Electrodynamics and relativity

Special theory of relativity

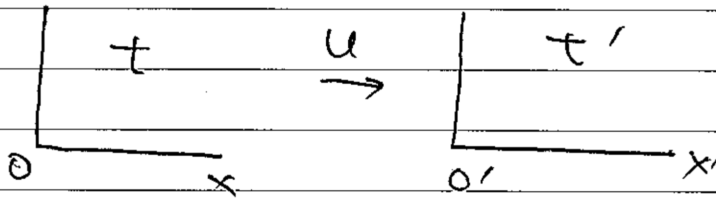
Principle of relativity in classical mechanics:

Before Einstein discovered the special relativity, it is known that dynamics of particles obeys the principle of relativity:

the same Newton's laws apply to any inertial frame.

Inertial frames are frames in which Newton's laws apply: e.g. if there is no force on a particle, the particle moves with constant velocity. i.e. there is no acceleration of inertial frames to others.

In particular, for any two inertial frames, their coordinates are related by Galilean transformations that obey the principle of relativity:



$$t = t'$$
$$x = x' + ut \quad \therefore \; a = a'$$
$$v = v' + u$$

What is an inertial frame ?

In practice, an inertial frame is often determined by the relative motion of the frame to some distant star.

In such frames, one finds Newton's laws apply approximately. There is no accelation in such frames.
There is no accelation of the inertial frame. But the question is : relative to which frame, there is no acceleration ?

Newton himself believes that there exists an absolute space and time. Relative to (which is at rest.) this frame, one has other inertial frames.

This line of thought dominated over minds of physicists when one tried to apply the principle of relativity to electrodynamics.

In that case, it was tempting to assume that similar to the propagation of other

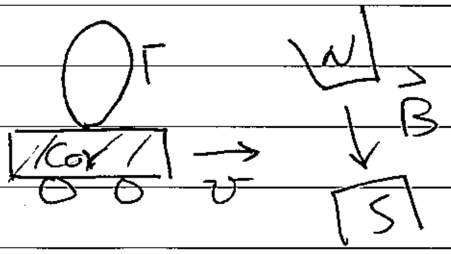
Waves (such as water waves, sound),

the propagation of EM waves also needs a medium, called ether.

Therefore, ether is associated with the absolute space and time that was believed by Newton.

Motional emf vs. induced emf

Under the above view, the equality of motional emf to induced emf is a coincidence. Consider a conducting loop Γ rest on a moving car passing a magnet. Shown in the below.



In the rest frame of the magnet, charges on loop Γ experiences a magnetic force $\vec{F} = q\vec{v}\times\vec{B}$, hence there is a motional emf $\vec{E} = \vec{v}\times\vec{B}$; $\mathcal{E} = \oint_{\Gamma} \vec{v}\times\vec{B}\cdot d\vec{l}$

$$= -\frac{d}{dt}\int \vec{B}\cdot d\vec{a} \quad = -\frac{d\Phi}{dt} \quad \text{(flux-rule)}$$

$$\left(\oint \vec{v}\times\vec{B}\cdot d\vec{l} = -\int \vec{B}\cdot d\vec{l}\times\vec{v} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a} \right)$$

In the rest frame of the car, the magnet is moving. According to the Faraday's law, an induced electric field gives rise to $\mathcal{E} = -\frac{d\Phi}{dt}$.

There is no magnetic force!

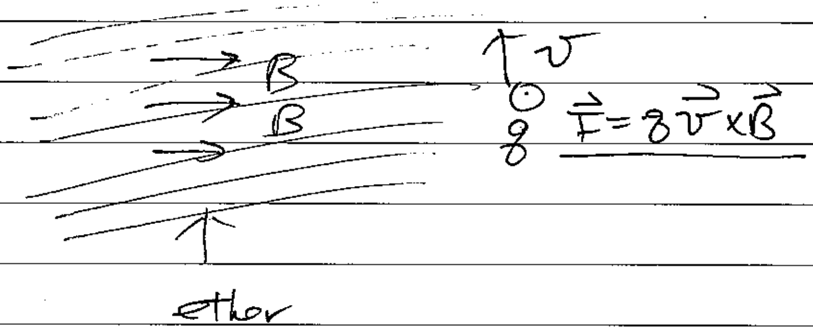
According to the Newton's view, there is a magnetic force. The interpretation of the rest frame of the car is wrong!

There is an absolute rest frame of ether in which one has the magnetic field. This is the frame when the source (the magnet) is at rest.

The laws of electromagnetism is valid only in the frame with ether.

In this view, in the ~~rest~~ frame of the Car, charges moves in ether with $\vec{B}$ and still experience the Lorentz force $q\vec{v}\times\vec{B}$ with $\vec{B}$ being associated with the ether and there is no electric field $\vec{E}$!

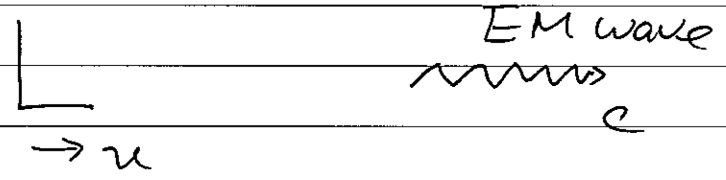
It is the relative motion $\vec{v}$ of the charge to the absolute frame with $\vec{B}$ that gives rise to the emf $\vec{v}\times\vec{B}$. ∴ The velocity of charge needs to be measured in the absolute frame.



Michelson-Morley expt. (1887)

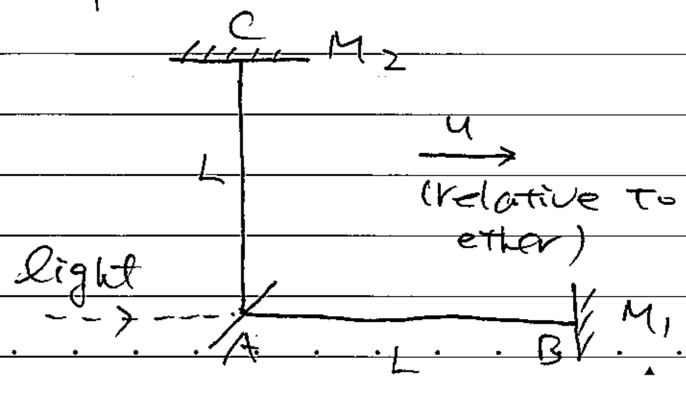
Even in the Newtonian view, the interpretation in the rest frame of magnet is also problematic: how do we know the observer in the rest frame of magnet is not moving relative to the ether?

If indeed there is an 'absolute frame with ether, the earth is also moving in the medium with ether. The EM wave / Maxwell's equations are only valid in the ether frame, One should be able to measure the difference for moving against or following the ether according to the Galilean transformation:



$$c' = c \mp u$$

Unfortunately, physicists in 19th century can't detect such differences. The most famous experiment is the Michelson-Morley experiment done in 1887. In this experiment



as shown in the left,

$\overline{AB} = \overline{AC} = L$

light is incident from the left on a

Semi-transparent mirror at point A: the
(beam-split)
light is splitted with half intensity going to $\overline{AB}$
& the other half intensity going to $\overline{AC}$.

If the earth is moving with velocity u relative to the ether, ∵ c is the speed of EM wave in the rest frame of ether, the

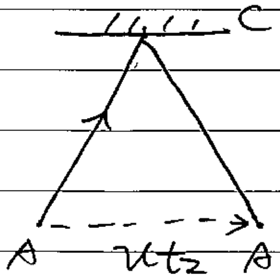
traveling time from A to B $= \frac{L}{c-u}$

B to A $= \frac{L}{c+u}$

∴ total traveling time in $\overline{AB}$

$$= t_1 = \frac{L}{c-u} + \frac{L}{c+u} \quad \cdots ①$$

For the traveling time t_2 between $\overline{AC}$, ∵ during the movement, point A moves ut_2,



$$\therefore \left(\frac{ct_2}{2}\right)^2 = L^2 + \left(\frac{ut_2}{2}\right)^2$$

$$t_2 = \frac{2L}{\sqrt{c^2-u^2}} = \frac{2L}{c}\frac{1}{\sqrt{1-u^2/c^2}} \quad \cdots ②$$

The difference in traveling time is

$$\Delta \equiv t_1 - t_2$$

$$\because \; t_1 = \frac{2L}{c}\,\frac{1}{1-u^2/c^2} = \frac{2L}{c}\left(1+\frac{u^2}{c^2}+\cdots\right)$$

$$t_2 = \frac{2L}{c}\left(1-u^2/c^2\right)^{-\frac{1}{2}} = \frac{2L}{c}\left(1+\frac{u^2}{2c^2}+\cdots\right)$$

$$\therefore \; \Delta = \frac{L}{c}\left(\frac{u}{c}\right)^2 + O\left(\left(\frac{u}{c}\right)^4\right)$$

Δ implies that two light beams that travel $\overline{AB}$ & $\overline{AC}$ combine at A have a phase shift $\omega t = \omega\Delta = \frac{2\pi\Delta}{T}$ ……… ③

It will induce an interference pattern.

In real experiment, $\overline{AB}$ will not be exactly the same as $\overline{CD}$. Hence eq③ may not be observable.

Michelson - Morley overcame this difficulty by rotating the whole equipment -90° so that roles $\overline{AB}$ & $\overline{AC}$ are switched.

That is, Rotation -90° : $\Delta \to -\Delta$

Which includes error's due to $\overline{AB} \neq \overline{AC}$.

$\therefore$ Total phase during the rotation

is $2\pi \frac{2\Delta}{T}$. The interference patten

should shift $\delta = \frac{2\Delta}{T}$ interference fringes $= \frac{2\Delta}{\lambda}c = \frac{2L}{\lambda}\left(\frac{u}{c}\right)^2$

L - - ④

For $L = 11$ m, $\lambda = 5900\,\text{Å}$ (Sodium light)

$u = 3\times10^{4}$ m/s, $\frac{u}{c} \sim 10^{-4}$

$\delta \cong \frac{22}{5.9\times10^{-5}} \times (10^{-4})^2 = 0.37 \sim \frac{1}{4}$ interference fringe

The accuracy of Michelson – Morley expt can detect 0.005 fringe but they did not find any shift of fringe when the apparatus is rotated 90°.

The result shows $\Delta_1 = \Delta_2$, i.e., the speed of light in going $A \to B$ & $A \to C$ is the same!

Furthermore, there is no ether. It indicates that any inertial frame is a suitable reference frame for application of Maxwell's equations. The velocity of charge does not have to be measured in the absolute frame.

Inspired by these observations, Einstein proposed his two famous postulates:

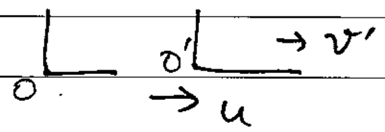
(i) The principle of relativity

The laws of physics apply in all inertial reference systems (with the same forms, i.e., there is no preferred inertial frame)

(ii) The universal speed of light

The speed of light in vacuum is the same for all inertial observers, regardless of the motion of sources

Clearly, (ii) implies the Galileo's velocity addition rule : $v = u + v'$



(v' = speed in O')
(v = speed in O)

is not correct!

($\because$ if $v'=c$, $v=u+c>c$)

As we shall see, the velocity addition rule in the special relativity becomes

$$v=\frac{u+v'}{1+\frac{uv'}{c^2}}$$

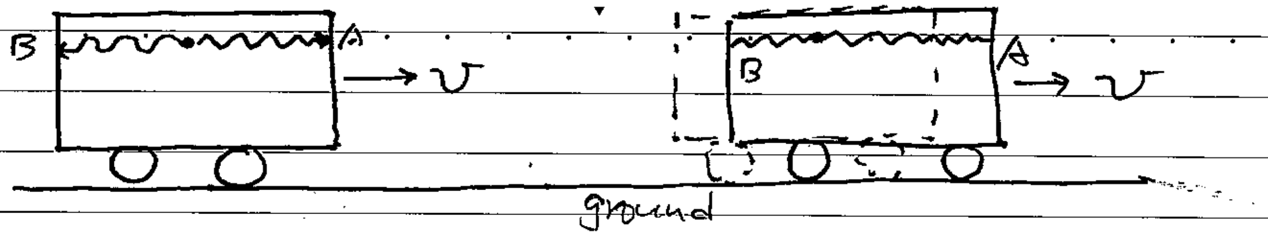
$\therefore$ if $v'=c$, $v=\frac{u+c}{1+\frac{u}{c}}=c$ is still : c !

Geometry of relativity

The postulates of Einstein lead to three remarkable consequences regarding space and time. We shall present them via thought (gedanken) experiments, to reveal their physical meanings.

(i) The relativity of simultaneity

The universality of speed for light implies that the concept of simultaneity needs to be revised. For this purpose, consider a car moving at a constant speed along a straight track as shown in below.



At the center of the car, light is switched on so that the light spreads out in both directions and hit frontal and back walls.

For the observer inside the car,

two events ① light hits the front end
② 〃 〃 〃 back end

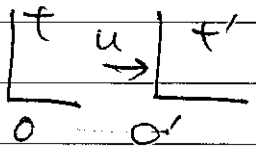
occur simultaneously !

However, since light travels with the same speed c for the frame of ground, light travels shorter when it reaches B (back end) as the car (back end) is moving forward and travels longer when it hits the front end A.

Clearly, for observers on the ground, light hitting the back end happens before light hitting the front end !

... Simultaneously occurring events in one inertial frame do not happen simultaneouly in the other frame!

The relativity of simultaneity implies that $t=t'$ in the Galilean transformation



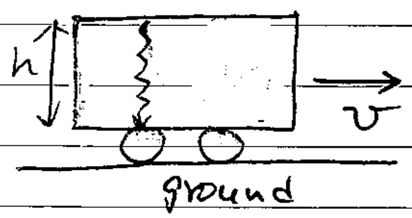
is no longer true.

We shall see that this will be replaced by a more general transformation known as the Lorentz transformation.

(ii) Time dilation

The universal speed of light also implies that the duration of time is also relative.

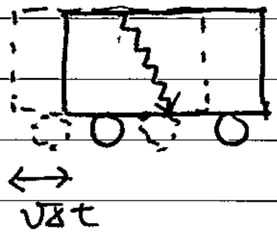
For this purpose, one considers a car carrying a light emitter emitting a light hitting directly $\underbrace{\text{in below}}_{\text{the floor}}$ as shown in the left figure.



For an observer inside the car,

the duration $\Delta t = \frac{h}{c}$

For an observer on the ground, however, because the height of emitter is the same, & the car is moving, he found (as we shall show later, $v \perp h$).



that if Δt = duration,

$$\Delta t = \frac{\sqrt{h^2 + (v\Delta t)^2}}{c}$$

$$\therefore \; \Delta t = \frac{h}{c}\frac{1}{\sqrt{1-v^2/c^2}}$$

$$\Delta\bar{t} = \sqrt{1-v^2/c^2}\;\Delta t \; < \Delta t$$

Hence the observer found that the duration Δt is longer than $\Delta\bar{t}$.

The time elapsed between two events ① light leaves emitter ② light hits the floor is thus different for observers in different frames. In fact, the duration $\Delta\bar{t}$ is shorter by the factor

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

Hence moving clocks run slower.

This is called time dilation.

The time dilation is the most spectacular prediction of the special relativity that has the most persuasive confirmation.

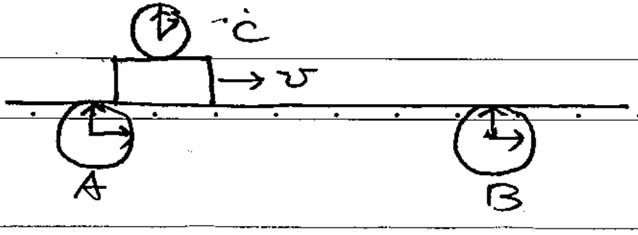
The experimental confirmation comes from the clocking of the life time of unstable elementary particles : life time of particle at rest = life time of the particle that moves $\times\sqrt{1-v^2/c^2}$.

time dilation & relativity

It may seem that there is an inconsistency of the time dilation with the principle of relativity : ① ground observer : train clock runs slow
② train observer : ground clock " "

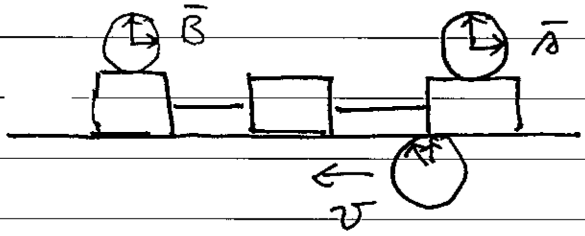
Who is right? Ans : both are correct.
Why?

To check ①, one performs the following measurement : one train clock (C) passing two ground clocks synchronized (A&B)



reading of A&B when C passes by $= t_A, t_B; \; \Delta t = t_B - t_A$

However, to clock ②, one has to use two train clocks synchronized ($\bar{A}$ & $\bar{B}$) and passes one ground clock ($\bar{C}$) to get reading of $\bar{A}$ & $\bar{B}$, $t_{\bar{A}}$, $t_{\bar{B}}$, $\Delta t = t_{\bar{B}} - t_{\bar{A}}$



Two measurements are different so there is no contraction. (not the same measurement but different conclusion!)

Note that clocks that are synchronized in one frame will not be synchronized in other frame. This is the relativity of simultaneity.

Hence moving clocks are not synchronized even if they are synchronized in their rest frame.

Therefore, it is essential that in the measurement, one compares clocks in the frame with only one clock in the moving frame. As a result, measurements ① & ② are different measurement.

No.
Date. 12-17

The twin paradox

If both statements in the above are correct, there would seem to be a paradox: twin brother: A & B

A takes off in a rocket ship to a far star and comes back to earth.

B remains rest on earth.

