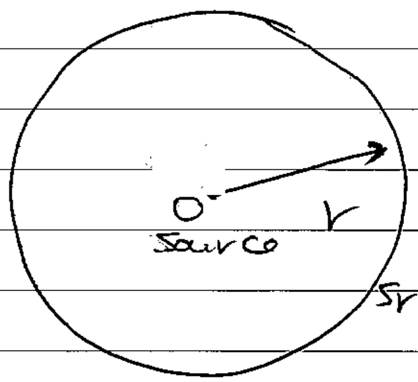
Radiation

We have learned that EM waves are solutions to the Maxwell's equations. However, how EM waves are generated is not discussed yet.

Clearly, the EM are generated by charges.

A process that generates EM waves so that energy can be transported to ∞ is call *radiation*

Specifically, we can examine the power passing the surface at $r=\infty$:



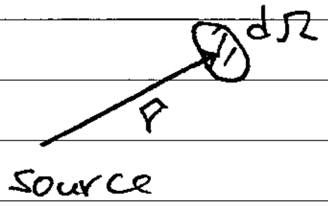
$$P(r,t) = \oint_{S_r} \vec{S}\cdot d\vec{a}$$

$$= \frac{1}{\mu_0}\oint_{S_r} (\vec{E}\times\vec{B})\cdot d\vec{a} \quad \text{--- ①}$$

The energy arrives at r at time t left the source at $t_0 = t - r/c$.

$\therefore$ Power radiated $= P_{rad}(t_0) = \lim_{r\to\infty} P(r, t_0 + r/c)$ --- ②

To get non-vanishing $P_{rad}(t_0)$,



One considers the energy flow out of the solid angle $d\Omega$ in $\hat{r}$ direction

$$dP(t) = \lim_{r\to\infty} \hat{r}\cdot\vec{S}(r,t)\, r^2 d\Omega$$

It implies $\vec{S} \propto \frac{\hat{r}}{r^2}$ as $r\to\infty$ --- ③

In this case $\frac{dP}{d\Omega} = r^2\hat{r}\cdot\frac{\vec{E}_{rad}\times\vec{B}_{rad}}{\mu_0}$

For static charges & currents

$E \sim \frac{1}{r^2}$ (Coulomb's law)

$B \sim \frac{1}{r^2}$ (Biot-Savart law)

$S \propto \frac{1}{r^4}$ $\therefore$ $P_{rad} = 0$

Therefore, static sources do not radiate energies.

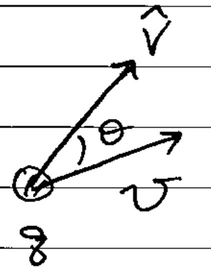
To radiate energy out, $\vec{E}$ & $\vec{B}$ has to go like $E, B \propto \frac{1}{r}$ -- ④

so that $S \sim \frac{1}{r^2}$, $P_{rad}(t_0) \neq 0$

There are two classes for $P_{rad}(t_0) \neq 0$:

(i) Charges accelerate

To escape from the static conditions, charges have to move. However, even when charges move with constant velocities,



$$\vec{E} = \frac{q}{4\pi\epsilon_0 \gamma^2 r^2 (1-\beta^2 \sin^2\theta)^{3/2}} \hat{r}$$

$$\propto \frac{1}{r^2}$$

$$\vec{B} = \frac{\mu_0 q v \sin\theta}{4\pi\gamma^2 r^2 (1-\beta^2 \sin^2\theta)^{3/2}} \hat{r} \propto \frac{1}{r^2}$$

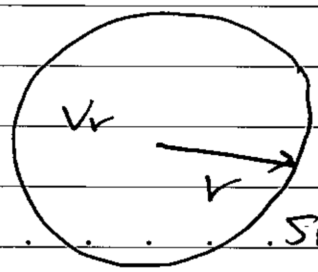
$S \propto \frac{1}{r^4}$, $P_{rad} = 0$

Hence only accelerating charges radiate!

