(i) CM frame



$$\begin{cases} \vec{r}_1 = \vec{R}_{CM} + \frac{m_2}{M}\vec{r} \\ \vec{r}_2 = \vec{R}_{CM} - \frac{m_1}{M}\vec{r} \end{cases}$$

write $\vec{r}_1$ & $\vec{r}_2$ in terms of $\vec{r}$ & $\vec{R}_{CM}$ $\quad \vec{R}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$

$$\psi_{inc} = e^{i\vec{k}_1\cdot\vec{r}_1} e^{i\vec{k}_2\cdot\vec{r}_2} = e^{i\vec{K}\cdot\vec{R}_{C.M.}} e^{i\vec{k}\cdot\vec{r}} \qquad \vec{r} = \vec{r}_1 - \vec{r}_2$$

$\vec{K} = \vec{k}_1 + \vec{k}_2$. $\quad \vec{k} = \frac{1}{M}(m_2\vec{k}_1 - m_1\vec{k}_2)$, $\hbar\vec{k}$ = momentum of reduced mass $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$

$$\hbar\vec{k} = \frac{1}{M}(m_2\hbar\vec{k}_1 - m_1\hbar\vec{k}_2)$$
$$= \frac{1}{M}(m_2 m_1)(\vec{v}_1 - \vec{v}_2) = \mu\vec{v}_{rel}$$

fictitious particle.

$e^{i\vec{K}\cdot\vec{R}_{CM}}$ is not changed during scattering, but

$$e^{i\vec{k}\cdot\vec{r}} \rightarrow e^{i\vec{k}\cdot\vec{r}} + \frac{f(\theta,\phi)}{r} e^{ikr}$$

$$\therefore \psi(r_1, r_2) \rightarrow e^{i\vec{K}\cdot\vec{R}_{C.M.}} \left[e^{i\vec{k}\cdot\vec{r}} + \frac{f(\theta,\phi)}{r} e^{ikr} \right]$$

In the C.M. frame, one has $\vec{K} = 0$, so $e^{i\vec{K}\cdot\vec{R}_{CM}} = 1$ can be dropped out. Since $\vec{r}_1 \parallel \vec{r}$, the rate for the projectiles scattered into $d\Omega$ is the same as that for the fictitious particles scattered into $d\Omega$!

i.e.

projectiles / per sec through $d\Omega$

$$= |f(\theta,\phi)|^2 \frac{\hbar k}{\mu} d\Omega \cdot \frac{1}{V}$$ (where we have used the result from fixed scattering center problem.)

In the CM frame. $\vec{k} = \frac{1}{M}(m_2\vec{k}_1 - m_1\vec{k}_2) \underset{\uparrow}{=} \vec{k}_1 = -\vec{k}_2$.

$\vec{k}_2 = -\vec{k}_1$

$$\text{relative incident flux} = \frac{\hbar \vec{k}_1}{m_1}\frac{1}{V} - \frac{\hbar \vec{k}_2}{m_2}\frac{1}{V}$$

$$= \frac{\hbar \vec{k}}{V}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

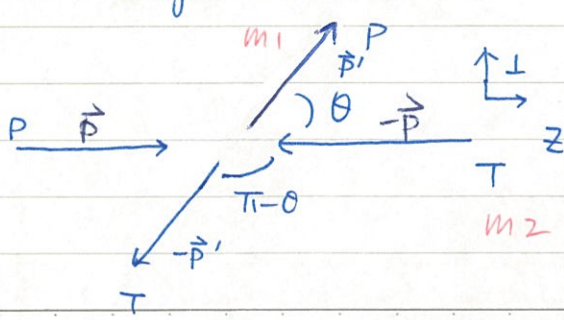
$$= \frac{\hbar \vec{k}}{\mu} \cdot \frac{1}{V}$$

∴ The whole problem is the same as the fixed scattering center problem except that one has to consider the fictitious particle with a reduced mass μ.

(ii) Lab frame

In the lab frame, the target is at rest initially.

The passage to the lab frame from the CM frame is straight forword:

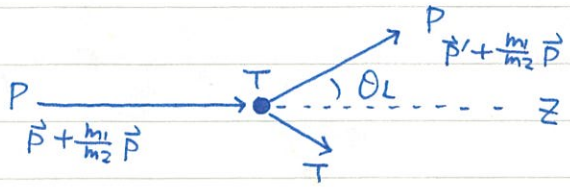


C.M. $\tan\theta = \frac{p_\perp'}{p_z'}$

$\tan\phi = \frac{p_y'}{p_x'}$

$|\vec{p}'| = |\vec{p}| = p$

The same scattering event occurs at different angles in the lab frame.