Who is older when A comes back?

It may seem that there is a paradox as B sees "A is moving", ∴ A's clock runs slow A is younger.

as A sees "B is moving", ∴ B's clock runs slow. B is younger.

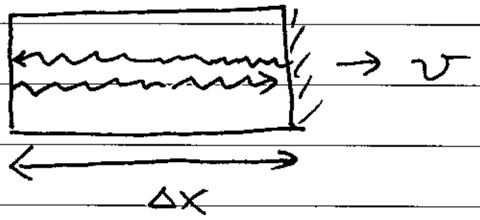
But actually, there is asymmetry in A & B: only A experiences acceleration! Two twins are not equivalent!

In fact, "A is younger" is correct as A experiences acceleration and is not in an inertial frame. Details will be investigated later (Problem 12-16).

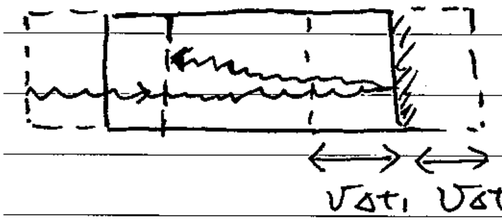
(iii) Lorentz Contraction

The length when properly measured also changes as a consequence of universal speed of light.

Consider that we have a lamp at one end & a mirror at the other end inside a car that is moving on a plane. (see below)



For the observer inside the car, the during Δt that light hits the mirror and reflection back $= 2\frac{\Delta x}{c}$ --- ①



For the observer of the ground, the during for light hitting the mirror is $\Delta t_1 = \frac{\Delta x + v\Delta t_1}{c}$.

The during for the light reflection from the mirror to the back end $\Delta t_2 = \frac{\Delta x - v\Delta t_2}{c}$

Hence

$$\Delta t_1 = \frac{\Delta x}{c-v}, \quad \Delta t_2 = \frac{\Delta x}{c+v}$$

$$\therefore \Delta t = \Delta t_1 + \Delta t_2 = 2\frac{\Delta x}{c}\frac{1}{1-v^2/c^2} \quad \cdots ⑥$$

Now the time dilation implies

$$\Delta \bar{t} = \sqrt{1-v^2/c^2}\,\Delta t \quad \cdots ⑦$$

$\therefore$ ⑤, ⑥ & ⑦ imply if $\Delta \bar{t} = \frac{2\Delta \bar{x}}{c}$

$$\Delta \bar{x} = \frac{\Delta x}{\sqrt{1-v^2/c^2}} \quad \cdots ⑧$$

The length measured by ground is shorter.

Hence moving objects are shortened by γ! This is called the Lorentz contraction.

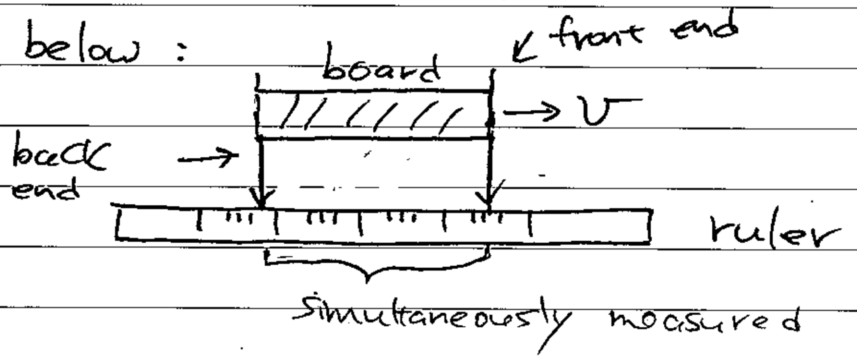
Relativity & the Lorentz contraction

Similar to the time dilation, there may seem to be an inconsistency of the Lorentz contraction with the principle of relativity

: ① ground observer : moving length in train is shorter
② train " " length of ground " "

Both are correct!

To see it, one first notes that to measure the length of a moving object, one has to do readings simultaneously as shown in below :

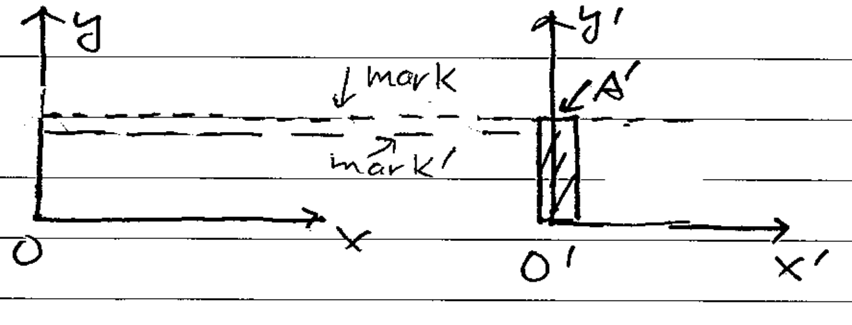


In the ruler's frame, measurements at front & back ends are done simultaneously. However, in the frame of the moving board, it's no longer simultaneous. The observer sees that front end is first measured and then the back end is measured. That is why the obtained length by ruler is shorter. The observer in the board thinks that the actual length is longer and the ruler is undersized (not long enough) so that the measurement of the back end is done later. Hence he found the ruler is shorter too! Both are thus correct!

Dimensions perpendicular to the velocity are not contracted

In deriving the time dilation, we assume that the height in perpendicular to the velocity is the same in both frames

To see why this is so, consider two frames O & O'. Before the two frames are set to more with relative velocity along x-axis, mark O frame the height y_0 and put a ruler of the same height along y-axis in O' frame as shown in below.



At $t=t'=0$, O & O' coincides

$t>0$, $O'=vt$

Since one end of the ruler is stick to O', one only needs to measure A'. This can be done, for example, by putting a marker in A' to introduce another line of marks when the two frames move relative to each other.

For the same event, if the observer in O finds marks with mark'<mark as O' is moving, the observer in O' seeing O frame is moving so he would conclude mark < mark'.

As this is the same event measured by O & O', the principle of relativity implies that both observers should find the same result. This is possible only when mark = mark'

Hence perpendicular dimensions are not contracted.

The Lorentz transformations

In the most general situation, one needs to find the difference of different many frames to describe the same event.

No.
Date. 12-23

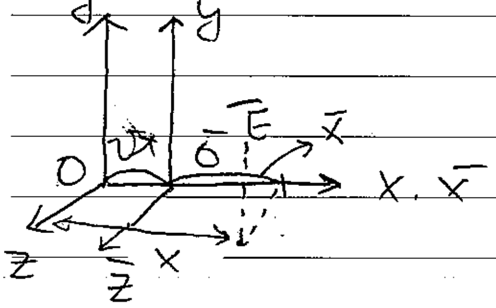
An event is something that takes place at a specific location (x, y, z) at a precise time t.

Hence $\underline{(x, y, z, t)}$ describes an event, say E, in one inertial frame S.

In another inertial frame, E is described by $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$.

What is the relation between (x, y, z, t) and $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$?

To describe the relation, one first adjust time so that at $t=0$ & $\bar{t}=0$, the origins O & $\bar{O}$ of two frames coincide with x & $\bar{x}$, y & $\bar{y}$ and z & $\bar{z}$ axes coincide as shown in the left figure.



Consider an event E in $\bar{O}$ with $\bar{x}$ coordinate relative to $\bar{O}$.

In Gailean transformation, one would conclude the coorinate x observed in O is $x = vt + \bar{x}$, i.e. $x - vt = \bar{x}$.

This, however, is not correct due to the Lorentz contraction.

If the observed length of $\bar{x}$ in O is d,

$$x = vt + d, \quad \therefore d = x - vt.$$

The Lorentz contraction implies $d = \frac{1}{\gamma}\bar{x}$

$$\gamma = \sqrt{1 - \frac{v^2}{c^2}}$$

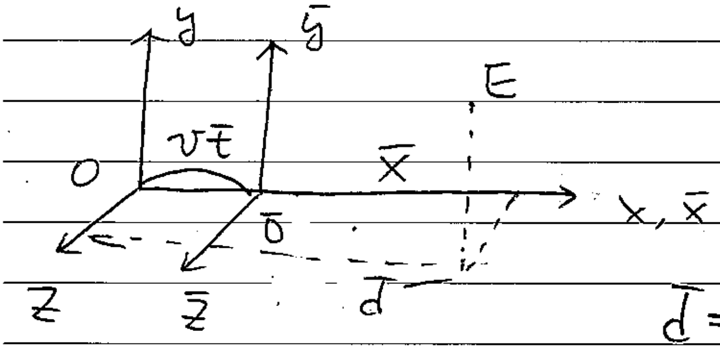
$$\therefore \quad \bar{x} = \gamma(x - vt) \quad \text{---} ⑨$$

On the other hand, for perpendicular directions, we have

$$\bar{y} = y \quad \text{--} ⑩$$

$$\bar{z} = z \quad \text{--} ⑪$$

We can ask the same thing for the same event E in O with x coordinate relative to O, which is observed as $\bar{d}$ in $\bar{O}$



$$\therefore \quad \bar{d} - v\bar{t} = \bar{x}$$

$$\bar{d} = \bar{x} + v\bar{t}$$

The Lorentz contraction implies $\bar{d} = \frac{1}{\gamma}x$

$$\therefore \quad x = \gamma(\bar{x} + v\bar{t}) \quad \text{---} ⑫$$

Which can be also obeaned by setting

$v \to -v$ and exchange x & $\bar{x}$ in eq. ⑨

Eliminating $\bar{x}$ by combing eqs. Ⓟ and ⑫,

one gets $x = \gamma\left[\gamma(x - vt) + v\bar{t}\right]$

$$\therefore \underbrace{(1-\gamma^2)}_{-\frac{v^2}{c^2}\gamma^2} x + \gamma^2 v t = \gamma v \bar{t}$$

$$\therefore \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right)$$

Therefore, we get the relations between (x, y, z, t) & $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$:

$$\begin{array}{lcl} \bar{x} = \gamma(x - vt) & & x = \gamma(\bar{x} + Ut) \\ \bar{y} = y & \Longleftrightarrow & y = \bar{y} \\ \bar{z} = z & & z = \bar{z} \\ \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right) & & t = \gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right) \end{array} \quad \leftarrow \text{-- ⑬}$$

Which is the famous Lorentz transformation.

Example : Using the Lorentz transformation, one can check

(i) Simultaneity : two events $A : (x_A = 0, t_A = 0)$
relativity of
$B : (x_B = b, t_B = 0)$

occur simultaneously in S frame.

In $\bar{S}$ frame,

A: $\bar{x}_A = \gamma(x_A - vt_A) = 0$

$\bar{t}_A = \gamma(t_A - \frac{v}{c^2}x_A) = 0$

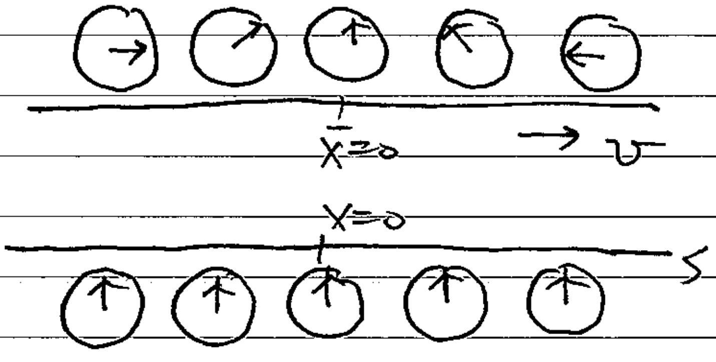
B: $\bar{x}_B = \gamma(x_B - vt_B) = \gamma b$

$\bar{t}_B = \gamma(t_B - \frac{v}{c^2}x_B) = -\frac{v}{c^2}\gamma b$

$\bar{t}_A \neq \bar{t}_B$, they are thus not simultaneous in $\bar{S}$ frame. B occurs earlier. The relativity of simultaneity follows from the Lorentz transformation.

In fact, if at $t=0$, S frame checks all clocks in $\bar{S}$, he finds that all clocks in $\bar{S}$ read differently:

$$\bar{t} = \gamma(0 - \frac{v}{c^2}x) = -\frac{v}{c^2}\gamma x$$



x>0, reading earlier (delay), x<0, ahead of time.

Similarly, from $\bar{S}$ frame, $\bar{S}$ would find the same thing

(ii) time dilation

Consider two events occuring in $\bar{S}$ at the same location $\bar{x}=a$ occuring at $\bar{t}_1$ & $\bar{t}_2$.

Then in S frame, one has

$$t_1 = \gamma\left(\bar{t}_1 + \frac{v}{c^2}a\right)$$

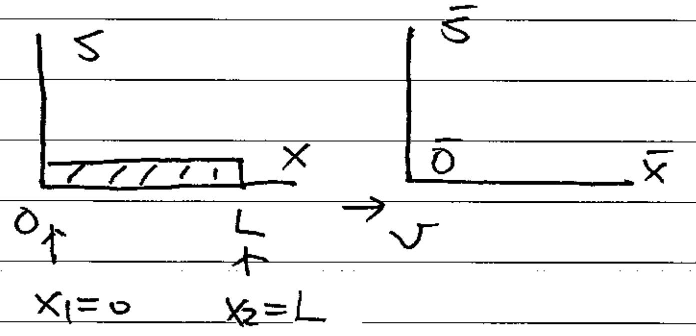
$$t_2 = \gamma\left(\bar{t}_2 + \frac{v}{c^2}a\right)$$

There fore $\Delta t = t_2 - t_1 = \gamma(\bar{t}_2 - \bar{t}_1) = \gamma \Delta \bar{t}$

$\Delta \bar{t} = \frac{1}{\gamma}\Delta t$. Clocks run slow in $\bar{S}$ as viewed from S.

(iii) Lorentz Contraction.

Consider a ruler at rest in S frame with length L.



In the $\bar{S}$ frame, two ends of ruler are described by

$$\bar{x}_1 = \gamma(x_1 - v t_1)$$
$$\bar{x}_2 = \gamma(x_2 - v t_2) \quad \cdots (14)$$

Where t_1 & t_2 are time of measurements done in $\bar{S}$ frame observed in S frame, The time of measurements in $\bar{S}$ frame are related to t_1 & t_2 by

$$\bar{t}_1 = \gamma\left(t_1 - \frac{v}{c^2}x_1\right)$$
$$\bar{t}_2 = \gamma\left(t_2 - \frac{v}{c^2}x_2\right)$$

To gaurantee that the measurements are simultaneous in $\bar{S}$ frame, one requires

$$\bar{t}_1 = \bar{t}_2 \text{ , hence } \gamma\left(t_1 - \frac{v}{c^2}x_1\right) = \gamma\left(t_2 - \frac{v}{c^2}x_2\right)$$

$$\therefore\ t_2 - t_1 = \frac{v}{c^2}(x_2 - x_1) \quad \text{-- ⑮}$$

From eq. ⑭, One obtains

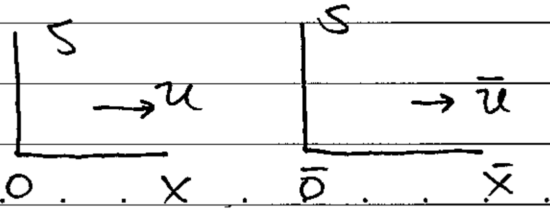
$$\bar{x}_2 - \bar{x}_1 = \gamma\left(x_2 - x_1 - v(t_2 - t_1)\right)$$
$$= \gamma\left[(x_2 - x_1) - \frac{v^2}{c^2}(x_2 - x_1)\right]$$

eq. ⑮

$$= \sqrt{1 - v^2/c^2}\,(x_2 - x_1)$$

Hence the length $\bar{L} = \sqrt{1 - v^2/c^2}\,L$ is contracted in $\bar{S}$ frame.

Velocity addition rule



Suppose that a particle moves in S

with velocity $u = \frac{dx}{dt}$

From $\bar{x} = \gamma(x - vt)$, during dt, the

change of $\bar{x}$ in $\bar{S}$ frame

$$d\bar{x} = \gamma(dx - v\,dt) \quad \text{---} ⑯$$

which occurs in the period $d\bar{t}$.

From $\bar{t} = \gamma(t - \frac{v}{c^2}x)$, one gets

$$d\bar{t} = \gamma(dt - \frac{v}{c^2}dx) \quad \text{--} ⑰$$

Hence eq ⑯ & ⑰ imply

$$\bar{u} = \frac{d\bar{x}}{d\bar{t}} = \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}}$$

$$= \frac{u - v}{1 - \frac{1}{c^2}uv} \quad \text{L} \cdot\cdot ⑰ - 1$$

This is the Einstein velocity addition rule. which can be put in a more transparent form by setting A = particle, B = S, C = $\bar{S}$

$$v = v_{CB} = -v_{BC}$$

$$\therefore v_{AC} = \frac{v_{AB} + v_{BC}}{1 + \frac{v_{AB}v_{BC}}{c^2}} \quad \cdots ⑱$$

Structure of space time

The Lorantz transformation characterizes the structure of space and time. It can be viewed as a rotation in space and time.

(i) Four-vectors

To reveal the rotational property of the Lorentz transformation, we first note that:

$$\because \bar{x} = \gamma(x - vt)$$

$$\therefore \bar{x}^2 = \frac{1}{1-v^2/c^2}(x^2 - 2xvt + v^2t^2)$$

$$\bar{t} = \gamma\left(t - \frac{v}{c^2}x\right)$$

$$\therefore c^2\bar{t}^2 = \frac{1}{1-v^2/c^2}\left(c^2t^2 - 2xvt + \frac{v^2}{c^2}x^2\right)$$

$$\therefore \bar{x}^2 - c^2\bar{t}^2 = \frac{1}{1-v^2/c^2}\left(x^2 - c^2t^2 - \frac{v^2}{c^2}(x^2 - c^2t^2)\right)$$

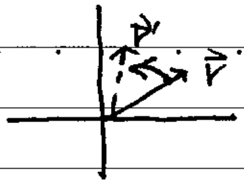
$$= x^2 - c^2t^2$$

$$\bar{r}^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2, \quad r^2 = x^2 + y^2 + z^2$$

$$\therefore \bar{r}^2 - c^2\bar{t}^2 = r^2 - c^2t^2 \cdots \text{(1P)}$$

i.e. $r^2 - c^2t^2$ is invariant under Lorantz transformation. This is similar to the rotation.