We shall illustrate the mechanism of radiation due to acceleration later.

(ii) Charge/current distributions oscillate.

From the Poynting theorem, one has



$$\oint_{S_r} d\vec{a} \cdot \vec{S} + \int_V d^3\vec{r}\, \vec{J} \cdot \vec{E}$$

$$= -\frac{d}{dt} \int_V d^3\vec{r} \left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \right).$$

Let $U_{EM} = \frac{1}{2}\int d^3r\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$

$$\therefore \quad \underbrace{\oint_{S_r} d\vec{a}\cdot\vec{E}}_{P(r,t)} + \int_{V_r} d^3r\, \vec{J}\cdot\vec{E} = -\frac{d}{dt}U_{EM}$$

For oscillating charges & currents,

$$\vec{J}(\vec{r},t) = \vec{J}(\vec{r},\omega)\, e^{-i\omega t}$$
$$\rho(\vec{r},t) = \rho(\vec{r},\omega)\, e^{-i\omega t}$$

are periodic in time.

$\therefore$ U_{EM} is also periodic in t with period T.

$$\therefore \int_t^{t+T} dt' \frac{d}{dt'}U_{EM} = U_{EM}(t+T) - U_{EM}(t)$$
$$= 0$$

$$\therefore \langle P \rangle \equiv \frac{1}{T}\int_t^{t+T} dt'\, P(\vec{r},t')$$

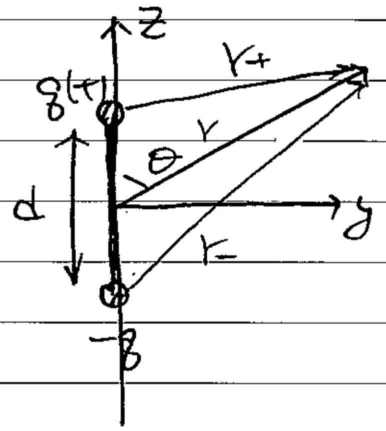
$$= \int_{V_r} d^3r \langle \vec{J}\cdot\vec{E}\rangle \neq 0 \quad \text{as } r\to\infty$$

The non-vanishing of averaged P implies that P itself doesn't vanish.

Hence when charge/current distributions oscillate, they yield radiation.

We shall start the investigation of radiation by the 2nd situation.

Electric dipole radiation



The simplest system with oscillating charges is the oscillating dipole as shown in the left figure:

two metal spheres separated by a distance d connected by a fine wire.

The charges on two spheres are opposite: $\pm q(t)$ with $q(t) = q_0 \cos\omega t$

The resulting dipole $\vec{p}(t) = p_0 \cos\omega t\,\hat{z}$ with $p_0 = q_0 d$ is oscillating.

The charge density for this oscillating dipole is $\rho(\vec{r},t) = q_0(t)\,\delta(\vec{r}-\frac{d}{2}\hat{z})$
$- q_0(t)\,\delta(\vec{r}+d/2\,\hat{z})$

L- (5)

∵ Positions of charges are fixed.

The retarded potential in this case can be easily evaluated as

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t_r)}{\mathscr{r}}\,d\tau' \qquad t_r = t - \frac{\mathscr{r}}{c}$$

$$\mathscr{r} = |\vec{r} - \vec{r}'|$$

$$= \frac{1}{4\pi\epsilon_0}\left\{ \frac{q_0 \cos\omega(t - \frac{r_+}{c})}{r_+} - \frac{q_0\cos\omega(t-\frac{r_-}{c})}{r_-} \right\}$$

$\leftarrow$ .. (6)

where $r_\pm^2 = r^2 \mp rd\cos\theta + (d/2)^2$

We shall consider the dipole as a point dipole by two approximations (i) $d \ll r$

