Homework 2
Due: November 9, 2023

Ex.1 Supercurrent density and effective penertation in nonlocal electrodynamic approximation
In contrast to the local electrodynamic approximation assumed in the London theory, the Pippard theory takes into account the nonlocal vector potential dependence of the supercurrent density $\vec{J}_s$ as

$$\vec{J}_s = -\frac{3}{4\pi\xi_0\Lambda c}\int d^3\vec{r}'\,\frac{\vec{R}(\vec{R}\cdot\vec{A}(\vec{r}'))}{R^4}e^{-R/\xi}, \tag{1}$$

where $\vec{R} = \vec{r} - \vec{r}'$, ξ_0 is the Pippard coherence length, $\vec{A}$ is the vector potential, $\Lambda = 4\pi\lambda_L^2/c^2$, λ_L is the London penetration depth, and $\xi^{-1} = \xi_0^{-1} + l^{-1}$, l is the electron mean free path.
(a) 10% In the Fourier space, $\vec{A}(\vec{r})$ is given by $\vec{A}(\vec{r}) = \sum_{\vec{q}}\vec{A}(\vec{q})e^{i\vec{q}\cdot\vec{r}}$ so that one can write the Fourier component of $\vec{J}_s$ as $\vec{J}(\vec{q}) = -\frac{c}{4\pi}K(q)\vec{A}(\vec{q})$. Find $K(q)$ and show that $\frac{K(q\to\infty)}{K(0)} = \frac{3\pi}{4q\xi}$.
(b) 5% In general, the nonlocal relation between the supercurrent and the vector potential can be writtern as

$$\vec{J}_s = -\frac{3}{4\pi\xi_0\Lambda c}\int d^3\vec{r}'\,\frac{\vec{R}(\vec{R}\cdot\vec{A}(\vec{r}'))}{R^4}F(R), \tag{2}$$

where $F(R)$ describes the spatial variation of the vector potential and may not be exactly the same form as $e^{-R/\xi}$. The effective peneration depth $\lambda_{eff}(T)$ is defined as

$$\lambda_{eff}^2(T) \equiv \lambda_L^2(T)\frac{K(0,T,l\to\infty)}{K(0,T,l)} = \lambda_L^2(T)\frac{\xi_0}{\int_0^\infty F(R,T)e^{-R/l}dR}, \tag{3}$$

where $\lambda_L(T) = \lambda_L(0)/\sqrt{1-(T/T_c)^4}$ is the London penetration depth at finite temperature T. If $F(R,T)$ is given by $F(R,T) = F(0,T)\exp[-F(0,T)R/\xi_0]$, find $\lambda_{eff}(T)$ in terms of $\lambda_L(T)$, l, ξ_0, and $F(0,T)$. In the BCS theory, $F(0,T) = 0$ for $T = 0$. Hence $\lambda_{eff}(T) = \lambda_L(0)\xi_0/\xi$.

Ex.2 10% Consider a flat superconducting slab of thickness d in a magnetic field H_a parallel to the slab. By solving the London equation $\nabla^2\vec{h} = \frac{1}{\lambda^2}\vec{h}$ for microscopic magnetic field $\vec{h}$, find the magnetization M inside the slab for cases when $d \gg \lambda$ and $d \ll \lambda$. From M, find the critical field (paralell to film) for superconducting thin films ($d \ll \lambda$) in terms of thermodynamic critical field H_c.

Ex.3 10% The critical current of a superconducting wire
A current I is injected into a long superconduting wire with radius R. Let λ be the penetration length.
In the London model, find the current density $J(r)$ for $r \le R$. (Express your answers in terms of r, R, λ, and modified Bessel functions). Find the critical current I_c when the magentic field at $r = R$ just becomes the critcal field H_c.

Ex. 4
(a) 5% Use Heisenberg's uncertainty principle to estimate the size, ξ_0, of a Cooper pair of electrons in a clean superconductor at zero temperature. Assume that the critical temperature, T_c, and the Fermi velocity, v_F, are known. Use the gap Δ and T_c relation: $\Delta = 1.76k_BT_c$ in the BCS theory.
(b) 5% Use the Heisenberg's uncertainty principle to estimate the size, $\xi(0)$, of a Cooper pair of electrons in a dirty or disordered superconductor. Assume that the elastic mean free path for single electrons, le, the cleanlimit coherence length, ξ_0, and the critical temperature in the clean-limit case (i.e., the same material, but without disorder), T_c, are known. Use the Anderson– Abrikosov–Gor'kov (AAG) theorem, which states that the critical temperature does not depend on the presence of disorder.

Ex 5 Two planar superconductors 1 and 2 are placed with their fiat faces in very good contact. Their critical temperatures are Tc_1 and Tc_2 respectively, with $Tc_2 > Tc_1$ and $Tc_2 - Tc_1 \ll Tc_1$. The system is cooled to a temperature T between Tc_1 and Tc_2 so that only superconductor 2 is superconducting.
(a) 5% Show that the Ginzburg-Landau equation for superconductor 1 can be written as $-\xi_1^2\frac{d^2\phi}{dx^2} + \phi + \phi^3 = 0$, where ϕ is the dimensionless wave function and ξ_1 is given by $\xi_1 = (\hbar^2/2m^*|\alpha_1|)^{1/2}$.
(b) 10% Making use of the fact that $|\phi| \ll 1$ in a normal metal so that the cubic term in the aboye equation can be neglected, show that the wave function decays according to $\phi = \phi_0 e^{-|x|/\xi_1}$ in superconductor 1, where $x = 0$ is at the interface between the two superconductors and superconductor 1 occupies the $x < 0$ region.
Outline a numerical scheme to find the value of ϕ_0.