Under rotation : $\underset{\substack{\uparrow \\ \text{after rotation}}}{\vec{r}'^{\,2}} = \vec{r}^{\,2}$



Rotation is most convenient described by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underset{\substack{\uparrow \\ \text{rotational matrix.}}}{R} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Similarly, one tries to rewrite the Lorentz transformation in the following way.

First, we define $x^0 = ct$, $x^1 = x$, $x^2 = y$, & $x^3 = z$, so that x^i are all in unit of length.

Let $\beta = v/c$. Eq⑬ become

$$\begin{aligned} \bar{x}^0 &= \gamma(x^0 - \beta x^1) \\ \bar{x}^1 &= \gamma(x^1 - \beta x^0) \\ \bar{x}^2 &= x^2 \\ \bar{x}^3 &= x^3 \end{aligned} \quad \cdots ⑳$$

In the matrix form

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\Lambda} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad \cdots ㉑$$

$\therefore$ Lorentz transformation is characterized by the "rotational" matrix Λ that only depends on $\underline{v}$!

More concisely, one rewrites eq. ㉑ by

$$\bar{X}^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} X^{\nu} \quad \cdots ㉒$$

Eq. ㉑ is for the case when $v \parallel x$.

For general direction of $\vec{v}$, Λ takes different form but eq. ㉒ is still valid.

Any event is now described by

$$\mathbb{X} = (X^0, X^1, X^2, X^3) = X^{\mu}, \quad \mu = 0, 1, 2, 3$$

is called $\underline{\text{a 4-vector}}$ as its transformation to other frame is described by eq. ㉒.

Eq. ⑫ implies "the length" of $\mathbb{X} = \left[-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2\right]^{1/2}$ is invariant.

It suggest that the inner product in space & time is $\mathbb{X}\cdot\mathbb{X} = -(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 \quad \cdots ㉓$

In general, any vector of 4 components $\mathbb{A} = (A^0, A^1, A^2, A^3) = A^{\mu}$

that transforms in the same way as

that eq. (22) : $\bar{A}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_{\ \nu} A^\nu$.

A is a four-vector.

$$\therefore \quad \bar{A}^0 = \gamma (A^0 - \beta A^1)$$
$$\bar{A}^1 = \gamma (A^1 - \beta A^0)$$
$$\bar{A}^2 = A^2$$
$$\bar{A}^3 = A^3$$

are the componets of A in another frame.

Clearly, the inner product of any two 4-vectors is invariant under the Lorentz transformation

$$A \cdot B = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3 \quad \cdots ㉔$$

Eq. ㉔ is call 4-dimensional scalar product.

Summation convention

To incoorparate "−" in the scalar product, one defines the covariant vector A_μ

such that $A_\mu = (A_0, A_1, A_2, A_3)$
$= (-A^0, A^1, A^2, A^3)$

The original vector A^μ is called

contravariant vector.

A^μ & A_μ are connected by the

Minkowski metric

$$\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\overset{|||}{\overleftrightarrow{g}} = \text{metric tensor}} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$$

or $A_\mu = \sum_{\nu=0}^{3} g_{\mu\nu} A^\nu$ -- ㉕

· A_μ transform with $(-v)$ under the Lorentz transform (see over)

g serves to lower down the index of A^ν!

The inner product of $\mathbb{A}\cdot\mathbb{B}$ can be

cast into the familiar form:

$$\mathbb{A}\cdot\mathbb{B} = \sum_{\mu=0}^{3} A^\mu B_\mu \quad (= -A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3)$$

or simply $\mathbb{A}\cdot\mathbb{B} = A^\mu B_\mu$

where repeated indices imply that a summation is

taken. (Known as Einstein summation convention)

Doppler effect & 4-vector

There are many 4-vectors. Here we shall illustrate one example.

EM waves are characterized by sinusoidal

$$\text{phase} = Kx - \omega t \quad \cdots \; ㉖$$

Which counts number of wavelengths.

# of wavelengths can not be changed when going into other frames

$$\therefore \quad Kx - \omega t = \text{invariant} \quad \cdots \; ㉗$$

Clearly, eq. ㉗ implies $\left(\frac{\omega}{c}, \underset{K^1}{\underset{\shortparallel}{K_x}}, \underset{K^2}{\underset{\shortparallel}{K_y}}, \underset{K^3}{\underset{\shortparallel}{K_z}}\right)$ -- ㉘

is also a 4-vector as ㉗ can be written as $-\frac{\omega}{c}(ct) + K^1x^1 + K^2x^2 + K^3x^3$

$= \text{invariant}$.

The 4-vector of $\left(\frac{\omega}{c}, K_x, K_y, K_z\right)$ implies in $\bar{S}$ frame

$$\bar{K}_x = \gamma\left(K_x - \frac{v}{c}\frac{\omega}{c}\right)$$
$$\bar{K}_y = K_y$$
$$\bar{K}_z = K_z$$
$$\frac{\bar{\omega}}{c} = \gamma\left(\frac{\omega}{c} - \frac{v}{c}K_x\right)$$

i.e. $\bar{k}_x = \gamma\left(k_x - \frac{v}{c^2}\omega\right)$

$\bar{k}_y = k_y$

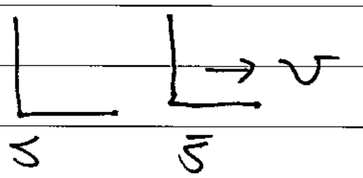
$\bar{k}_z = k_z$

$\bar{\omega} = \gamma\left(\omega - v k_x\right)$ ... ㉙

Eqs. ㉙ are the transformation rules of angular frequency ω & wavevector $\vec{k}$.

They mclude the Doppler effect as follows:

For light, $\frac{\bar{\omega}}{\bar{k}} = c$, if it is travelling in x direction, $k_y = k_z = 0$

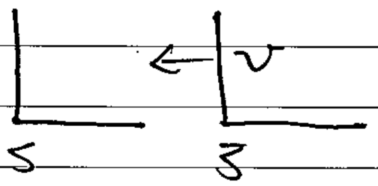


$\therefore \quad \omega = \gamma\bar{\omega}\left(1 + v\frac{\bar{k}}{\bar{\omega}}\right)$

$= \gamma\bar{\omega}\left(1 + \frac{v}{c}\right)$ ($\bar{S}$ moving away from S)

or $\omega = \gamma\bar{\omega}\left(1 - v\bar{k}/\bar{\omega}\right)$

$= \gamma\bar{\omega}\left(1 - v/c\right)$



(approaching to S)

$\therefore \quad \omega = \sqrt{\frac{1\pm\beta}{1\mp\beta}}\,\bar{\omega}$

+ : c & v approaching
− : c & v moving away

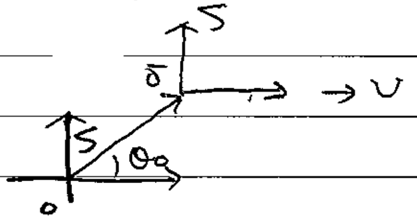
In the non-relativistic limit, $\beta \ll 1$

$\omega \doteqdot \sqrt{1\pm2\beta}\,\bar{\omega} \approx (1\pm\beta)\bar{\omega}$ in agree with the non-relativistic Doppler effect.

Transverse Doppler effect

The difference between non-relativistic & relativistic Doppler can be best exhibited in the transverse Doppler effect.

There are two ways for transverse Doppler effect:



Suppose o = source

$\bar{o}$ = observer

For light on $\bar{S}$, its propagation angle is $\theta_{\bar{o}}$ in S frame.

$$\therefore \quad \omega = \gamma(\bar{\omega} + v\bar{k}_x)$$

$$\bar{k}_x = \frac{\bar{\omega}}{c}\cos\theta_o \quad \left(\frac{\bar{\omega}}{c} = \bar{k}\right)$$

$$\therefore \quad \omega = \gamma\bar{\omega}\,(1+\beta\cos\theta_{\bar{o}})$$

Where $\bar{\omega}$ = frequency measured by observer.

ω = " " " source

(i) ∴ If the light is perpendicular to source,

$\theta_{\bar{o}} = \pi/2$, $\quad \omega = \gamma\bar{\omega}$, $\quad \bar{\omega} = \sqrt{1-\beta^2}\,\omega$

i.e. ν_o (observer's frequency) $= \sqrt{1-\beta^2}\,\nu_s$ (source)

(ii) From the point view of $\bar{o}$,

$$\bar{\omega} = \gamma(\omega - vk_x). \qquad k_x = \frac{\omega}{c}\cos\theta_s$$

θ_o = angle measured in $\bar{S}$ frame. $\quad \therefore \bar{\omega} = \gamma\omega((1+\beta\cos\theta_s)$

$\theta_o = \pi/2$, one gets $\bar{\omega} = \gamma\omega$, $\therefore \nu_o = \dfrac{\nu_s}{\sqrt{1-\beta^2}}$

which is for the case when light is perpendicular to the observer.

Charge conservation & $(c\rho, \vec{J})$

Another example of 4-vector is based on the invariance of charge q.

Experimentally, it's found that electric charge is invariant, i.e, the same in all frames.

Therefore, in all frame, one has

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad \cdots ㉚$$

Where ρ = charge density at (ct, x, y, z)

and $\vec{J}$ = current density at (ct, x, y, z).

Now, in $\bar{S}$ frame, the operator

$$\frac{\partial}{\partial \bar{x}} = \frac{\partial x}{\partial \bar{x}}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \bar{x}}\frac{\partial}{\partial t}$$

$$\underset{\uparrow}{=} \gamma\frac{\partial}{\partial x} + \gamma\frac{v}{c^2}\frac{\partial}{\partial t} = \gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)$$

$$x = \gamma(\bar{x} + v\bar{t})$$
$$t = \gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right)$$

$$\frac{\partial}{\partial \bar{y}} = \frac{\partial y}{\partial \bar{y}}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial z} \quad \cdots ㉛$$

$$\frac{\partial}{\partial \bar{t}} = \frac{\partial x}{\partial \bar{t}}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \bar{t}}\frac{\partial}{\partial t}$$

$$= \gamma v\left(\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial t}\right)$$

$$= \gamma\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)$$

Clearly, eqs ㉛ imply $\left(\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ transforms as a covariant vector.

$\therefore \frac{\partial}{\partial x^{\mu}} = \partial_{\mu}$ (lower index)

It must combine with a contravariant vector to become an invariant, Therefore, one can write eq ㉚ as an invariant: $\partial_{\mu} J^{\mu} = 0$ --- ㉜

With

$$\partial_{\mu} J^{\mu} = \frac{1}{c}\frac{\partial J^0}{\partial t} + \frac{\partial J^1}{\partial x} + \frac{\partial J^2}{\partial y} + \frac{\partial J^3}{\partial z}$$

$\therefore J^0 = c\rho,\ J^1 = J^x,\ J^2 = J^y,\ J^3 = J^z$

$J^{\mu} = (c\rho, \vec{J})$ is a 4-vector. --- ㉝

Physically, $\because \vec{J} = \rho \cdot \vec{u}$, due to the Lorentz contraction,

$\rho = \frac{Q}{V} = \frac{Q}{\Delta x \Delta y \Delta z} = \frac{Q}{\Delta x_0 \Delta y \Delta z}\frac{1}{\sqrt{1-\beta^2}} = \frac{\rho_0}{\sqrt{1-\beta^2}}$ (ρ_0 = charge density in rest frame)

$\vec{J} = \frac{\rho_0 \vec{u}}{\sqrt{1-\beta^2}}$

(ii) Invariant intervals

The scalar product of a 4-vector with itself can be either positive, zero or negative.

$$A^\mu A_\mu = -(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2$$

if $A^\mu A_\mu > 0$, A^μ is called space like

$A^\mu A_\mu < 0$, " .. time like

$A^\mu A_\mu = 0$, - - - light like

∵ $A^\mu A_\mu$ is invariant. The above classification is also invariant: a spacelike 4-vector can not be turned into a time-like 4-vector under Lorentz transformation.

For any two events A & B: $(X_A^0, X_A^1, X_A^2, X_A^3)$
$(X_B^0, X_B^1, X_B^2, X_B^3)$,

∵ X_A & X_B are 4-vectors, their difference $\Delta X = X_A - X_B$ is also a 4-vector

• Hence $$\underset{\substack{\parallel \\ (S_{AB})^2}}{\Delta X^\mu \Delta X_\mu} = -(\Delta X^0)^2 + (\Delta X^1)^2 + (\Delta X^2)^2 + (\Delta X^3)^2$$

is invariant and is called invariant interval

The invariance of $(S_{AB})^2$ implies relations of any two events can be classified into

① time-like

$S_{AB}^2 < 0$. For this case, it's possible to find a frame in which two events occur at the same location but different times: $(\Delta\tilde{x}^1)^2 + (\Delta\tilde{x}^2)^2 + (\Delta\tilde{x}^3)^2 = 0$

$$(\tilde{S}_{AB})^2 = (S_{AB})^2 = -(\Delta\tilde{x}^0)^2 < 0$$

$\Delta\tilde{x}^0 = c\Delta\tilde{t}$, $\Delta\tilde{t}$ is called the proper time.

② space-like

$S_{AB}^2 > 0$. One con find a frame in which $\Delta\tilde{t} = 0$. Two events occur simultaneously but at different locations.

$$\therefore (\tilde{S}_{AB})^2 = (S_{AB})^2 = (\Delta\tilde{x}_1)^2 + (\Delta\tilde{x}_2)^2 + (\Delta\tilde{x}_3)^2 > 0$$

is the distance between two events

③. light-like

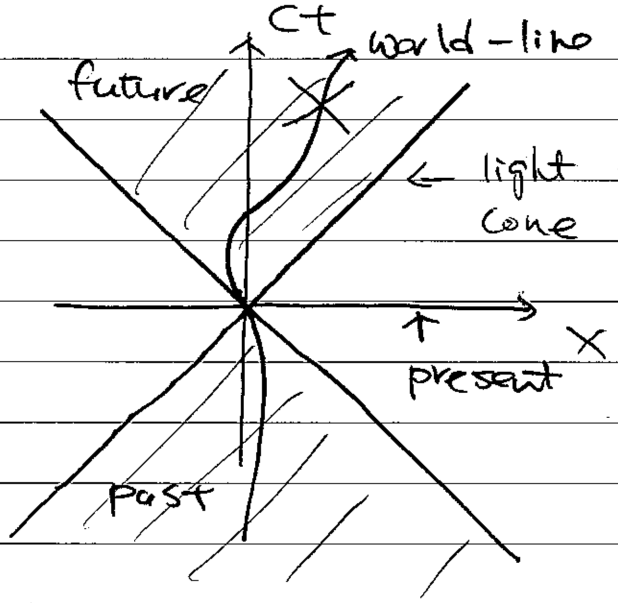
$S_{AB} = 0$. $\Delta r = c\Delta t$

which are events occurring to light.

(iii) space-time diagram

To present the motion of a particle, one usually plot the position versus time.

For this purpose, one use $x^0 = ct$ as the vertical axis, and $x^1 = x$ as the transverse axis to present the motion of the particle. This is known as the Minkowski diagram. In this diagram, the trajectory of a particle is called a world line, ∵ light travels at speed of c. Lines of slope ± 1 are light cones that are trajectories of light rays.



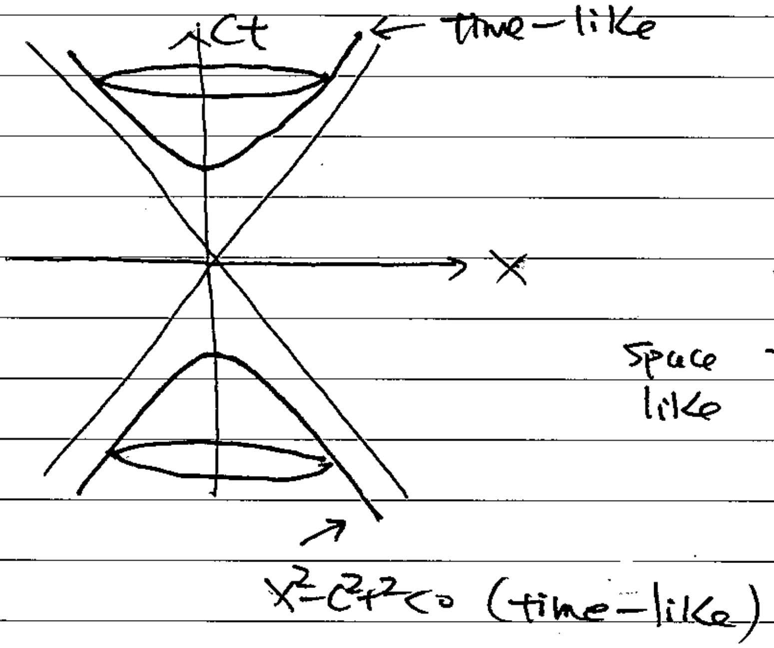
As shown in the above, a particle starts to move at the origin at $t=0$, traces out a world line. Since the speed of particle is less than c, the world line can only be inside the light cone. (slope > 1)

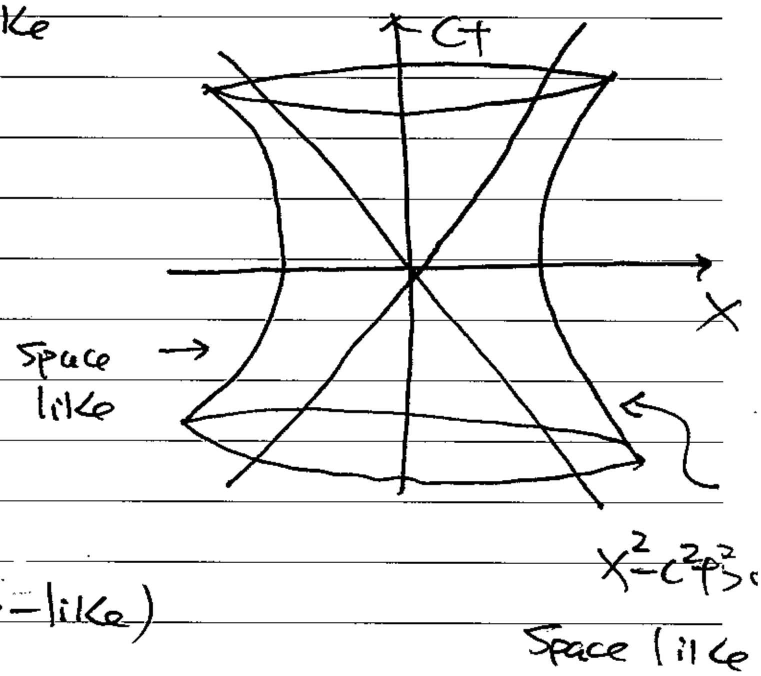
Causality For an event (x,t) which is time-like to the event at $x=0$, $t=0$, one has.