(ii) $d \ll \frac{c}{\omega}$ $(d \ll \lambda)$

(i) implies $r_\pm = \sqrt{r^2 \mp rd\cos\theta + (d/2)^2}$

$$\simeq r\left(1 \mp \frac{d}{2r}\cos\theta\right)$$

$$\therefore \frac{1}{r_\pm} \simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) \quad \text{-- (7)}$$

$$\cos\left[\omega\left(t - \frac{r_\pm}{c}\right)\right] \simeq \cos\left[\omega(t - r/c) \pm \frac{\omega d}{2c}\cos\theta\right]$$

$$= \cos[\omega(t-r/c)]\cos\left(\frac{\omega d}{2c}\cos\theta\right)$$

$$\mp \sin[\omega(t-r/c)]\sin\left(\frac{\omega d}{2c}\cos\theta\right) \text{ - (8)}$$

(ii) implies that $d \ll \lambda$. $\cos\left(\frac{\omega d}{2c}\cos\theta\right) \simeq 1$

$\sin\left(\frac{\omega d}{2c}\cos\theta\right) \simeq 0$

$$\therefore \cos\omega(t-\frac{r_\pm}{c}) \simeq \cos[\omega(t-r/c)]$$

$$\mp \frac{\omega d}{2c}\sin[\omega(t-r/c)]\cos\theta$$

Eq (6) becomes

$$V(\vec{r},t) = \frac{q_0}{4\pi\epsilon_0}\left\{\frac{1}{r}(1+\frac{d}{2r}\cos\theta)\left[\cos\omega(t-r/c)-\frac{\omega d}{2c}\cos\theta\sin\omega(t-r/c)\right]\right.$$

$$\left.-\frac{1}{r}(1-\frac{d}{2r}\cos\theta)\left[\cos\omega(t-r/c)+\frac{\omega d}{2c}\cos\theta\sin\omega(t-r/c)\right]\right\}$$

$$=\frac{q_0}{4\pi\epsilon_0}\left\{\frac{d}{r^2}\cos\theta\cos[\omega(t-r/c)]-\frac{\omega d}{rc}\cos\theta\sin[\omega(t-r/c)]\right\}$$

$$=\frac{P_0\cos\theta}{4\pi\epsilon_0 r}\left\{-\frac{\omega}{c}\sin[\omega(t-r/c)]+\frac{1}{r}\cos[\omega(t-r/c)]\right\}$$

$$=\frac{1}{4\pi\epsilon_0}\left[\frac{\dot{\vec{p}}(t-r/c)\cdot\vec{r}}{cr^2}+\frac{\vec{p}\cdot\vec{r}}{r^3}\right] \quad \text{---} \quad (9)$$

Clearly, when $\omega\to 0$, $\vec{p}(t-r/c)\to P_0\hat{z}$

$\dot{\vec{p}}=0$ $\therefore$ $V\to \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$, one recovers

$$=\frac{1}{4\pi\epsilon_0}\frac{P_0\cos\theta}{r^2}$$

the potential of a static dipole

Eq (9) can be also obtained by taking $d\to 0$ from the beginning.

Close to the point dipole $\vec{p}$ at $r=0$,

One can neglect the retardation effect.

$\therefore$ $$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}(t)\cdot\vec{r}}{r^3} = \frac{-1}{4\pi\epsilon_0}\vec{p}(t)\cdot\nabla\frac{1}{r} \quad \text{for } r\to 0$$

$\therefore$ The charge density $\rho(\vec{r},t)$ for a point dipole satisfies

$$\frac{-\rho(\vec{r},t)}{\varepsilon_0} = \nabla^2 V(\vec{r},t)$$

$$= \frac{1}{4\pi\epsilon_0}\nabla^2\vec{p}\cdot\left(-\nabla\frac{1}{r}\right)$$

$$= -\frac{1}{4\pi\epsilon_0}\vec{p}\cdot\nabla\left(\nabla^2\frac{1}{r}\right)$$

