Homework 3
Due: December 21, 2023

Ex.1 10% A thin nanowire of constant diameter is connected into a loop. The radius of this closed circular loop is $R = 10^{-7}$m (so it can be called a nanoloop). The loop contains one superconducting vortex. Find the magnitude of the superfluid velocity v_s in the wire. No external magnetic field is applied. Assume the wire forming the loop is so thin that the magnetic field produced by the supercurrentis negligible. Comment on how v_s would change if the loop is deformed from the circular to some arbitrary shape, assuming that the length of the wire forming the loop remains unchanged. Electric capacitance of the loop is assumed negligibly small and the system is in local equilibrium.

Ex.2 10%
In the construction of the solution of Ginzburg-Landau equation for the Abrikisiv lattice near H_{c2}, the solution takes the form

$$\psi_L(x,y) = \sum_{n=-\infty}^{\infty} C_n e^{inky} e^{-(x-x_n)^2/(2\xi^2)}, \tag{1}$$

where $x_n = n\hbar c/(e^* H_{c2})$. Find the condition on C_n such that ψ_L describes a triangle lattice.

Ex.3 10%
The displacement field $\vec{u}_\alpha$ of a vortex lattice is defined as $\vec{u}_\alpha = \vec{r}_\alpha - \vec{R}_\alpha$, where $\vec{r}_\alpha$ ($\vec{R}_\alpha$) is the real (ideal) position of a flux line. Note that $\vec{u}_\alpha$ are two-dimensional vectors. Assume that the magnetic field is applied along z-axis, and there is no pinning, then the thermal average of the mean displacement is given by

$$\langle u^2 \rangle = \frac{\int d|\vec{u}(r)|^2 e^{-\delta G(u)/k_B T}}{\int du(r) e^{-\delta G(u)/k_B T}}, \tag{2}$$

where $\delta G(u)$ is the excess Gibbs free energy due to the elastic response to the distortion of the vortex lattice. The elastic matrix $\phi_{\alpha\beta}$ (α, $\beta = x$, y) is given by

$$\phi_{\alpha\beta}(k) = [C_{11}(k) - C_{66}]k_\alpha k_\beta + \delta_{\alpha\beta}[C_{66}k_\perp^2 + C_{44}k_z^2],] \tag{3}$$

where $k_\perp^2 = k_x^2 + k_y^2$, and C_{11}, C_{44}, C_{66} are the compression, tilt, and shear moduli, respectively. From Eqs.(1) and (2), show that

$$\langle u^2 \rangle = \int \frac{d^3k}{(2\pi)^3} \left[\frac{k_B T}{C_{66}k_\perp^2 + C_{44}k_z^2}\right] + \left[\frac{k_B T}{C_{11}k_\perp^2 + C_{44}k_z^2}\right] \tag{4}$$

Ex.4 The complex conductivity $\sigma(\omega)$ of a superconductor can be generally expressed in terms of superfluid density ρ_s and the frequency ω by the following formula

$$\sigma(\omega) \sim \frac{\rho_s}{-i\omega + \epsilon}, \tag{5}$$

where $\epsilon \to 0^+$, $\rho_s \sim \xi^{2-d}$ in the critical regime, with d being the sample dimensionality, $xi \sim |T - T_M(H)|^{-\nu}$ being the vortex cpherence length, and ν being the static exponent. One also defines a characteristic relaxation time $\tau \sim \xi^z$, where z is the dynamical exponent. $T_M(H)$ is the vortex lattice melting temperature at external field H and we are are interested in the transport p[roperies neat T_M. Let us consider the conductivity as a funciton of the applied current density (J), H, and the frequency of applied currents (ω), i.e., $\sigma = \sigma(x, y, u)$, where $x = J\xi^{d-1}\Phi_0/(k_B T)$, $y = H\xi^d/\Phi_0$, $u = \omega\xi^z$ with Φ_0 being the flux quantum.
(a) 10% Show that in the presence of a constant magnetic field (H), where $H_{c1} < H < H_{c2}$, the electrical field E in the limit of $\omega \to 0$ and $T \to T_M(H)$ is given by

$$E(J,T) = J\sigma^{-1} \sim J\xi^{d-2-z} F_\pm(x), \tag{6}$$

and that

$$E(J,T) = J^{(z+1)/(d-1)}, \;\; T = T_M(H), \tag{7}$$

where $F_\pm$ are the universal functions for $T > T_M^+$ and $T < T_M^-$, respectively.
(b) 5% For $J \to 0$, and $H_{c1} < H \ll \frac{1}{2}H_{c2}$ (so that $T_M(H) \to T_c^-$), show that the resistivity $\rho \sim \sigma^{-1}$ has the following magnetic field dependence

$$\rho(H,T) = \xi^{d-2-z} R_\pm(y), \tag{8}$$

and that $\rho(H,t) \sim H^{(z+2-d)/2}$, $T \to T_c^-$. What is the functional form of $R_+(y)$ as $y \to 0$?
(c) 5% In the linear resistivity regime, show that the conductivity at a constant magnegtic field has the following frequency dependence

$$\sigma(\omega,T) \sim \xi^{2-d+z} S_\pm(u), \tag{9}$$

where $S_\pm(u)$ are the universal functions for $T > T_M^+$ and $T < T_M^-$, respectively. Show that $\sigma(\omega,T) \sim \omega^{-(z+2-d)/z}$, $T \to T_M(H)$, and that $S_+(u) \to$ real constant and $S_-(u) \to 1/(-iu)$ for $u \to 0$.