$x^2-c^2t^2<0$ which is a hyperboloid.

While for the event (x,t) to be space like, $x^2-c^2t^2>0$

The trajectories are shown in below





It's clear from the above figures that for time-like events, it stays on one of hyperboloid branch. Therefore, either it is in the future $(t>0)$ or in the past. The ordering can't be changed under Lorentz transformation, It is thus consistent with the notion of causality: it's not possible to reverse the causality relation : A causes B

This is because the influence of A on B can't propagate faster than c.

Therefore, the displacement between causality related events is always timelike

On the other hand, for space like events, there is no definite ordering of time.

It can happen in the future $(t>0)$ and becomes in the past $(t<0)$ in another frame.

This is because spacelike events are not causality related and there is no binding that there is no requirement which one must precede the other.

Relativistic Mechanics

proper time and proper velocity

On the world line of a particle, the interval Δs

is time-like, so one can write $(\Delta s)^2 = -c^2(\Delta \tau)^2$

τ is the proper time which is the time measured in the rest frame of the particle

For other frame, $(\Delta s)^2 = c^2 \Delta t^2 - (d\vec{r})^2$

$$\therefore (\delta \tau)^2 = (\Delta t)^2 \left[1 - \frac{(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2}{c^2} \right]$$

$$= (\Delta t)^2 \sqrt{1 - u^2/c^2}$$ u = velocity of the particle in the frame.

$\therefore \Delta t = \dfrac{\Delta \tau}{\sqrt{1 - u^2/c^2}}$ which reflects the time dilation.

Clearly, the proper time is an quantity defined in the rest frame of the particle and hence it is not changed as the frame is changed. (invariant)

Therefore, $X = (x^0, x^1, x^2, x^3)$ is 4-vector, so is $\frac{dX}{d\tau}$. known as the 4-velocity We shall denote

$$\eta \equiv \frac{dX}{d\tau} \quad \text{as the 4-velocity} \qquad \text{--- (34)}$$

∵ $\frac{d\tau}{dt} = \sqrt{1-u^2/c^2}$

∴ $\vec{\eta} = \frac{d\vec{x}}{dt}\frac{1}{\sqrt{1-u^2/c^2}} = \frac{\vec{u}}{\sqrt{1-u^2/c^2}}$

with $\eta^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = \frac{c}{\sqrt{1-u^2/c^2}}$

$\eta^i = \frac{dx^i}{d\tau} = \frac{u^i}{\sqrt{1-u^2/c^2}}$

∴ $\vec{\eta}\cdot\vec{\eta} = -(\eta^0)^2+(\eta^1)^2+(\eta^2)^2+(\eta^3)^2$

$= -c^2$

The 4-velocity transforms exactly the same as (ct, x^1, x^2, x^3)

∴ $\bar{\eta}^0 = \gamma(\eta^0 - \beta\eta^1)$

$\bar{\eta}^2 = \eta^2$

$\bar{\eta}^3 = \eta^3$

$\bar{\eta}^1 = \gamma(\eta^1 - \beta\eta^0)$ --- ㉟

While the real velocity of the particle transforms in more complicated way:

$\bar{u}_x = \frac{d\bar{x}}{d\bar{t}} = \frac{u_x - v}{1 - vu_x/c^2}$ (velocity addition rule)

$\bar{u}_y = \frac{d\bar{y}}{d\bar{t}} = \frac{u_y}{\gamma(1 - vu_x/c^2)}$ (see eq. ⑲, ⑲-1)

$\bar{u}_z = \frac{d\bar{z}}{d\bar{t}} = \frac{u_z}{\gamma(1 - vu_x/c^2)}$ --- ㊱

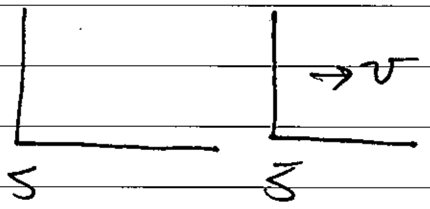
Relativistic energy and momentum

In Newtonian mechanics, the momentum

$$\vec{p} = m_0\vec{u}, \quad \text{with mass} = m_0$$

being an invariant quantity (the same in all frames). When generalized to relativity, this is no longer true.

The main reason is that if the momentum is conserved in one frame, it will not be conserved in other frames:



$$m_A^0 \bar{u}_A + m_B^0 \bar{u}_B = m_C^0 \bar{u}_C + m_D^0 \bar{u}_D \quad \text{in } \bar{S} \text{ frame}$$

In S frame, $u_i = \dfrac{\bar{u}_i + v}{1 + \bar{u}_i v/c^2}$

$$m_A^0\left(\frac{\bar{u}_A + v}{1 + \bar{u}_A v/c^2}\right) + m_B^0\left(\frac{\bar{u}_B + v}{1 + \bar{u}_B v/c^2}\right)$$

$$= m_C^0\left(\frac{\bar{u}_C + v}{1 + \bar{u}_C v/c^2}\right) + m_D^0\left(\frac{\bar{u}_D + v}{1 + \bar{u}_D v/c^2}\right)$$

is generally not correct!

Hence it is necessary to generalize the relation $\vec{p} = m_0\vec{u}$.

A plausible candidate is the 4-velocity when being multiplied by the mass of the particle in its rest frame — i.e. the rest mass m_0: relativistic momentum ≡

$$\mathbb{P} = m_0 \vec{\eta} \quad \text{--} \ (37)$$

∵ m_0 = rest mass is invariant in all frames, $\vec{p}$ is a 4-vector all.

Therefore one has

$$\bar{p}^0 = \gamma(p^0 - \beta p^1)$$
$$\bar{p}^2 = p^2$$
$$\bar{p}^3 = p^3$$
$$\bar{p}^1 = \gamma(p^1 - \beta p^0) \quad \text{---} \ (38)$$

Explicitly. $\mathbb{P} = \left(\frac{m_0 c}{\sqrt{1-(\frac{u}{c})^2}}, \frac{m_0 \vec{u}}{\sqrt{1-(\frac{u}{c})^2}}\right)$ -- (39)

In the limit, $\frac{m_0\vec{u}}{\sqrt{1-(\frac{u}{c})^2}}$ reduces to $m_0\vec{u}$ hence it is a reasonable generalization.

If eq (39) is the correct expression for relativistic momentum, the force that acts on the particle is $\vec{F} = \frac{d}{dt}\left(\frac{m_0\vec{u}}{\sqrt{1-(u/c)^2}}\right)$ --- (40)

12-48

The work done by the force is

$$\int_0^x F\cdot dx = \int_0^t F v \, dt_1 \qquad F=\frac{dp}{dt}$$

$$= up - \int_0^u p \, du_1$$

$$= \frac{m_0 u^2}{\sqrt{1-(\frac{u}{c})^2}} - \int_0^u \frac{m_0 u_1 \, du_1}{\sqrt{1-(\frac{u_1}{c})^2}}$$

$$= \frac{m_0 u^2}{\sqrt{1-(\frac{u}{c})^2}} + m_0 c^2 \sqrt{1-(\frac{u_1}{c})^2}\,\Big|_0^u$$

$$= \frac{m_0 c^2}{\sqrt{1-(\frac{u}{c})^2}} - m_0 c^2$$

$\because \int_0^x F dx$ = change of energy of the particle $= E_K$

$$\therefore \; E_K = \frac{m_0 c^2}{\sqrt{1-(\frac{u}{c})^2}} - m_0 c^2$$

$$\underset{u\to 0}{\xrightarrow{=}} m_0 c^2 \left(1+\frac{u^2}{2c^2}\right) - m_0 c^2 \simeq \frac{m_0}{2} u^2$$

Clearly, from the above consideration, one identifies $\frac{m_0 c^2}{\sqrt{1-(\frac{u}{c})^2}}$ as the total energy of the particle E and $m_0 c^2$ = rest energy of the particle | Hence $\mathbb{P} = (\frac{E}{c}, \vec{p})$ is a 4-vector! ---- (41)

The experimental facts indicate that the above identified relativistic momentum & energy are conserved for closed systems. (mass is not conserved!) L-(42)

Therefore, for any reactions (or collisions)

$$1+2+3\cdots \rightarrow 1'+2'+3'+\cdots$$

relativistic momentum & energy are conserved and can be summarized as:

$$\mathbb{P}_1+\mathbb{P}_2+\mathbb{P}_3+\cdots=\mathbb{P}_1'+\mathbb{P}_2'+\mathbb{P}_3'+\cdots$$

L-(43)

On the other hand,

$$\mathbb{P}^2=-\left(\frac{E}{c}\right)^2+(\vec{p})^2$$

$$=-\left(\frac{m_0c}{\sqrt{1-u^2/c^2}}\right)^2+\left(\frac{m_0\vec{u}}{\sqrt{1-(\frac{u}{c})^2}}\right)^2$$

$$=\frac{m_0^2(u^2-c^2)}{1-u^2/c^2}=-m_0^2c^2$$ is invariant L-(43)-1

in all frames.

Hence $-\left(\sum_{i=1}^{N}\frac{E_i}{c}\right)^2+\left(\sum_{i}^{N}\vec{p}_i\right)^2$ is Lorentz invariant -- (44)

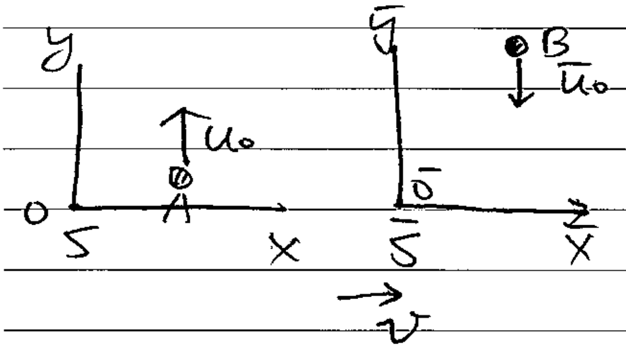
Example : Relativistic mass

The difference between relativistic momentum and $m_0\vec{u}$ is the mass :

$$m = \frac{m_0}{\sqrt{1-(\frac{u}{c})^2}}$$

Which is called relativistic mass and it indicates that the mass of the particle depends on its velocity and goes to ∞ as $u \to c$

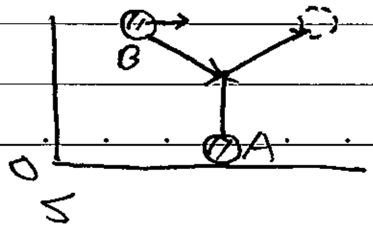
To see the necessity of relativistic mass, Consider collision between two identical ball of mass m (m S & $\bar{S}$ respectively) : A & B.



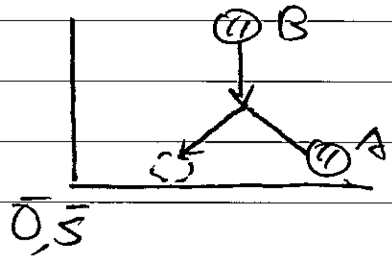
Relative to S, A is moving in +y direction with speed u_0 While relative to $\bar{S}$ B is moving in $-y$ direction with speed u_0

Assuming that there is no friction.

By symmetry, the collision in S frame is as shown in the right



(while for $\bar{S}$ frame, one observes



)

Clearly, in S frame, the change of momentum $\Delta P_A = -2mu_0$ ... ㊺

(m = mass of A in S)

While in S frame, $u_B^x = v$

$$u_B^y = \frac{\bar{u}_B^y}{\gamma\left(1+\frac{\bar{u}_B^x v}{c}\right)} \quad (\bar{u}_B^x = 0)$$

$$= \frac{-u_0}{\gamma} = -u_0\sqrt{1-v^2/c^2}$$

By symmetry, the change of momentum

$$\Delta P_B = \underbrace{2u_0\sqrt{1-v^2/c^2}}_{\Delta u_B} \cdot \underline{\text{mass of B in S}} \quad \text{...} ㊻$$

Now, if mass of B in S = m, clearly

$\Delta P_A + \Delta P_B \neq 0$

To make $\Delta P_A + \Delta P_B = 0$, mass of B in S $\neq m$!

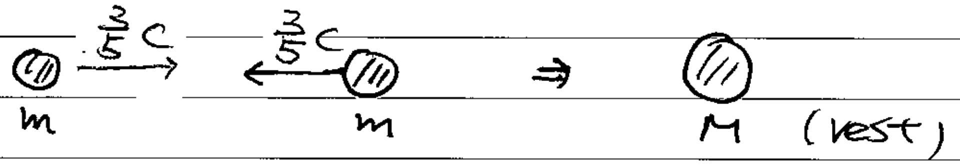
For small u_0, $m \simeq m_0$ $\therefore \Delta P_A = -2m_0u_0$

$\therefore \Delta P_A + \Delta P_B = 0$ then implies

m = mass of B in S $\Rightarrow 2m\sqrt{1-v^2/c^2} = 2m_0$

Hence $m = \frac{m_0}{\sqrt{1-(\frac{u}{c})^2}}$ is essential to keep the momentum conservation.

<u>Example</u> Rest mass is not conserved



find M

Solution :

energy for each $m = \frac{mc^2}{\sqrt{1-(\frac{3}{5})^2}} = \frac{5}{4}mc^2$

energy conservation : $Mc^2 = \frac{5}{4}mc^2 \times 2$

$\therefore M = \frac{5}{2}m > 2m$.

<u>Example</u> Threshold energy

Find the minimum Kinetic energy of 1 to start the following reaction

$$1 + 2\,(\text{target, rest})$$
$$\rightarrow 1' + 2' + 3' + 4' + \cdots$$

Solution : The conservation of momentum and energy implies

No.
Date.

12-53

$$\mathbb{P}_1+\mathbb{P}_2=\mathbb{P}_1'+\mathbb{P}_2'+\mathbb{P}_3'+\cdots$$

$$\therefore \left(\frac{E_1+E_2}{c}\right)^2-(\vec{p}_1+\vec{p}_2)^2$$

$$=\left(\sum_i\frac{E_i'}{c}\right)^2-\left(\sum_i\vec{p}_i'\right)^2 \cdots ㊼$$

Now, using that $\left(\sum_i\frac{E_i'}{c}\right)^2-\left(\sum_i\vec{p}_i'\right)^2$ is Lorentz invariant, one chooses the frame of center of momentum (COM) in which $\sum_i\hat{\vec{p}}_i=0$

$$\therefore \left(\sum_i\frac{E_i'}{c}\right)^2-\left(\sum_i\vec{p}_i'\right)^2=\left(\sum_i\frac{\hat{E}_i}{c}\right)^2$$ ($\hat{E}_i$ = energy in the frame of COM)

Clearly $\hat{E}_i\ge m_i'c^2$, m_i' = rest mass of ith particle

$$\therefore \left(\sum_i\frac{E_i'}{c}\right)^2-\left(\sum_i\vec{p}_i'\right)^2\ge\left(\sum_i\frac{m_i'c^2}{c}\right)^2=\left(\sum_i m_i'c\right)^2 \cdots ㊽$$

Combining eqs. ㊼ & ㊽, one gets

$$\left(\frac{E_1+E_2}{c}\right)^2-(\vec{p}_1+\vec{p}_2)^2\ge\left(\sum_i m_i'c\right)^2$$

Now, $\vec{p}_2=0$, $\vec{p}_1=\vec{p}$, $E_2=m_2c^2$. $\therefore \left(\frac{E_1+E_2}{c}\right)^2-(\vec{p}_1+\vec{p}_2)^2$

$=\left(\frac{E_1+m_2c^2}{c}\right)^2-p^2=\left(\frac{E_1}{c}\right)^2-p^2+(m_2c)^2+2E_1m_2=(m_1c)^2+(m_2c)^2$

$+2(m_1c^2+K_1)m_2$, where K_1 = kinetic energy of particle 1

$$\therefore 2K_1m_2\ge\left[\left(\sum_i m_i'\right)^2-(m_1+m_2)^2\right]c^2$$

$$K_1\ge\frac{1}{2m_2}\left[\left(\sum_i m_i'\right)^2-(m_1+m_2)^2\right]c^2$$

$$\therefore K_1 \geq \frac{1}{2m_2}\left(\sum_{i'} m_i' + m_1 + m_2\right)\left(\sum_{i'} m_i' - m_1 - m_2\right)c^2$$

let $Q = m_1 + m_2 - \sum_{i'} m_i'$

$M = m_1 + m_2 + \sum_{i} m_i'$ = total mass

$$\therefore K_1 \geq -\frac{Q}{2m_2} Mc^2$$

if $Q < 0$, K_1 (kinectic energy of incident particle) neads to be greator than $\underbrace{\frac{|Q|}{2m_2}Mc^2}$ to start the reaction. = threshold energy

On the other hand, if $Q > 0$, the reaction can happen spotoneausly.