$\because$ $\nabla^2\frac{1}{r} = -4\pi\delta(\vec{r})$

$\therefore$ $$\rho(\vec{r},t) = -\vec{p}(t)\cdot\vec{\nabla}\delta(\vec{r}) = -\vec{\nabla}\cdot\left[\vec{p}(t)\delta(\vec{r})\right] \quad \text{--- (10)}$$

Clearly, to satisfy $\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$,

the current density for a point dipole

satisfies $\vec{\nabla}\cdot\vec{J} = \dot{\vec{p}}(t)\cdot\vec{\nabla}\delta(\vec{r}) = \vec{\nabla}\cdot\left(\dot{\vec{p}}(t)\delta(\vec{r})\right)$

$\therefore$ $$\vec{J}(\vec{r},t) = \dot{\vec{p}}(t)\,\delta(\vec{r}) \quad \text{--- (11)}$$

$$= -p_0\omega\sin\omega t\,\delta(\vec{r})\,\hat{z}$$

Using eq. ⑩, we obtain

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t_r)}{\gamma}\,d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{-\vec{p}(t_r)\cdot\vec{\nabla}'\delta(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{-\vec{\nabla}'\cdot(\vec{p}(t_r)\,\delta(\vec{r}')) + (\vec{\nabla}'\cdot\vec{p}(t_r))\,\delta(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\tau'$$

$$\int \frac{\vec{\nabla}'\cdot(\vec{p}(t_r)\delta(\vec{r}'))}{|\vec{r}-\vec{r}'|}\,d\tau' = \int\left[\vec{\nabla}'\cdot\left(\frac{\vec{p}(t_r)\delta(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) - \left(\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|}\right)\vec{p}(t_r)\delta(\vec{r}')\right]d\tau'$$

$$= \cancel{\oint_{S_{r=\infty}} \frac{\vec{p}(t_r)\delta(\vec{r}')}{|\vec{r}-\vec{r}'|}\cdot d\vec{a}'} \to 0 + \int \frac{-(\vec{r}-\vec{r}')\cdot\vec{p}(t_r)}{|\vec{r}-\vec{r}'|^3}\,\delta(\vec{r}')\,d\tau'$$

$$= \frac{-\vec{p}(t-r/c)\cdot\vec{r}}{r^3}$$

$$\vec{\nabla}'\cdot\vec{p}(t_r) = \frac{\partial P_x}{\partial x'} + \frac{\partial P_y}{\partial y'} + \frac{\partial P_z}{\partial z'}$$

$$\frac{\partial P_i(t_r)}{\partial x_i'} = \frac{d P_i(t_r)}{d t_r}\frac{\partial t_r}{\partial x_i'} = \frac{-1}{c}\frac{\partial|\vec{r}-\vec{r}'|}{\partial x_i}\frac{d P_i}{d t_r} = \frac{x_i - x_i'}{c|\vec{r}-\vec{r}'|}\dot{P}_i(t_r)$$

$$\frac{dP_i}{dt_r} = \frac{d}{dt}P_i$$

$$\therefore\ \vec{\nabla}'\cdot\vec{p}(t_r) = \frac{(\vec{r}-\vec{r}')\cdot\dot{\vec{p}}(t_r)}{c|\vec{r}-\vec{r}'|}$$

$$\int \frac{(\vec{\nabla}'\cdot\vec{p}(t_r))\,\delta(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\tau' = \frac{\dot{\vec{p}}(t-r/c)\cdot\vec{r}}{c r^2}$$

$$\therefore V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\left[\frac{\dot{\vec{p}}(t-r/c)\cdot\vec{r}}{cr^2} + \frac{\vec{p}\cdot\vec{r}}{r^3}\right]$$

Which recovers eq (9).