In the lab frame, T is at rest initially. So we shift the momentum of projectiles by $\vec{p}'+\frac{m_1}{m_2}\vec{p}$

$$\therefore \tan\theta_L = \frac{p_\perp'}{p_z'+\frac{m_1}{m_2}p} \underset{|p'|=|p|}{=} \frac{p_\perp'/p'}{\frac{p_z'}{p'}+\frac{m_1}{m_2}} = \frac{\sin\theta}{\cos\theta + m_1/m_2}$$

$$\tan\phi_L = p_y'/p_x' = \tan\phi, \quad \phi_L = \phi$$

$$\because \frac{d\sigma}{d\Omega_L}d\Omega_L = \frac{d\sigma}{d\Omega}d\Omega \quad \text{--- (15)}$$

$$\therefore \frac{d\sigma}{d\Omega_L} = \frac{d\sigma}{d\Omega}\frac{d\Omega}{d\Omega_L} = \frac{d\sigma}{d\Omega}\frac{\sin\theta\, d\theta\, \cancel{d\phi}}{\sin\theta_L\, d\theta_L\, \cancel{d\phi_L}}$$

By expressing θ in terms of θ_L, we get

$$\frac{d\sigma}{d\Omega_L}(\theta_L).$$

For example, $m_1=m_2$, $\because \frac{\sin\theta}{\cos\theta+1} = \tan(\theta/2) \quad \therefore \theta_L = \theta/2$

$$\frac{\sin\theta\, d\theta}{\sin\theta_L\, d\theta_L} = 2\frac{\sin 2\theta_L}{\sin\theta_L} = 4\cos\theta_L \quad \therefore \frac{d\sigma}{d\Omega_L} = 4\cos\theta_L \frac{d\sigma}{d\Omega}(2\theta_L)$$

Note that because of eq. (15), σ_{total} is the same in both frames.

4/1/97

Scattering of identical particles

In the above discussion, the calculated cross section is only for the projectiles.

If the detector is sensible only to the projectile, $\frac{d\sigma}{d\Omega}$ is the cross section one measures.

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2$$

If the detector just counts # of particles that enter into it, we have

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 + |f(\pi-\theta)|^2$$ for

non-identical particles.

For identical particles, one needs to consider receiving target particles automatically no matter what kind of sensor one uses.

For spin-zero bosons, we need to symmetrize the total wave function.

$$\therefore \ \Psi_{init.} = e^{i\vec{K}\cdot\vec{R}_{C.M}}\left(e^{i\vec{k}\cdot\vec{r}} + e^{-i\vec{k}\cdot\vec{r}}\right)$$

$$\Psi(r\to\infty) = \underbrace{e^{i\vec{k}\cdot\vec{r}} + e^{-i\vec{k}\cdot\vec{r}}} + \left[f(\theta,\phi) + f(\pi-\theta,\phi+\pi)\right]\frac{e^{ikr}}{r}$$

actually, no need for symmetrization

as initially they are well seperated and are

$\therefore \quad \frac{d\sigma}{d\Omega} = |f(\theta,\phi) + f(\pi-\theta, \phi+\pi)|^2$ $\qquad \frac{\hbar \vec{k}_1}{m}\frac{1}{V} - \frac{\hbar \vec{k}_2}{m_2}\frac{1}{V}$

(note that the incident flux is still the same because initially projectiles & targets are well seperated, and thus distinguishable!)

$$= \underbrace{|f(\theta,\phi)|^2 + |f(\pi-\theta,\phi+\pi)|^2}_{\text{distinguishable case}} + 2Re\left[f(\theta,\phi) f^*(\pi-\theta,\phi+\pi)\right]$$

* One sees that in addition to $|f(\theta,\phi)|^2 + |f(\pi-\theta,\phi+\pi)|^2$, one gets interference term $2Re\left[f(\theta,\phi) f^*(\pi-\theta,\phi+\pi)\right]$!

If $f = f(\theta)$, then at $\theta = \pi/2$, the cross section $= 4|f(\pi/2)|^2$ is enhanced by 2 times!

* To find the total cross section σ, we integrate over only 2π not 4π:

$$\sigma = \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} d\phi \, \sin\theta \, \frac{d\sigma}{d\Omega}.$$

This is <u>not to double count</u> <u>each event twice</u>!

For Fermions, the total wave function needs to be antisymmetrized.

For example, consider scattering of two identical electrons. If the incident beam is <u>unpolarized</u>, one has $\frac{1}{4}$ contribution coming from singlet, $\frac{3}{4}$ from triplet.

Therefore,

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \left| f(\theta, \phi) + f(\pi-\theta, \phi+\pi) \right|^2$$

↑
symmetric in space part
singlet

$$+ \frac{3}{4} \left| f(\theta, \phi) - f(\pi-\theta, \phi+\pi) \right|^2$$

↑
antisymmetric in space part
triplet.