Example: $p + p \rightarrow \bar{p} + p + p + p$
↑
target

$$p^2 - \left(\frac{E + mc^2}{c}\right)^2 \leq -(4m_pc)^2$$

$= \left(\frac{E}{c}\right)^2 - (m_pc)^2$ $\therefore 2Em_p + 2(m_pc)^2 \geq 16(m_pc)^2$

$E \geq 7m_pc^2$ $\quad E = K + m_pc^2$

$K \geq 6m_pc^2$

12-53

$$\mathbb{P}_1 + \mathbb{P}_2 = \mathbb{P}_1' + \mathbb{P}_2' + \mathbb{P}_3' + \cdots$$

$$\therefore \left(\frac{E_1+E_2}{c}\right)^2 - (\vec{P_1}+\vec{P_2})^2$$

$$= \left(\sum_{i=1} \frac{E_i'}{c}\right)^2 - \left(\sum_i \vec{P_i}'\right)^2$$

The conservation of momentum : $\left(\sum_i \vec{P_i}'\right)^2 = (\vec{P_1}+\vec{P_2})^2 = P^2$

$\vec{P}$ = momentum of particle I.

Massless particle

In classical mechanics, there is no massless particle : $m=0$ because its momentum $m\vec{u}$ and its kinetic energy would be zero, one can not apply a force $F=ma$ on the particle as a would be ∞ !

If one examines the relation

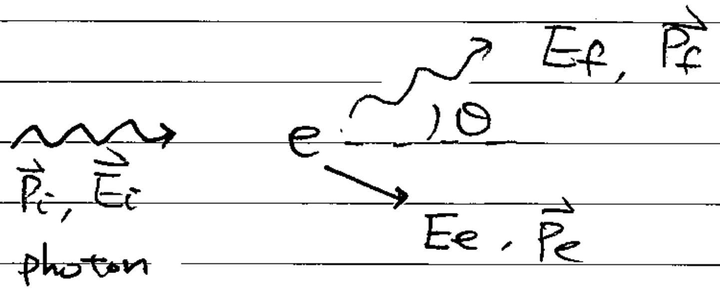
$$\vec{p} = \frac{m_0\vec{u}}{\sqrt{1-u^2/c^2}} \quad \& \quad E = \frac{m_0c^2}{\sqrt{1-u^2/c^2}} \quad \leftarrow ④P$$

, one would tend to conclude $\vec{p}$ & E also vanish when $m_0=0$. Hence there is no massless particle. However, this is no longer true if $u=c$. In this case, $\vec{p}$ & $\vec{E}$ appear to be indeterminate ($0/0$). Therefore, special relativity allows massless particles as long as their velocities are c. In this case, although eqs. ④P can't determine E & $\vec{p}$, Eq. ㊸-1 : $\vec{p}^2 - E^2/c^2 = -m_0^2c^2$ implies

$$E = pc \text{ for massless particles.} \quad -- ㊿$$

In particulars, photons (quantum of EM field) are known be massless with speed $=c$, It obeys. eq. ㊿.

Example. Compton effect



Conservation of 4-vector:

$$\left(\frac{E_i}{c}, \vec{P_i}\right) + \left(\frac{m_e c^2}{c}, 0\right) = \left(\frac{E_f}{c}, \vec{P_f}\right) + \left(\frac{E_e}{c}, \vec{P_e}\right)$$

$$\therefore \left(\frac{E_e}{c}, \vec{P_e}\right) = \left(\frac{E_f - E_i + m_e c^2}{c}, \vec{P_i} - \vec{P_f}\right)$$

$$P_e^2 - \left(\frac{E_e}{c}\right)^2 = (\vec{P_i} - \vec{P_f})^2 - \left(\frac{E_f - E_i + m_e c^2}{c}\right)^2$$

$$\Downarrow$$

$$-m_e^2 c^4 = -(E_f - E_i + m_e c^2)^2 + (\vec{P_i} - \vec{P_f})^2 c^2$$

$$= -E_f^2 + P_f^2 c^2 - E_i^2 + P_i^2 c^2 - m_e^2 c^4 \quad \cdots (51)$$

$$-2 m_e c^2 (E_f - E_i) + 2 E_f E_i - 2 P_i P_f c^2 \cos\theta$$

$\because$ For photons, $E = pc$, $\therefore -E_f^2 + P_f^2 c^2 = 0$. $-E_i^2 + P_i^2 c^2 \Rightarrow$

$\therefore$ Eq.(51) becomos

$$m_e c^2 (E_f - E_i) = E_f E_i (1 - \cos\theta)$$

$$\therefore \frac{1}{E_i} - \frac{1}{E_f} = \frac{1}{m_e c^2}(1 - \cos\theta)$$

In quantum mechanics, it is shown $E = h\nu = \frac{hc}{\lambda}$ for photons. Hence $\lambda_i - \lambda_f \equiv \Delta\lambda = \frac{h}{m_e c}(1 - \cos\theta)$

The scattering light's wave length changes. This is known as the Compton effect.

Relativistic dynamics

Force

The ordinary force in Newton's law $\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\frac{m_0\vec{u}}{\sqrt{1-u^2/c^2}}\right)$ --- (52)

is still valid in relativistic mechanics even though one can define "proper" force by

$$\mathbb{K} = \frac{d\mathbb{P}}{d\tau}$$

$$= \left(\frac{dp^0}{d\tau}, \vec{K}\right) \quad --- \quad (53)$$

with $\vec{K} = \frac{d\vec{p}}{d\tau}$.

Clearly, $\vec{K} = \frac{dt}{d\tau}\frac{d\vec{p}}{dt} = \frac{1}{\sqrt{1-u^2/c^2}}\vec{F}$ --- (54)

$$K^0 = \frac{dp^0}{d\tau} = \frac{1}{c}\frac{dE}{d\tau}.$$

$\vec{K}$ is related to $\vec{F}$. and $\mathbb{K}$ is known as Minkowski force. A natural question arises: which one ($\vec{F}$ or $\vec{K}$) is related to the force law? For EM fields, That is

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{u}\times\vec{B})$$

or

$$\frac{d\vec{p}}{d\tau} = q(\vec{E} + \vec{u}\times\vec{B})$$

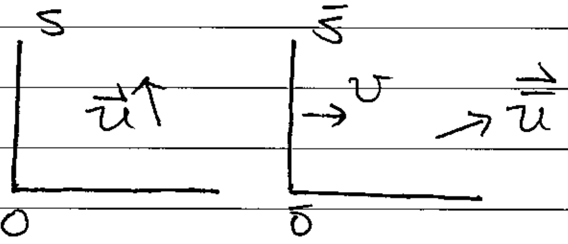
is the appropriate generalization to the regime of special relativity?

It turns out that the ordinary force

$$\vec{F} = \frac{d\vec{p}}{dt}$$ is consistent with experiments.

However, the Minkowski force transforms as a 4-vector and can help to formulate relativistic dynamics in a neater way.

Transformation of ordinary force $\vec{F}$



Consider a particle which moves with Velocity $\vec{u}$ & $\vec{\bar{u}}$ in S & $\bar{S}$ respectively.
Since $\mathbb{K}$ is a 4vector, we have

$$\bar{K}_x = \gamma(k_x - \beta K^0)$$
$$\bar{K}_y = K_y$$
$$\bar{K}_z = K_z$$
$$\bar{K}_0 = \gamma(K_0 - \beta K_x)$$ L - - (55)

where $\gamma = \frac{1}{\sqrt{1-(\frac{v}{c})^2}}$

$\beta = v/c$.

$$\because K^0 = \frac{1}{c}\frac{dE}{dt}\frac{dt}{d\tau} = \frac{1}{c}\vec{u}\cdot\vec{F}\frac{1}{\sqrt{1-(\frac{u}{c})^2}}$$

$$\vec{K} = \frac{1}{\sqrt{1-(\frac{u}{c})^2}}\vec{F}$$

$$\therefore \frac{1}{\sqrt{1-(\frac{\bar{u}}{c})^2}}\bar{F}_x = \frac{1}{\sqrt{1-(\frac{v}{c})^2}}\left[\frac{F_x}{\sqrt{1-(\frac{u}{c})^2}} - \frac{v}{c^2}\frac{1}{\sqrt{1-(\frac{u}{c})^2}}\vec{F}\cdot\vec{u}\right]$$ L - - (56)

Now, because the momentum transforms as

$$\bar{P}_x = \gamma (P_x - \frac{v}{c^2} E)$$

$$\therefore \frac{\cancel{m_0} \bar{u}_x}{\sqrt{1-(\frac{\bar{u}}{c})^2}} = \frac{1}{\sqrt{1-(\frac{v}{c})^2}} \left[\frac{\cancel{m_0} u_x}{\sqrt{1-(\frac{u}{c})^2}} - \frac{\cancel{m_0} v}{\sqrt{1-(\frac{u}{c})^2}} \right] \cdots (57)$$

Togeth with the velocity addition rule

$$\bar{u}_x = \frac{u_x - v}{1-\frac{u_x v}{c^2}} \quad \cdots (58)$$

eqs. (57) & (58) imply

$$\frac{1}{\sqrt{1-(\frac{\bar{u}}{c})^2}} = \frac{1-\frac{u_x v}{c^2}}{\sqrt{1-(\frac{v}{c})^2}\sqrt{1-(\frac{u}{c})^2}} \quad \cdots (59)$$

Combing eqs. (59) & (56), we obtain

$$\bar{F}_x = \frac{F_x - \frac{v}{c^2}\vec{F}\vec{u}}{1-\frac{u_x v}{c^2}} \quad \cdots (60)$$

Similarly, $\bar{K}_y = K_y$ implies

$$\frac{1}{\sqrt{1-(\frac{\bar{u}}{c})^2}} \bar{F}_y = \frac{1}{\sqrt{1-(\frac{u}{c})^2}} F_y$$

Using eq. (59),

$$\bar{F}_y = \sqrt{1-(\frac{v}{c})^2}\, \frac{F_y}{1-\frac{u_x v}{c^2}} = \frac{F_y}{\gamma(1-\beta\frac{u_x}{c})} \cdots (61)$$

$$\bar{F}_z = \frac{F_z}{\gamma(1-\beta\frac{u_x}{c})} \quad \cdots (62)$$

Eqs (60)-(62) are transformations of ordinary force.

Work-energy theorem

As shown before, $\int_0^x F\,dx = E_k = \frac{m_0c^2}{\sqrt{1-(\frac{u}{c})^2}} - m_0c^2$

In general, work done by $\vec{F}$

$$\omega = \int \vec{F}\cdot d\vec{r}$$

$$= \int \frac{d\vec{p}}{dt}\cdot d\vec{r} = \int \frac{d\vec{p}}{dt}\cdot\frac{d\vec{r}}{dt}\,dt$$

$$= \int \frac{d\vec{p}}{dt}\cdot\vec{u}\,dt = \int \vec{F}\cdot\vec{u}\,dt \quad \text{L -- ⑥③}$$

$\because E = \frac{m_0c^2}{\sqrt{1-(\frac{u}{c})^2}}$, $\frac{dE}{dt} = \frac{d}{dt}\frac{m_0c^2}{\sqrt{1-(\frac{u}{c})^2}}$

$$= \frac{m_0}{(1-(\frac{u}{c})^2)^{3/2}}\,\vec{u}\cdot\frac{d\vec{u}}{dt}$$

On the other hand, $\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\frac{m_0\vec{u}}{\sqrt{1-(\frac{u}{c})^2}}$

$$= \frac{m_0}{\sqrt{1-(\frac{u}{c})^2}}\frac{d\vec{u}}{dt} + m_0\vec{u}\,\frac{1}{(1-(\frac{u}{c})^2)^{3/2}}\,\frac{1}{c^2}\,\vec{u}\cdot\frac{d\vec{u}}{dt}$$

$$\therefore \vec{F}\cdot\vec{u} = \vec{u}\cdot\frac{d\vec{u}}{dt}\left[\frac{m_0}{\sqrt{1-(\frac{u}{c})^2}} + \frac{m_0(\frac{u}{c})^2}{(1-(\frac{u}{c})^2)^{3/2}}\right]$$

$$= \frac{m_0\vec{u}}{(1-(\frac{u}{c})^2)^{3/2}}\cdot\frac{d\vec{u}}{dt}$$

$\therefore \frac{dE}{dt} = \vec{F}\cdot\vec{u}$ (in agreement with classical expression) --- ⑥④

$$\therefore \omega = \int \frac{dE}{dt} dt \qquad \frac{d\omega}{dt} = \frac{dE}{dt}.$$

$$\omega = \int \frac{dE}{dt} dt = \Delta E = \text{change of } \frac{m_0 c^2}{\sqrt{1-(\frac{u}{c})^2}}.$$

L - (65)

This is the work-energy theorem.

Example Motion under a constant force

A particle of rest mass m_0 is subject to a constant force F at $t \geq 0$.

Find $x(t)$

Solution. For relativistic dynamics, $\because$ $m = \frac{m_0}{\sqrt{1-(\frac{u}{c})^2}}$ depends on u, it is easier to find p first. From p, find u.

$$\because \frac{dp}{dt} = F \quad \text{for } t \geq 0$$

$$\therefore p = Ft \quad \therefore \frac{m_0 u}{\sqrt{1-(\frac{u}{c})^2}} = Ft$$

$$\therefore u = \frac{F/m\, t}{\sqrt{1+(\frac{Ft}{mc})^2}}$$ is the velocity.

One sees that for $\frac{Ft}{m} << c$, $u \simeq F/m\, t$ is the result of Newtonian mechanics. $t \to \infty$, $u \to c$
$u < c$ is always true.

To find x(t), one sets $u=\frac{dx}{dt}$

$$\therefore \quad x(t)=\frac{F}{m}\int_0^t \frac{t'}{\sqrt{1+(\frac{Ft'}{mc})^2}}\,dt'$$

$$=\frac{mc^2}{F}\sqrt{1+(\frac{Ft'}{mc})^2}\Big|_0^t$$

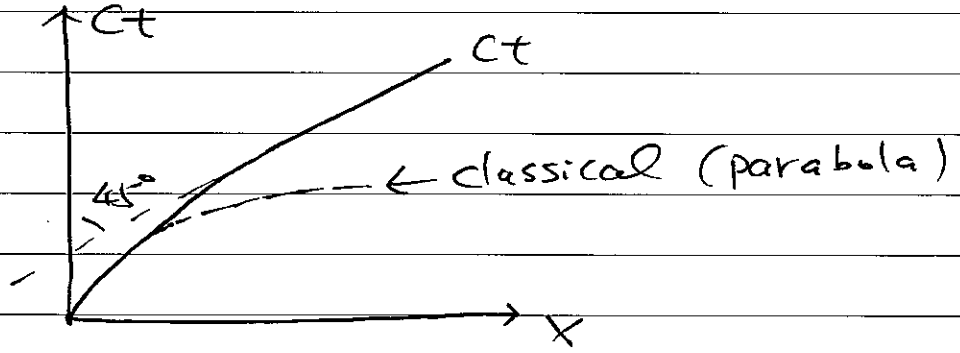
$$=\frac{mc^2}{F}\left(\sqrt{1+(\frac{Ft}{mc})^2}-1\right)$$

for $\frac{Ft}{m}<<c$, $x(t)\approx\frac{mc^2}{F}\left(1+\frac{1}{2}(\frac{Ft}{mc})^2-1\right)$

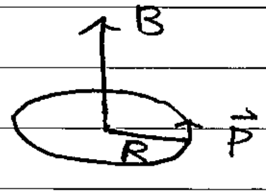
$$=\frac{F}{2m}t^2$$

Which is a hyperbola. $t\to\infty$, $x(t)\to ct$

In general, the trajectory is a hyperbolic motion



Example Cyclotron motion



$$\frac{d\vec p}{dt}=\vec F=q\,\vec u\times\vec B \quad \text{--} \ ⑯$$

$\because\ \vec p=\frac{m_0\vec u}{\sqrt{1-(\frac{u}{c})^2}}$, $\therefore$ (66) implies $\frac{d\vec p}{dt}\cdot\vec p=0$

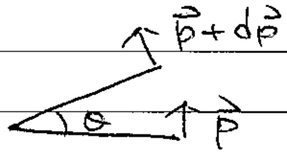
Hence $\frac{dp^2}{dt}=0$ $\therefore$ p = constant for relativistic cyclotron motion

Hence P plays the role of m_0u in non-relativistic cyclotron motion (m_0u = Const).

For non-relativistic cyclotron motion,

$$quB = \frac{m_0u^2}{R} = \frac{pu}{R}$$

For relativistic case, $\frac{dp}{dt} = \frac{p\,d\theta}{dt}$ as well



$$\frac{d\theta}{dt} = \frac{u}{R}$$

∴ $quB = \frac{pu}{R}$ is correct as well

∴ $p = qBR$ which is correct for both relativistic & non-relativistic cyclotron motion. except that $p = \frac{m_0u}{\sqrt{1-(\frac{u}{c})^2}} = mu$ for relativistic cyclotron motion

<u>Newton's third law</u>

The Newton's third law does not extend to relativistic domain.