Similarly, the vector potential for a point dipole is given by

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}', t_r)}{|\vec{r}-\vec{r}'|}d\tau'$$

$$\underset{\text{eq. (11)}}{=} \frac{\mu_0}{4\pi}\int\frac{\dot{\vec{p}}(t_r)\delta(\vec{r}')}{|\vec{r}-\vec{r}'|}d\tau'$$

$$= \frac{\mu_0}{4\pi}\frac{\dot{\vec{p}}(t-r/c)}{r} \quad \cdots \quad (12)$$

Radiation Zone

Let $\omega = \frac{2\pi}{T}$, T = characteristic time of the system

For $r \ll cT$ (i.e. $r \ll \frac{c}{\omega}$, $r \ll \lambda$) - near zone

$$\frac{\dot{\vec{p}}(t-r/c)\cdot\vec{r}}{cr^2} \sim \frac{\omega}{c}\frac{p}{r}$$

$$\ll \frac{\vec{p}\cdot\vec{r}}{r^3} \sim \frac{p}{r^2} \quad \therefore\ V \sim \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$$

$\simeq$ static potential

For $r \gg cT$, i.e. $r \gg c/\omega$, i.e. $r \gg \lambda$

(far zone, or radiation zone)

$$\frac{\dot{\vec{p}}(t-r/c)\cdot\vec{r}}{cr^2} \gg \frac{\vec{p}\cdot\vec{r}}{r^3}$$

$$\therefore V(\vec{r},t) = \frac{-p_0\omega}{4\pi\epsilon_0 c}\frac{\cos\theta}{r}\sin[\omega(t-r/c)]$$

$$\left(= \frac{1}{4\pi\epsilon_0}\frac{\dot{\vec{p}}(t-r/c)\cdot\vec{r}}{cr^2}\right) \quad \text{--- (13)}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{\dot{\vec{p}}(t-r/c)}{r}$$

$$= \frac{-\mu_0 p_0\omega}{4\pi\, r}\sin[\omega(t-r/c)]\,\hat{z} \quad \text{--- (14)}$$

From eqs (13) & (14), one can compute $\vec{E}$ & $\vec{B}$ by computing

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta}$$

$$= \frac{-p_0\omega}{4\pi\epsilon_0 c}\left\{\cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\sin[\omega(t-r/c)]\right)\hat{r} - \frac{\sin\theta}{r^2}\sin[\omega(t-r/c)]\,\hat{\theta}\right\}$$

$$\because \frac{\partial}{\partial r}\left(\frac{1}{r}\sin[\omega(t-r/c)]\right)$$

$$= \frac{-1}{r^2}\sin[\omega(t-r/c)] - \frac{\omega}{cr}\cos[\omega(t-r/c)]$$

$\therefore$ the leading term of ∇U

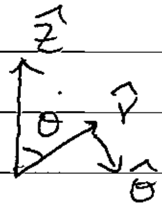
is $\nabla U(r,t) \simeq \frac{P_0 \omega \cos\theta}{4\pi\epsilon_0 c^2 r} \cos[\omega(t-r/c)]$ --- (15)

$\frac{1}{\epsilon_0 c^2} = \mu_0$

Similarly,

$$\frac{\partial \vec{A}}{\partial t} = \frac{-\mu_0}{4\pi}\frac{P_0\omega}{r}\frac{\partial}{\partial t}\sin[\omega(t-r/c)]\,\hat{z}$$

$$= -\frac{\mu_0}{4\pi}\frac{P_0\omega^2}{r}\cos\omega[t-r/c]\,\hat{z}$$



$$= -\frac{\mu_0 P_0 \omega^2}{4\pi r}\cos[\omega(t-r/c)]\,(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$

--- (16)

$$\therefore \vec{E} = -\nabla U - \frac{\partial \vec{A}}{\partial t}$$

$$= -\frac{\mu_0 P_0 \omega^2}{4\pi}\frac{\sin\theta}{r}\cos[\omega(t-r/c)]\,\hat{\theta} \quad \text{--- (17)}$$