For example, the force $\vec{F}(t)$ and its reaction force $-\vec{F}(t)$ occurs simultaneously in one frame. In the other frame, they no longer occur simultaneously:

$\vec{F_a} = \vec{F}'(t_1)$, $-\vec{F}'(t_2) = \vec{F_r}$ (due to they act on different objects)

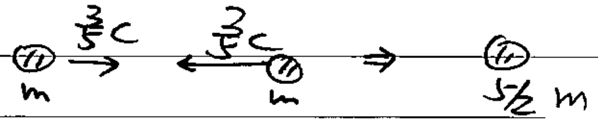
Hence generally $\vec{F_a}(t) + \vec{F_r}(t) \neq 0$ unless both forces apply at the same point (i.e. they are contact forces). In this case, they always occur at the same space & time

$\vec{F_a} + \vec{F_r} = 0$ is obeyed.

Center of energy

The relation $E = mc^2 = \frac{m_0 c^2}{\sqrt{1-(\frac{u}{c})^2}}$ has a profound implication that any energy, including thermal energy & potential energy, would contribute the mass $m = \frac{E}{c^2}$.

In the example,



Classically, we attribute the collision as an inelastic collision: kinetic energy is not conserved, loss of energy of center of mass goes to internal energy U (heat + ...).

Therefore, the resulting lump is hotter.

This is still true (temperature is higher after collision) in relativistic point of view.

However, the <u>total energy is conserved</u>.

The collision is <u>elastic relativistically</u>.

The only thing difference is that since $v=0$ after collision, all the energies go in to the mass. Hence mass becomes $\frac{5}{2}m$.

<u>The mass is thus no longer conserved!</u>

Once energy also possesses the property of mass $m=E/c^2$, it also carries momentum $\frac{E}{c^2}\vec{u}$!

Therefore, in calculating the total momentum of a system, one <u>needs to include</u> <u>the contribution of all energies</u> that are either in the form of masses (particles, ---) or not in the form of masses (such as EM fields). As a result, the concept for center of mass needs to be revised.

Center of energy theorem

In Newton's mechanics, when the center of mass of an isolated system is stationary, $\vec{P}_{total} = 0$.

This is no longer true once energy without mass also carries momentum.

It has to be generalized.

For the case of EM waves & massive particles,

if u_{EM} = field energy density, $\mathcal{E}_i = \underbrace{\text{ith particle}}_{\text{mechanical energy of}}$

One defines the center of energy by

$$\vec{R}_E \equiv \frac{\int d\tau \left[u_{EM}(\vec{r}) \cdot \vec{r} + \sum_i \mathcal{E}_i \vec{r}_i \right]}{\int d\tau \left(u_{EM} + \sum_i \mathcal{E}_i \right)}$$

Let $_{tot} = \int d\tau \left(u_{EM} + \sum_i S_i \right)$ be the total energy.

$$\vec{P}_{total} = \vec{P}_{EM} + \sum_i \frac{v_i}{c^2} \mathcal{E}_i$$

the center of energy theorem states

$$E_{tot} \frac{d\vec{R}_E}{dt} = c^2 \vec{P}_{total}$$

Pf.: We start from energy conservation eq.

$$\frac{\partial u_{EM}}{\partial t} + \vec{\nabla} \cdot \vec{S} = - \vec{J} \cdot \vec{E} \quad — \text{(67)}$$

and the momentum conservation eq:

$$\frac{\partial \vec{g}}{\partial t} - (\vec{\nabla}\cdot\overleftrightarrow{T}) = -f_{mech} = -(\rho\vec{E}+\vec{J}\times\vec{B})$$

└ (68)

(67)$\cdot\vec{r}$ ⇒ $\frac{\partial}{\partial t}(u_{EM}\vec{r}) + (\vec{\nabla}\cdot\vec{S})\vec{r} = -\vec{J}\cdot(\underbrace{\vec{E}+\vec{v}\times\vec{B}}_{\overset{\|}{0}})\vec{r}$

$$[(\vec{\nabla}\cdot\vec{S})\vec{r}]_i = (\vec{\nabla}\cdot\vec{S})x_i$$

$$= \vec{\nabla}\cdot(\vec{S}x_i) - \vec{S}\cdot(\nabla x_i) = \vec{\nabla}\cdot(\vec{S}x_i) - S_i$$

$(\vec{\nabla}\cdot\vec{S})\vec{r} = \vec{\nabla}\cdot(\vec{S}\vec{r}) - \vec{S}$ where $(\vec{S}\vec{r})_i = \vec{S}x_i$

∴ (67)$\cdot\vec{r}$ − (68)$\cdot c^2 t$ becomes

$$\frac{\partial}{\partial t}(\vec{r}u_{EM} - c^2 t g) + \vec{\nabla}\cdot(\vec{S}\vec{r} + c^2 t\overleftrightarrow{T}) - \vec{S} + c^2\vec{g}$$

$$= -\vec{J}\cdot(\vec{E}+\vec{v}\times\vec{B})\vec{r} + \rho(\vec{E}+\vec{J}\times\vec{B})c^2 t$$

∵ $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \epsilon_0\frac{1}{\mu_0\epsilon_0}\vec{E}\times\vec{B} = c^2\vec{g}$

∴ $\frac{\partial}{\partial t}(\vec{r}u_{EM} - c^2 t g) + \vec{\nabla}\cdot(\vec{S}\vec{r} + c^2 t\overleftrightarrow{T})$

$= -\vec{J}(\vec{E}+\vec{v}\times\vec{B})\vec{r} + \rho(\vec{E}+\vec{J}\times\vec{B})c^2 t$ --- (69)

For point charges, we have

$$\rho = \sum_i q_i\delta(\vec{r}-\vec{r}_i), \quad \vec{J} = \sum_i q_i\vec{v}_i\delta(\vec{r}-\vec{r}_i)$$

$\int d\tau$ (69) then picks up q_i at r_i, we obtain

$$\frac{d}{dt}\int (\vec r\, u_{EM} - c^2 t\, \vec g)\, d\tau + \sum_i \Big[\vec r_i\, \vec v_i \cdot \underbrace{\vec F_i}_{\uparrow\; q_i(\vec E(r_i) + \vec v_i \times \vec B(\vec r_i))} - c^2 t\, \vec F_i\Big] = 0 \quad \cdots \quad (70)$$

$\because\ \vec v_i \cdot \vec F_i = \frac{d\varepsilon_i}{dt}$ $\quad\varepsilon_i$ = mechanical energy of ith particle

$\vec F_i = \frac{d\vec p_i}{dt}$ $\quad\vec p_i$ = mechanical momentum of ith particle.

$\therefore$ (70) can be rewritten as

$$\frac{d}{dt}\int d\tau\, \vec r\, u_{EM} + \sum_i \vec r_i \frac{d\varepsilon_i}{dt} = c^2 \int d\tau\, \vec g + c^2 t\, \frac{d}{dt}\underbrace{\Big[\int d\tau\, \vec g + \sum_i \vec p_i\Big]}_{\vec P_{total}}$$

For isolated systems,

$$\frac{d}{dt}\vec P_{total} = \oint_{S\to\infty} d\vec a \cdot \overleftrightarrow{T} = 0$$

∴ We obtain

$$\frac{d}{dt}\left(\int dz\, \vec{r}\, u_{EM}\right) + \sum_i \vec{r}_i \frac{d\varepsilon_i}{dt} = c^2 \underbrace{\int dz\, \vec{g}}_{\vec{P}_{EM}}$$

$$\because \quad \vec{r}_i \frac{d\varepsilon_i}{dt} = \frac{d}{dt}(\vec{r}_i \varepsilon_i) - \vec{v}_i \varepsilon_i$$

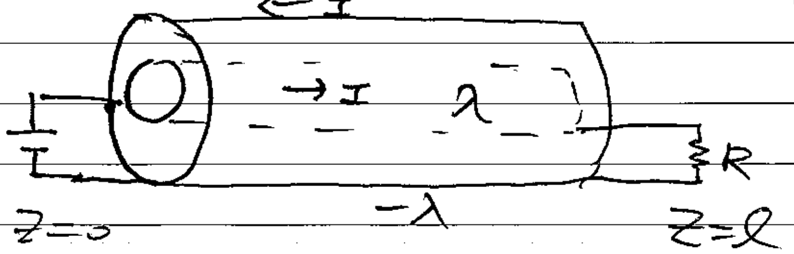
$$\therefore \quad \frac{d}{dt}\left(\int dz\, \vec{r}\, u_{EM} + \sum_i \vec{r}_i \varepsilon_i\right)$$

$$= c^2\Big(\vec{P}_{EM} + \underbrace{\frac{1}{c^2}\sum_i v_i \varepsilon_i}_{\text{special relativity } = \sum_i \vec{P}_i}\Big)$$

$$\therefore \quad = c^2 \vec{P}_{total}$$

$$\therefore \quad E_{tot} \frac{d\vec{R}_E}{dt} = c^2 \vec{P}_{total}$$

Example: In the co-axial cylinder example,



the setup can be set with z=0 being the battery and z=l being the position for the resistor R.

The system can be divided into three parts:

12-69

① the resistor : energy = E_R , $\vec{r} = l\hat{z}$

② the battery : energy = E_b , $\vec{r} = 0$

and ③ the rest of the system (EM fields + many electrons

: energy = E_0 $\vec{r}$ = Center of mass = $\vec{R_0}$

∴ $\vec{R_E} = \frac{1}{E}(0 \cdot E_b + E_0 R_0 + E_R l \hat{z})$ $E = E_0 + E_R + E_b$

$= \frac{1}{E}(E_0 \vec{R_0} + E_R l \hat{z})$ = constant

$\vec{R_E}$ is changing because

(i) the energy of the resistor E_R is changing (for example, it is getting hotter due to Joule heating)

(ii) the energy of the rest is

static : U_{EM} = static = fixed

current I = constant

∴ Kinetic energy of electrons = constant

E_0 = Const $\vec{R_0}$ = Const

(iii) E_b is decreasing

∴ $\frac{d\vec{R_E}}{dt} = \frac{1}{E} \frac{dE_R}{dt} l \hat{z}$

$\frac{dE_R}{dt} = IV$ ∴ $\frac{d\vec{R_E}}{dt} = \frac{IVl}{E}\hat{z}$

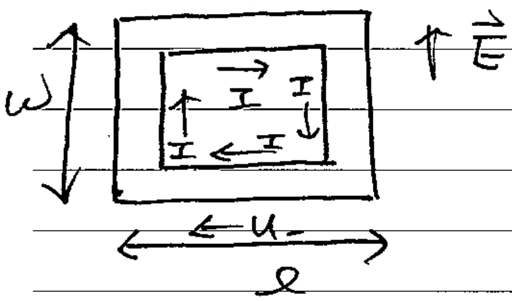
∵ $\frac{d\vec{R_E}}{dt} = \frac{c^2}{E} \vec{P}_{total}$ ∴ $\vec{P}_{total} = \frac{IVl}{c^2}\hat{z}$

in agreement with $\vec{P}_{EM} = \epsilon_0 \mu_0 \int S dz = \frac{IVl}{c^2}\hat{z}$

∴ $\vec{P}_{total}$ is carried by EM fields

(matters are fixed in space)

Example : Hidden momentum of a magnetic dipole in an electric field



Consider a magnetic dipole modelled by a rectangular loop of wire carrying a steady current I.

Suppose I is carried by non-interacting positive charges q. A uniform field $\vec{E}$ is applied so that charges accelerate in the left segment and decelerate in the right segment. Find the total momentum of all charges in the loop.

Solution : The momentum of EM fields in this problem is done in problem 8.20:

$$\vec{P}_{EM} = -\mu_0\epsilon_0 \vec{m}\times\vec{E} = \frac{-1}{c^2}\vec{m}\times\vec{E}.$$

Together with EM fields, the whole system is isolated. Hence one expects that

$$\vec{P}_{total} = \vec{P}_{mech} + \vec{P}_{EM} = 0$$

However, naively if one uses Newtonian mechanics, one would conclude

$$\sum_i \text{current} = \sum_i q \vec{v_i} = 0$$

$$\therefore \sum_i \vec{v_i} = 0, \quad \sum_i M \vec{v_i} = 0, \quad \therefore \vec{P}_{mech} = 0$$

It would appear $\vec{P}_{mech} + \vec{P}_{EM} \neq 0$

The missing momentum to counter $\vec{P}_{EM}$ is called the Hidden momentum.

In this problem, the hidden momentum lies in relativistic momentum.

Suppose that there are N_+ charges in the upper segment, going with speed u_+ to the right.

There are N_- charges in the lower segment going with speed u_- to the left.

$$\therefore I = \underbrace{\frac{q N_+}{\ell}}_{\lambda_+} u_+ = \underbrace{\frac{q N_-}{\ell}}_{\lambda_-} u_- \quad \text{(Current is conserved.)}$$

The relativistic momentum of charges

$$p = \gamma_+ M N_+ u_+ - \gamma_- M N_- u_- \qquad M = \text{rest mass of charge}$$

$$\gamma_\pm = \frac{1}{\sqrt{1-(\frac{u_\pm}{c})^2}} \quad \text{(to the right)}$$

$$\because \; N_+ u_+ = N_- u_- = \frac{\ell I}{q}$$

$$\therefore \; p = \frac{M \ell I}{q} (\gamma_+ - \gamma_-)$$

Now, when a charge moves up from the bottom to the left segment to the upper segment, the energy gain $\Delta(\gamma M c^2) = qEw$ = work done by the electric field.

$$\therefore \; \gamma_+ M c^2 - \gamma_- M c^2 = qEw$$

$$\gamma_+ - \gamma_- = \frac{qEw}{Mc^2}$$

$$p = \frac{M \ell I}{q} \times \frac{qEw}{Mc^2} = \frac{I \ell E w}{c^2} = \frac{mE}{c^2} \text{ (to the right)}$$

$$\therefore \; \vec{P}_{mech} = \frac{1}{c^2} \vec{m} \times \vec{E}$$

which counters $\vec{P}_{EM} = \frac{-1}{c^2} \vec{m} \times \vec{E}$ exactly!

$\therefore$ A magnetic dipole at rest in an electric field carries linear momentum even though it is not moving! This is the hidden momentum that balances $\vec{P}_{EM}$!

No.
Date. 12-13.

Relativistic electrodynamics

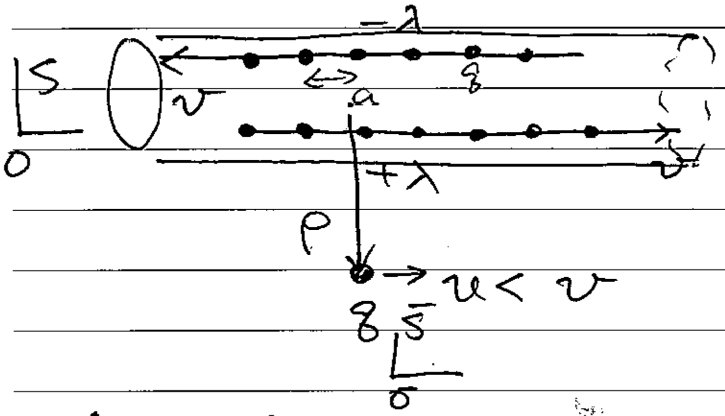
Magnetism as a relativistic phenomenon

As indicated in the beginning, the special theory of relativity is developed to incorporate classical electrodynamics. Therefore, classical electrodynamics is already consistent with special relativity. What we have demonstrated is that Newtonian mechanics has to be corrected. Having corrected Newtonian mechanics, we now develop a more complete formulation of electrodynamics to exhibit its relativistic nature.

We shall first point out that given electrostatics, the special relativity automatically implies the existence of magnetism.

Emergent magnetic field

Consider in S, one has two lines of (overlapped) charges with charge density (line density) λ in a wire as shown in below : ∵ $-\lambda$ moves to left with speed v



λ moves to right with speed v

∴ net current $I = 2\lambda v$ ---(72)

A point charge q moves with velocity $u < v$ outside the wire

∵ net charge $= 0$, ∴ $\vec{F}_E = 0$ on q ---⑦③

Now, we move to the frame $\bar{S}$ that goes with q so that q is at rest in $\bar{S}$.