$\frac{1}{\epsilon_0 c^2} = \mu_0$

$o(\hat{r})$ gets concelled

On the other hand, $\because \hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$,

$$\nabla \times \vec{A} = \frac{1}{r\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & 0 \end{vmatrix}$$

$A_r = \frac{-\mu_0 P_0 \omega}{4\pi r}\sin[\omega(t-r/c)] \times \cos\theta$

$A_\theta = \frac{\mu_0 P_0 \omega}{4\pi r}\sin[\omega(t-r/c)] \times \sin\theta$

$$= \frac{1}{r}\left(\frac{\partial r A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\hat{\phi}$$

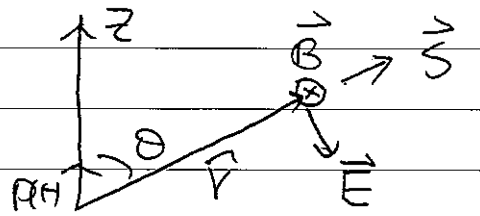
$$= \frac{\mu_0 p_0 \omega}{4\pi}\left\{\frac{1}{r}\frac{\partial}{\partial r}\sin[\omega(t-r/c)]\sin\theta + \frac{1}{r^2}\sin[\omega(t-r/c)]\underbrace{\frac{\partial}{\partial\theta}\cos\theta}_{-\sin\theta}\right\}\hat{\phi}$$

$$= -\frac{\mu_0 p_0 \omega}{4\pi r}\left\{\frac{\omega}{c}\sin\theta\cos[\omega(t-r/c)] + \frac{\sin\theta}{r}\sin[\omega(t-r/c)]\right\}\hat{\phi}$$

$$\to -\frac{\mu_0 p_0 \omega^2}{4\pi c}\frac{\sin\theta}{r}\cos[\omega(t-r/c)]\,\hat{\phi} \quad \cdots (18)$$

$$\therefore \vec{B}(\vec{r},t) = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\frac{\sin\theta}{r}\cos[\omega(t-r/c)]\,\hat{\phi} \quad \cdots (19)$$

Clearly, Eqs (17) & (19) show that both $\vec{E}$ & $\vec{B} \to O(\frac{1}{r})$ as $r\to\infty$, hence an & $E/B = c$ is satisfied
oscillating electric dipole radiates!



$(\cos\omega(t-r/c) < 0)$

The Poynting vector $\vec{S}$ determines the power radiated.
by

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\frac{\mu_0 P_0 \omega^2}{4\pi} \times \frac{\mu_0 P_0 \omega^2}{4\pi c}$$

$$\times \left(\frac{\sin\theta}{r}\right)^2 \cos^2[\omega(t - r/c)]\ \hat{r}$$

$$= \frac{\mu_0 P_0^2 \omega^4}{16\pi^2 c}\frac{\sin^2\theta}{r^2}\cos^2[\omega(t-r/c)]\ \hat{r}$$

$$\therefore \langle \vec{S} \rangle = \frac{\mu_0 P_0^2 \omega^4}{32\pi^2 c}\frac{\sin^2\theta}{r^2}\ \hat{r} \quad \text{--- (20)}$$

which decays as $\frac{1}{r^2}$ as required

Note that $\langle \vec{S} \rangle = 0$ at $\theta = 0$, $\therefore$ the intensity $I = \langle S \rangle$ profile takes the form of a donut.

The total average power $\langle P \rangle = \int_{S_{r=\infty}} \langle \vec{S} \rangle \cdot d\vec{a}$

$$= \frac{\mu_0 P_0^2 \omega^4}{32\pi^2 c}\int \frac{\sin^2\theta}{\cancel{r^2}}\cancel{r^2}\underbrace{\sin\theta\, d\theta\, d\phi}_{2\pi \times \frac{4}{3}} = \frac{\mu_0 P_0^2 \omega^4}{12\pi c} \propto \omega^4$$

--- (21)

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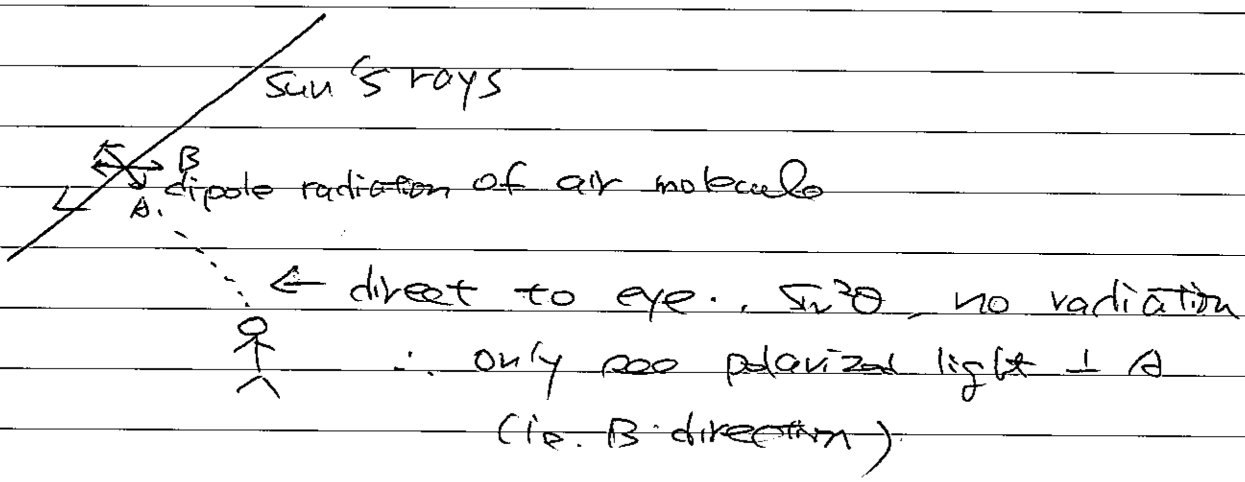
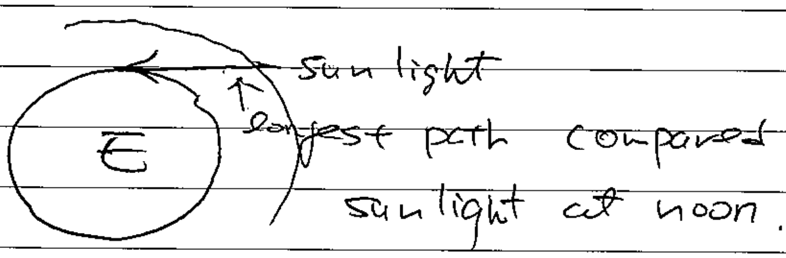
Example. When we look into the sky, it is re-radiated light of sun light that we see.

Since $\langle p \rangle \propto \omega^4$, it's more intense in the blue. That's why we see a blue sky:



On the other hand, during the sun set, the sun light that can reach us is directly from the sun, it is red because it scatters less and can reach to us (blue is removed.



Hertz's scenario of radiation

How the E & B fields go from near zone to the radiation zone is not exhibited in the above analysis. The configuration change of E & B field lines enables us to understand how the radiation is started!

For this purpose, Hertz re

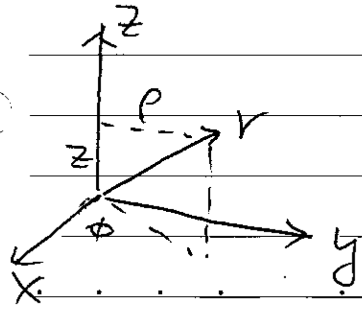
the $\vec{E}$ field via the Ampère - Maxwell's equation

$$\mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t} = \vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$$