The velocities of $\pm\lambda$ charges become

$$v_{\pm} = \frac{v \mp u}{1 \mp vu/c^2} \quad \text{---} \quad (74)$$

∴ $v_- > v_+$

The Lorentz contraction : $a \to \sqrt{1-(\frac{v_\pm}{c})^2}\, a$ implies

$$\lambda_\pm = \frac{\pm q}{a} = \pm\frac{\lambda_0}{\sqrt{1-(\frac{v_\pm}{c})^2}} \equiv \pm\gamma_\pm \lambda_0 \quad \text{-} (75)$$

$\lambda_0 =$ line charge density in rest frame of charges (positive or negative)

(Note that $\lambda = \frac{\lambda_0}{\sqrt{1-(\frac{v}{c})^2}}$) ---(76)

Combining eqs. ⑭ & ⑮, we have

$$\gamma_\pm = \frac{1}{\left(1-\frac{1}{c^2}(v\mp u)^2\left(1\mp\frac{vu}{c^2}\right)^{-2}\right)^{1/2}}$$

$$= \frac{c^2\mp uv}{\sqrt{(c^2\mp uv)^2-c^2(v\mp u)^2}}$$

$$= \frac{c^2\mp uv}{\sqrt{(c^2-v^2)(c^2-u^2)}} = \underbrace{\frac{1}{\sqrt{1-(\frac{v}{c})^2}}}_{\gamma}\,\frac{1\mp\frac{uv}{c^2}}{\sqrt{1-u^2/c^2}}$$

$$\therefore\ \lambda_{total} = \lambda_+ + \lambda_- = \lambda_0(\gamma_+ - \gamma_-) = \frac{-2\lambda uv}{c^2\sqrt{1-u^2/c^2}} \quad \cdots (77)$$

Therefore, a electric neutral line in one frame with current becomes charged in another frame. This reflects the 4-vector nature of $(c\rho, \vec{J})$: $c\bar{\rho} = \gamma(c\underset{\swarrow 0}{\cancel{\rho}} - \frac{v}{c}J_x)$ $\therefore \bar{\rho} \neq 0$

Check : $c\lambda_{total} = -\underbrace{(2\lambda v)}_{I}\frac{u}{c}\underbrace{\frac{1}{\sqrt{1-u^2/c^2}}}_{\gamma}$

in agree with $c\bar{\rho} = \gamma(c\rho - \frac{v}{c}J_x)$

λ_{total} gives rise an electric field

$$E = \frac{\lambda_{total}}{2\pi\epsilon_0\rho} \quad \cdots (78)$$

Hence there is an electric force on $\bar{S}$ (y-direction)

$$\bar{F}_y = qE = -\frac{\lambda v}{\pi\epsilon_0 c^2 \rho}\frac{qu}{\sqrt{1-(\frac{u}{c})^2}} \quad \cdots \text{(79)}$$

According to the transformation law for forces, $\bar{F}$ implies that there is a force F_y on on q in $\bar{S}$ frame:

$$\underset{\underset{\text{eq. (61)}}{\uparrow}}{\bar{F}_y} = \gamma\frac{F_y}{1-\beta\frac{u_y}{c}} \quad \cdots \text{(80)}$$

where $u_y = 0$ ∴ $F_y = \sqrt{1-(\frac{u}{c})^2}\,\bar{F}_y$

$$= -\frac{\lambda v}{\pi\epsilon_0 c^2}\frac{qu}{\rho} \quad - \text{(81)}$$

Therefore, even though $F_E = 0$ in S, to be consistent with $\bar{S}$, there must be a force $F_y = -\frac{I}{2\pi\epsilon_0 c^2}\frac{qu}{\rho}$ due to the presence of current I.

$$\because \frac{1}{\epsilon_0 c^2} = \mu_0 \quad \therefore F_y = -\left(\frac{\mu_0 I}{2\pi\rho}\right)\times q\times u$$

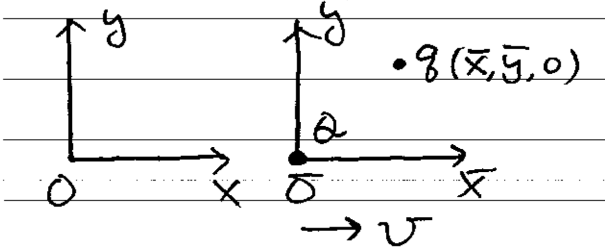
which is consistent with $\vec{F} = q\,\vec{u}\times\vec{B}$

with $\vec{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\theta}$ being the magnetic field yielded by a long straight wire. Hence it automatically implies the existence of magnetic field.

Fields of a charge at constant velocity

The appearance of magnetic field can be best demonstrated by considering moving charges at constant velocities.

(i) Two charges Q & q at rest in $\bar{S}$ frame



Let positions of Q & q be at $\bar{O}$ & $(\bar{x},\bar{y},0)$ in $\bar{S}$ frame.

At $t=\bar{t}=0$, O & $\bar{O}$ coincide.

$$\therefore\quad \bar{x}=\gamma(x-\underset{0}{\underline{v\cancel{t}}})=\gamma x$$

$$\bar{y}=y \qquad \text{--}\ ⑧②$$

Force on q in $\bar{S}$ frame is

$$\bar{F}_x=\frac{qQ}{4\pi\epsilon_0}\frac{\bar{x}}{(\bar{x}^2+\bar{y}^2)^{3/2}}$$

$$\bar{F}_y=\frac{qQ}{4\pi\epsilon_0}\frac{\bar{y}}{(\bar{x}^2+\bar{y}^2)^{3/2}}$$

$$\bar{F}_z=0$$

In S frame, the force on q, F is

$$F_x\underset{\underset{\text{eq.⑥⓪}\ (\bar{u}=0)}{\uparrow}}{=}\frac{\bar{F}_x+\frac{v}{c^2}\vec{\bar{F}}\cdot\vec{\bar{u}}}{1+\frac{\bar{u}_x v}{c^2}}=\bar{F}_x$$

$$\bar{F}_y=\gamma F_y,\quad \bar{F}_z=\gamma F_z$$

Here $\bar{F}_x$, $\bar{F}_y$, $\bar{F}_z$ are evaluated at the same event $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ or expressed in $(x, y, z, \overset{=0}{t})$ using eq. (82)

$$\therefore \quad \bar{F}_x \rightarrow \frac{qQ}{4\pi\epsilon_0} \frac{\gamma x}{[\gamma^2 x^2 + y^2]^{3/2}}$$

$$\bar{F}_y \rightarrow \frac{qQ}{4\pi\epsilon_0} \frac{y}{[\gamma^2 x^2 + y^2]^{3/2}}$$

$$\bar{F}_z = 0$$

$$\therefore \quad F_x = \frac{\gamma q Q x}{4\pi\epsilon_0 [\gamma^2 x^2 + y^2]^{3/2}}$$

$$F_y = \frac{\bar{F}_y}{\gamma} = \frac{qQ \quad y}{4\pi\epsilon_0 \gamma [\gamma^2 x^2 + y^2]^{3/2}} = \frac{\gamma q Q y}{4\pi\epsilon_0 [\gamma^2 x^2 + y^2]^{3/2}} \times \left[1 - \left(\frac{v}{c}\right)^2\right]$$

$$F_z = 0 \quad --- \quad (83)$$

This is the force acting on q due to Q at $t=0$ when Q is at O and be rewritten in a vector form.

$$\vec{F}_q = \frac{\gamma q Q \vec{r}}{4\pi\epsilon_0 [\gamma^2 x^2 + y^2]^{3/2}} - \frac{v^2}{c^2} \frac{\gamma q Q y \hat{y}}{4\pi\epsilon_0 (\gamma^2 x^2 + y^2)^{3/2}}$$

$$= \frac{\gamma q Q}{4\pi\epsilon_0 [\gamma^2 x^2 + y^2]^{3/2}} \left\{ \vec{r} - \frac{v^2}{c^2} y \hat{y} \right\}$$

$$= q \left\{ \frac{\gamma Q \vec{r}}{4\pi\epsilon_0 [\gamma^2 x^2 + y^2]^{3/2}} + \vec{v} \times \frac{\gamma Q v y \hat{z}}{4\pi\epsilon_0 c^2 (\gamma^2 x^2 + y^2)^{3/2}} \right]$$

$\llcorner$ (84)

Which can be identified as

$$\vec{F_q} = q(\vec{E} + \vec{v} \times \vec{B})$$ in the form of Lorentz force

With $$\vec{E} = \frac{\gamma Q \vec{r}}{4\pi\epsilon_0 (\gamma^2 x^2 + y^2)^{3/2}} \quad \text{--- (85)}$$

$$\vec{B} = \frac{\mu_0 \gamma Q v y \hat{z}}{4\pi (\gamma^2 x^2 + y^2)^{3/2}} \quad \text{--- (86)}$$

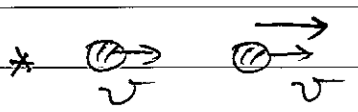
$$\uparrow$$

$$\frac{1}{\epsilon_0 c^2} = \mu_0$$

We see that a magnetic field is generated in S frame.

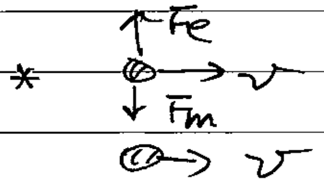
The electric field is in the radial direction for a charge moving at constant velocity.

Eq. (84) implies the following extreme cases.

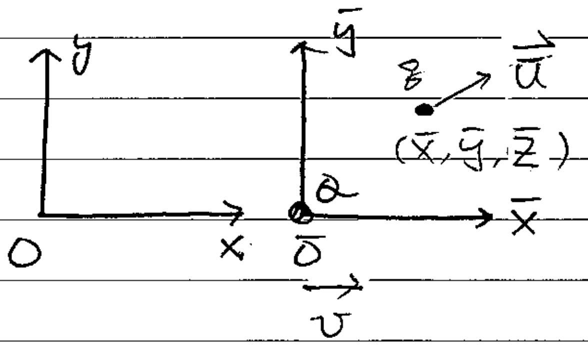
* 

$\vec{F} = q\vec{E} = \vec{F_e}$

$(y=0, \vec{B}=0)$

* 

(ii) Two charges q & Q are moving at different velocities. One can choose the velocity of $Q = v\hat{x}$ and set the velocity of q be $\vec{\bar{u}}$ in $\bar{S}$. Hence Q is at rest at $\bar{O}$ in $\bar{S}$ as shown in below.



Again, assuming at $t = \bar{t} = 0$, O & $\bar{O}$ coincide ~~the event $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$~~ occurs at (x, y, z, t)

In $\bar{S}$ frame, Q is at rest and hence it only generates an electric field. Hence the force on q is still the electric force due to Q.

$$\therefore \quad \bar{F}_x = \frac{qQ}{4\pi\epsilon_0}\frac{\bar{x}}{\bar{r}^3}, \quad \bar{r}^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2$$
$$= \gamma^2(x - vt)^2 + y^2 + z^2 \quad \text{--- (87)}$$

$$\bar{F}_y = \frac{qQ}{4\pi\epsilon_0}\frac{\bar{y}}{\bar{r}^3}$$

$$\bar{F}_z = \frac{qQ}{4\pi\epsilon_0}\frac{\bar{z}}{\bar{r}^3} \quad \text{--- (88)}$$

Using eq. (60) with $\bar{F}$ & F exchanged, we have

$$F_x = \frac{\bar{F}_x + \frac{v}{c^2}\vec{\bar{F}}\,\vec{\bar{u}}}{1 + \frac{\bar{u}_x v}{c^2}} = \bar{F}_x + \frac{v}{c^2 + \bar{u}_x v}(\bar{u}_y \bar{F}_y + \bar{u}_z \bar{F}_z) \quad \text{--- (89)}$$

$$= \frac{q\theta}{4\pi\epsilon_0 \bar{r}^3}\left[\bar{x} + \frac{\bar{u}_y v}{c^2+\bar{u}_x v}\bar{y} + \frac{\bar{u}_z v}{c^2+\bar{u}_x v}\bar{z}\right]$$

Using the velocity transformation rule (eq ㊱)

$$u_y(u_z) = \frac{\bar{u}_y(\bar{u}_z)}{\gamma\left(1+\frac{\bar{u}_x v}{c^2}\right)}$$

$$\therefore \frac{\bar{u}_y v}{c^2+u_x v} = \frac{1}{c^2}\frac{\bar{u}_y v}{1+\frac{\bar{u}_x v}{c^2}} = \frac{\gamma u_y v}{c^2} \quad (y \to z \text{ as well})$$

$$\therefore \bar{x} + \frac{\bar{u}_y v}{c^2+\bar{u}_x v}\bar{y} + \frac{\bar{u}_z v}{c^2+\bar{u}_x v}\bar{z}$$

$$= \gamma(x-vt) + \gamma\frac{u_y v}{c^2}y + \gamma\frac{u_z v}{c^2}z \quad -- (90)$$

Similarly

$$F_y = \frac{1}{\gamma}\frac{\bar{F}_y}{1+\frac{\bar{u}_x v}{c^2}}$$

$$\left(\text{eq (59)} \rightarrow \frac{1}{\sqrt{1-(\frac{u}{c})^2}} = \frac{1+\frac{\bar{u}_x v}{c^2}}{\sqrt{1-(\frac{v}{c})^2}\sqrt{1-(\frac{\bar{u}}{c})^2}}\right)$$

$$= \frac{8\theta}{4\pi\epsilon_0 \bar{r}^3\gamma}\frac{\bar{y}}{1+\frac{v}{c^2}\bar{u}_x}$$

$$F_z = \frac{8\theta}{4\pi\epsilon_0 \bar{r}^3\gamma}\frac{\bar{z}}{1+\frac{v}{c^2}\bar{u}_x} \quad -- (91)$$

Using the velocity addition rule & eq (59), one has

$$\frac{1}{\sqrt{1-(\frac{\bar{u}}{c})^2}} = \frac{1-\frac{u_x v}{c^2}}{\sqrt{1-(\frac{v}{c})^2}\sqrt{1-(\frac{u}{c})^2}} \quad -- (92)$$

$$\frac{1}{\sqrt{1-(\frac{u}{c})^2}} = \frac{1+\frac{\bar{u}_x v}{c^2}}{\sqrt{1-(\frac{v}{c})^2}\sqrt{1-(\frac{\bar{u}}{c})^2}} \quad \text{--- } \textcircled{\small P3}$$

$$\textcircled{\small P2}\times\textcircled{\small P3} \Rightarrow \left(1+\frac{\bar{u}_x v}{c^2}\right)\left(1-\frac{u_x v}{c^2}\right) = 1-\left(\frac{v}{c}\right)^2$$

$$= \frac{1}{\gamma^2} \quad \text{--} \textcircled{\small P3}\text{-1}$$

$$\therefore \frac{\bar{y}}{1+\frac{v\bar{u}_x}{c^2}} = \gamma^2\left(1-\frac{u_x v}{c^2}\right) y$$

$$\frac{\bar{z}}{1+\frac{v\bar{u}_x}{c^2}} = \gamma^2\left(1-\frac{u_x v}{c^2}\right) z \quad \text{---} \textcircled{\small P4}$$

Combining (SP), (P0), (P1) & (P4), we obtain

$$\vec{F}_q = \frac{\gamma q Q}{4\pi\epsilon_0 r^3}\left\{\vec{r} + \frac{v}{c^2}\left[(u_y y + u_z z)\hat{x} - u_x y\,\hat{y} - u_x z\,\hat{z}\right]\right\}.$$

where $\vec{r} = (x-vt)\hat{x} + y\,\hat{y} + z\,\hat{z}$ = the position vector from Q to q.

Now, $\frac{v}{c^2}\left[(u_y y + u_z z)\hat{x} - u_x y\,\hat{y} - u_x \dot{z}\,\hat{z}\right]$

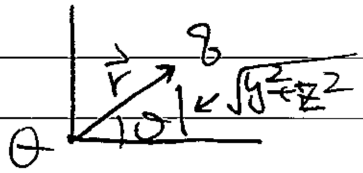
$$= \frac{1}{c^2}\vec{u}\times(\underbrace{\vec{v}\times\vec{r}}_{v\hat{x}\times\vec{r} = vy\hat{z} - vz\hat{y}})$$

Check: $\vec{u}\times(\vec{v}\times\vec{r}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_x & u_y & u_z \\ 0 & -vz & vy \end{vmatrix}$

$$= v(y u_y + z u_z\,;\, -y u_x\,;\, -u_x z)$$

$$\therefore\ \vec{F}_q = \frac{\gamma q Q}{4\pi\epsilon_0 \bar{r}^3}\left\{\vec{r} + \frac{1}{c^2}\vec{u}\times(\vec{v}\times\vec{r})\right\} \quad \cdots \text{⑨⑤}$$

Now $\bar{r}^2 = \gamma^2(x-vt)^2 + y^2 + z^2$



$$= \gamma^2\left[(x-vt)^2 + (y^2+z^2)(1-\beta^2)\right]$$

$$= \gamma^2\left[r^2 - \beta^2\underbrace{(y^2+z^2)}_{r^2\sin^2\theta}\right]$$

$$= \gamma^2 r^2(1-\beta^2\sin^2\theta)$$

$$\therefore\ \vec{F}_q = \frac{qQ}{4\pi\epsilon_0\gamma^2 r^2(1-\beta^2\sin^2\theta)^{3/2}}\left[\hat{r} + \frac{1}{c^2}\vec{u}\times(\vec{v}\times\hat{r})\right]$$

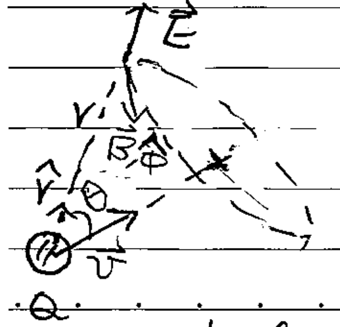
$\llcorner$ -- ⑨⑥

is the force acting on q when the position of Q is $(vt, 0, 0)$.

From the form of Lorentz force

$$\vec{F}_q = q(\vec{E} + \vec{u}\times\vec{B}),$$

One can identify (see the left figure)

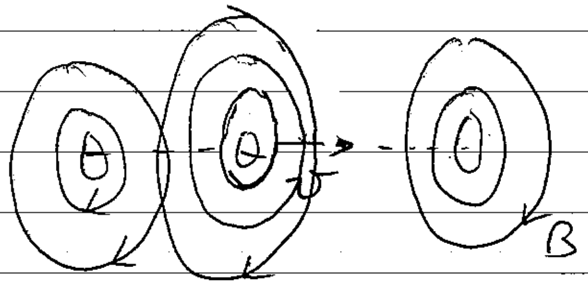


$$\vec{E} = \frac{Q}{4\pi\epsilon_0\gamma^2 r^2(1-\beta^2\sin^2\theta)^{3/2}}\hat{r} \quad \cdots \text{⑨⑦}$$

$$\vec{B} = \frac{\mu_0 Q v\sin\theta}{4\pi\gamma^2 r^2(1-\beta^2\sin^2\theta)^{3/2}}\hat{\phi} \quad \cdots \text{⑨⑧}$$

Note that setting $t=0$, $z=0$ eqs. ⑨⑦ & ⑨⑧ recover ⑧⑤ & ⑧⑥

Clearly, for a particle moving at constant velocity, the electric field observed by a stationary observer is radial, while the magnetic field for circular lines centered on the trajectory as shown in below.

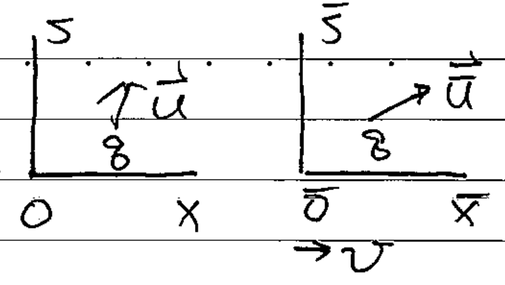


Transformation of E & B fields

From the above transformation of force for a single charge, it is clear that the most general form of the force that acts on a charge particle can be written as

$$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$$ — the Lorentz force

The transformation of fields, $\vec{E}$ & $\vec{B}$ among different frames can thus be found by transformation of the Lorentz force.



Consider two frams S & $\bar S$. A charged particle, q, has different velocities $\vec u$ & $\vec{\bar u}$ in O & $\bar O$.

q is invariant. Therefore, the forces on q in S & $\bar S$ are

$$\vec F = q(\vec E + \vec u \times \vec B)$$

$$\text{and} \quad \vec{\bar F} = q(\vec{\bar E} + \vec{\bar u} \times \vec{\bar B}) \quad \text{--- (99)}$$

$$\therefore \quad F_x = q[E_x + (u_y B_z - u_z B_y)]$$

$$F_y = q[E_y + (u_z B_x - u_x B_z)]$$

$$F_z = q[E_z + (u_x B_y - u_y B_x)] \quad \text{-- (100)}$$

From $\vec F$, the transformation of $\vec F$ determines $\vec{\bar F}$:

$$\bar F_x = F_x - \frac{v}{c^2 - u_x v}(u_y F_y + u_z F_z) \quad \text{-- (101)}$$

(eg. 88)

$$\bar F_y = \frac{1}{\gamma} \frac{F_y}{1 - \frac{u_x v}{c^2}} \quad \text{-- (102)}$$

$$\bar F_z = \frac{1}{\gamma} \frac{F_z}{1 - \frac{u_x v}{c^2}} \quad \text{-- (103)}$$

where $\vec u$ has to be replaced by $\vec{\bar u}$ via

$$u_x = \frac{\bar{u}_x + v}{1 + \frac{\bar{u}_x v}{c^2}}$$

$$u_y = \frac{\bar{u}_y}{\gamma\left(1 + \frac{\bar{u}_x v}{c^2}\right)}$$

$$u_z = \frac{\bar{u}_z}{\gamma\left(1 + \frac{\bar{u}_x v}{c^2}\right)} \quad \& \quad \frac{1}{1 - \frac{u_x v}{c^2}} = \gamma^2\left(1 + \frac{\bar{u}_x v}{c^2}\right)$$

(eq ③-1)

Starting from eq. (102), one gets

$$\bar{F}_y = \gamma\left(1 + \frac{\bar{u}_x v}{c^2}\right) F_y$$

$$= q\gamma\left(1 + \frac{\bar{u}_x v}{c^2}\right)\left(E_y + u_z B_x - u_x B_z\right)$$

$$= q\left[\gamma E_y + \frac{\gamma \bar{u}_x v}{c^2} E_y + \bar{u}_z B_x - \gamma(\bar{u}_x + v) B_z\right]$$

$$= q\left[\gamma(E_y - vB_z) + \bar{u}_z B_x - \bar{u}_x \gamma\left(B_z - \frac{v}{c^2} E_y\right)\right]$$

Which is in the same form

of $\bar{F}_y = q(\bar{E}_y + \bar{u}_z \bar{B}_x - \bar{u}_x \bar{B}_z]$

$\therefore \quad \bar{E}_y = \gamma(E_y - vB_z)$ --- (104)

$\bar{B}_z = \gamma\left(B_z - \frac{v}{c^2} E_y\right)$ --- (105)

$\bar{B}_x = B_x$ --- (106)

Similarly, eq. (103)

$$\bar{F}_z = \gamma\left(1 + \frac{\bar{u}_x v}{c^2}\right) F_z$$

$$= q\gamma\left(1+\frac{\bar{u}_x v}{c^2}\right)\left(E_z + u_x B_y - u_y B_x\right)$$

$$= q\left[\gamma E_z + \frac{\gamma \bar{u}_x v}{c^2} E_z + \gamma(\bar{u}_x + v) B_y - \bar{u}_y B_x\right]$$

$$= q\left[\gamma(E_z + v B_y) + \bar{u}_x \gamma\left(B_y + \frac{v}{c^2} E_z\right) - \bar{u}_y B_x\right]$$

$$= q\left[\bar{E}_z + \bar{u}_x \bar{B}_y - \bar{u}_y \bar{B}_x\right]$$

$$\therefore \quad \bar{E}_z = \gamma(E_z + v B_y) \quad --- ⑩⑦$$

$$\bar{B}_y = \gamma\left(B_y + \frac{v}{c^2} E_z\right) \quad -- ⑩⑧$$

There is still one relation for $\bar{E}_x$ & E_x that is needed. This can be found in eq.(101). To find it, one notices that u_y & u_z in (101) $\propto$ $\bar{u}_y$ & $\bar{u}_z$ respectively.

Hence, $-\frac{v}{c^2 - u_x v}(u_y F_y + u_z F_z)$

$\propto \bar{u}_y(\cdots) + \bar{u}_z(\cdots)$ belong to $\vec{\bar{u}} \times \vec{\bar{B}}$ term. and is not related to $\vec{\bar{E}}$.

$\therefore$ We can set $u_y = u_z = 0$.

Then eq.(101) $\Rightarrow$ $\bar{F}_x = F_x = q E_x$ $\quad \therefore \bar{E}_x = E_x$

$$\bar{F}_x = q(\bar{E}_x + \underset{0}{\cancel{\bar{u}_y}} \bar{B}_z - \underset{0}{\cancel{\bar{u}_z}} \bar{B}_y) \quad \llcorner -- (109)$$

12-88

Hence the complete set of transformation rules is

$$\bar{E}_x = E_x, \quad \bar{E}_y = \gamma(E_y - vB_z), \quad \bar{E}_z = \gamma(E_z + vB_y)$$

$$\bar{B}_x = B_x, \quad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \quad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

$\leftarrow$ (110)

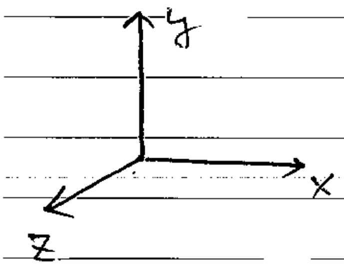
(Reversely, one replaces $v \to -v$)

Example:

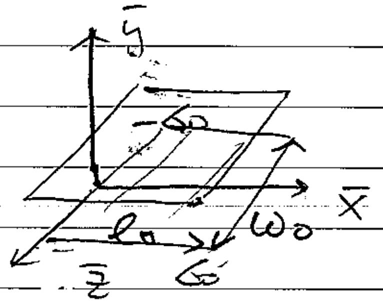
One can check

$$B^2 - \frac{\bar{E}^2}{c^2} = B^2 - \frac{E^2}{c^2}$$

$\leftarrow$ -- (111)



S



$\bar{S} \to v$

A paralell plate is at rest in $\bar{S}$ frame (as shown in above)

$$\vec{E}_0 = \frac{\sigma_0}{\epsilon_0}\hat{y} = \frac{\theta_0}{\epsilon_0 l_0 w_0}\hat{y}$$

∵ θ_0, is invariant $= \theta_0$ in S

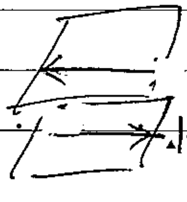
$l_0 \to l_0\sqrt{1 - v^2/c^2}$ in S , $w_0 = w_0$ in S

$$\therefore \text{In } S, \quad \vec{E} = \frac{\theta_0}{\epsilon_0 l_0 w_0 \sqrt{1 - v^2/c^2}}\hat{y} = \frac{\gamma\sigma_0}{\epsilon_0}\hat{y} = \frac{\sigma}{\epsilon_0}\hat{y}$$

$$\therefore \vec{E}^{\perp} = \gamma\vec{E}_0^{\perp} \quad (\perp \text{ means } \vec{E} \perp \vec{v}) \quad (\sigma = \gamma\sigma_0)$$

$\leftarrow$ -- (113)

In addition to $\vec{E}$, there is a surface $K_\pm = \pm\sigma v\hat{x}$ current in S :



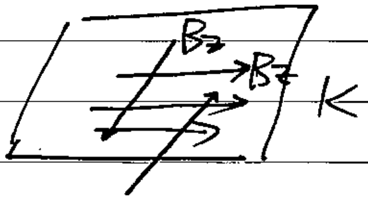
$K_- = -\sigma v\hat{x}$

$K_+ = \sigma v\hat{x}$

12-89

For a single sheet of surface current,

the Ampere's law implies



$$2B_z l = \mu_0 l K$$

$$\therefore B_z = \frac{\mu_0 K}{2}$$

$$\therefore \vec{B} = \frac{\mu_0 K}{2}\hat{z}$$

For a pair of sheets, one gets

$\vec{B} = 0$ outside the capacitor

$$= \mu_0 K \hat{z} = \mu_0 \sigma v \hat{z}$$

$\llcorner$ - (113)

From eqs. (113) & (112), we have

In $\bar{S}$, $\bar{E}_y = \frac{\sigma_0}{\epsilon_0}$, $\bar{E}_x = \bar{E}_z = 0$, $\vec{\bar{B}} = 0$

In S, $E_y = \gamma(\bar{E}_y + v\underset{0}{\cancel{\bar{B}_z}}) = \gamma \bar{E}_y$ (eq. (111))

$E_x = \bar{E}_x = 0$, $E_z = \gamma(\underset{0}{\cancel{\bar{E}_z}} - v\underset{0}{\cancel{\bar{B}_y}}) = 0$

$B_x = \bar{B}_x = 0$

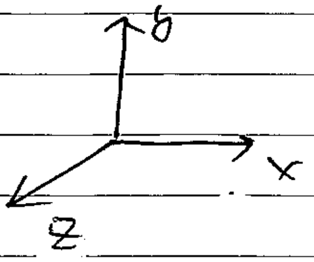
$B_y = \gamma(\overset{0}{\cancel{\bar{B}_y}} - \frac{v}{c^2}\overset{0}{\cancel{\bar{E}_z}}) = 0$

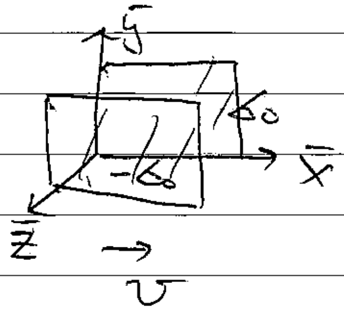
$B_z = \gamma(\underset{0}{\cancel{\bar{B}_z}} + \frac{v}{c^2}\bar{E}_y) = \gamma\frac{v}{c^2}\frac{\sigma_0}{\epsilon_0} = \frac{\sigma v}{\epsilon_0 c^2}$

$= \sigma v \mu_0$

(eq. (113))

Example





S　　　　S̄

In S̄ frame, $\vec{\bar{E}} = \frac{\sigma_0}{\epsilon_0}\hat{z}$　　$\vec{\bar{B}} = 0$

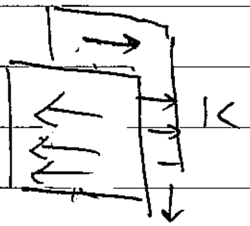
In S fram, $\vec{E} = \frac{\gamma\sigma_0}{\epsilon_0}\hat{z} = \frac{\sigma}{\epsilon_0}\hat{z}$.

$$\therefore E_z = \gamma(\bar{E}_z - v\cancel{\bar{B}_y}^{0}) = \frac{\gamma\sigma_0}{\epsilon_0}$$

$$E_y = \gamma(\cancel{\bar{E}_y} + v\cancel{\bar{B}_z})^{0} = 0$$

$$E_x = \bar{E}_x = 0$$

Current



$\vec{K} = \sigma v\hat{x}$

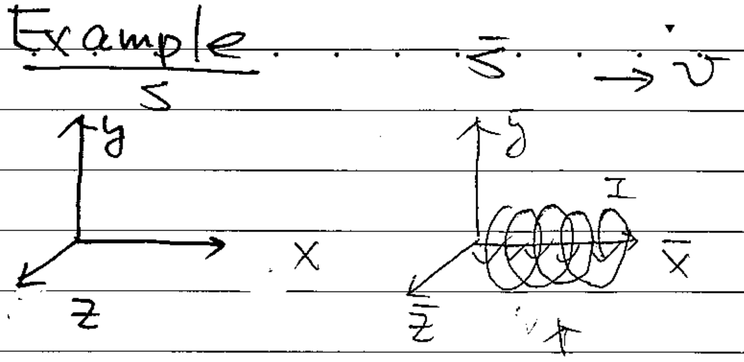
$\vec{B} = -\mu_0\sigma v\hat{y}$

$$B_y = \gamma(\cancel{\bar{B}_y}^{0} - \frac{v}{c^2}\bar{E}_z) = -\frac{\gamma v\sigma_0}{\epsilon_0 c^2} = -\mu_0\sigma v$$

$$B_z = \gamma(\cancel{\bar{B}_z}^{0} + \frac{v}{c^2}\cancel{\bar{E}_y})^{0} = 0$$

$$B_x = \bar{B}_x = 0$$

Example



$\bar n$ = # of torns per length

In $\bar S$ frame, $\bar B_x = \mu_0 \bar n I$ $\quad \bar B_y = \bar B_z = 0$

$\bar{\vec E} = 0$

In S frame, due to contraction of length

$$\therefore n = \frac{\bar n}{\sqrt{1-v^2/c^2}} = \gamma \bar n$$

Time dilation, $I = \frac{dQ}{dt} = \sqrt{1-(\frac{v}{c})^2}\,\bar I = \frac{\bar I}{\gamma}$

$$\therefore B_x = \mu_0 n I = \mu_0 \bar n \bar I = \bar B_x$$

in agreement with $\bar B_x = B_x$

Two special cases:

(i) $\bar{\vec B} = 0$ in $\bar S$

then $B_x = \bar B_x = 0$

$$B_y = \gamma(\overset{0}{\cancel{\bar B_y}} - \frac{v}{c^2}\bar E_z) = -\gamma \frac{v}{c^2} \bar E_z$$

$$B_z = \gamma(\overset{0}{\cancel{\bar B_z}} + \frac{v}{c^2}\bar E_y) = \gamma \frac{v}{c^2} \bar E_y$$

Now, $E_y = \gamma(\bar{E}_y + v\cancelto{0}{\bar{B}_z})$

$E_z = \gamma(\bar{E}_z - v\cancelto{0}{\bar{B}_y})$

∴ $B_y = -\frac{v}{c^2}E_z = \frac{1}{c^2}(v\hat{x}\times\vec{E})_y$ $\quad B_z = \frac{v}{c^2}E_y = \frac{1}{c^2}(v\hat{x}\times\vec{E})_z$

∴ $\vec{B} = \frac{1}{c^2}\vec{v}\times\vec{E}$ --- (114)

Hence a moving electric field generates $\vec{B}$!

(ii) $\vec{E}=0$ in $\bar{S}$

then $E_x = \bar{E}_x = 0$

$E_y = \gamma(\cancelto{0}{\bar{E}_y} + v\bar{B}_z) = \gamma v\bar{B}_z$

$E_z = \gamma(\cancelto{0}{\bar{E}_z} - v\bar{B}_y) = -\gamma v\bar{B}_y$

Now, $B_y = \gamma(\bar{B}_y - \frac{v}{c^2}\cancelto{0}{\bar{E}_z}) = \gamma\bar{B}_y$

$B_z = \gamma(\bar{B}_z + \frac{v}{c^2}\cancelto{0}{\bar{E}_y}) = \gamma\bar{B}_z$

∴ $E_y = vB_z = -(v\hat{x}\times\vec{B})_y$

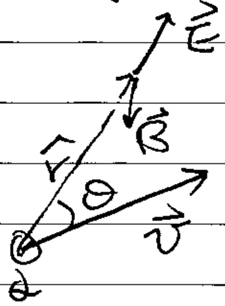
$E_z = -vB_y = -(v\hat{x}\times\vec{B})_z$

∴ $\vec{E} = -\vec{v}\times\vec{B}$ --- (115)

Hence a moving magnetic field generates an electric field $\vec{E} = -\vec{v}\times\vec{B}$

Example.

According to eqs. (97) & (98), a moving charge Q generates



$$\vec{E} = \frac{Q}{4\pi\epsilon_0\gamma^2 r^2(1-\beta^2\sin^2\theta)^{3/2}}\hat{r}$$

$$\vec{B} = \frac{\mu_0 Q v\sin\theta}{4\pi\gamma^2 r^2(1-\beta^2\sin^2\theta)^{3/2}}\hat{\phi}$$

$$\because \mu_0 = \frac{1}{\epsilon_0 c^2} \quad \vec{v}\times\hat{r} = v\sin\theta\,\hat{\phi}$$

$$\therefore \vec{B} = \frac{1}{c^2}\vec{v}\times\vec{E}$$

in agreement of eq. (114) ($\because$ in $\bar{S}$ frame, the rest frame of Q, $\vec{B}=0$)

For $v^2 \ll c^2$, one may neglect β^2

$\gamma^2 \simeq 1$

$$\therefore \vec{B} = \frac{\mu_0}{4\pi} Q \frac{\vec{v}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{\vec{J}\times\hat{r}}{r^2} \quad \cdots (116)$$

Which is exactly the formed one would apply the Biot-Savart law.