$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

Using eq. (14) $\because \vec{A} \doteq \frac{\mu_0}{4\pi}\frac{\dot{p}(t-r/c)}{r}\hat{z}$

$$\therefore \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t} = \frac{\mu_0}{4\pi}\left[\vec{\nabla}\frac{\partial}{\partial z}\left(\frac{\dot{p}}{r}\right) - \hat{z}\,\nabla^2\frac{\dot{p}}{r}\right]_{t-r/c\ (\text{retarded})} \quad \text{...(22)}$$



Integrating over t, one gets

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left[\vec{\nabla}\frac{\partial}{\partial z}\left(\frac{p}{r}\right) - \hat{z}\nabla^2\frac{p}{r}\right]_{ret}$$

In the cylindrical coordinate, $\because \frac{\partial P}{\partial \phi} = 0$

$$\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{\partial}{\rho \partial \phi} + \hat{z}\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

$$\therefore \vec{\nabla}\frac{\partial}{\partial z}\left(\frac{P}{r}\right) - \hat{z}\nabla^2\left(\frac{P}{r}\right)$$

$$= \hat{\rho}\frac{\partial}{\partial \rho}\frac{\partial}{\partial z}\left(\frac{P}{r}\right) + \cancel{\hat{z}\frac{\partial^2}{\partial z^2}\left(\frac{P}{r}\right)}$$

$$- \hat{z}\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\frac{P}{r}\right) + \cancel{\frac{\partial^2}{\partial z^2}\left(\frac{P}{r}\right)}\right]$$

$$\therefore \vec{E}(\rho, z, t) = \frac{1}{4\pi\epsilon_0}\left[\hat{\rho}\frac{\partial}{\partial z}\frac{\partial}{\partial \rho}\left(\frac{P}{r}\right) - \hat{z}\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\frac{P}{r}\right)\right] \quad \text{--- (23)}$$

Let $R(t) \equiv -\rho\frac{\partial}{\partial \rho}\left(\frac{P(t-r/c)}{r}\right)$, we can write

$$\vec{E} = \frac{1}{4\pi\epsilon_0 \rho}\left[\frac{\partial R}{\partial \rho}\hat{z} - \frac{\partial R}{\partial z}\hat{\rho}\right] \quad \text{-- (24)}$$

However, $\because \nabla R = \frac{\partial R}{\partial \rho}\hat{\rho} + \frac{\partial R}{\partial z}\hat{z}$,

eq. (24) implies $\vec{E} \cdot \vec{\nabla} R = 0$

$\because \nabla R \perp R(\rho, z) = \text{constant}$,

$\therefore \vec{E}$ lies in the tangent plane of R.

i.e. $R(\rho, z) =$ constant are tangent to $\vec{E}$ lines!

We can use $R(\rho, z) = \text{const}$ to see the electric field lines.

Now, $p(t - r/c) = p_0 \cos\omega[t - r/c]$

$$= p_0 \cos(kr - \omega t), \quad k = \frac{\omega}{c}$$

$$\frac{\partial r}{\partial \rho} = \frac{\rho}{r} = \frac{r\sin\theta}{r} = \sin\theta, \quad r = \sqrt{\rho^2 + z^2}$$

$$\therefore R = -\rho \frac{\partial}{\partial \rho} \frac{p(t - r/c)}{r}$$

$$= -\rho \frac{\partial r}{\partial \rho} \frac{\partial}{\partial r} \left(\frac{p_0 \cos(kr - \omega t)}{r} \right)$$

$$= -r\sin^2\theta \left[\frac{-kp_0 \sin(kr - \omega t) r - p_0 \cos(kr - \omega t)}{r^2} \right]$$

$$= p_0 \sin^2\theta \frac{k\sin(kr - \omega t) r + \cos(kr - \omega t)}{r}$$

$\therefore$ **The surfaces of $R = \text{const}$** essentially describes $\vec{E}$ field lines and are given by $R = \text{const}$.

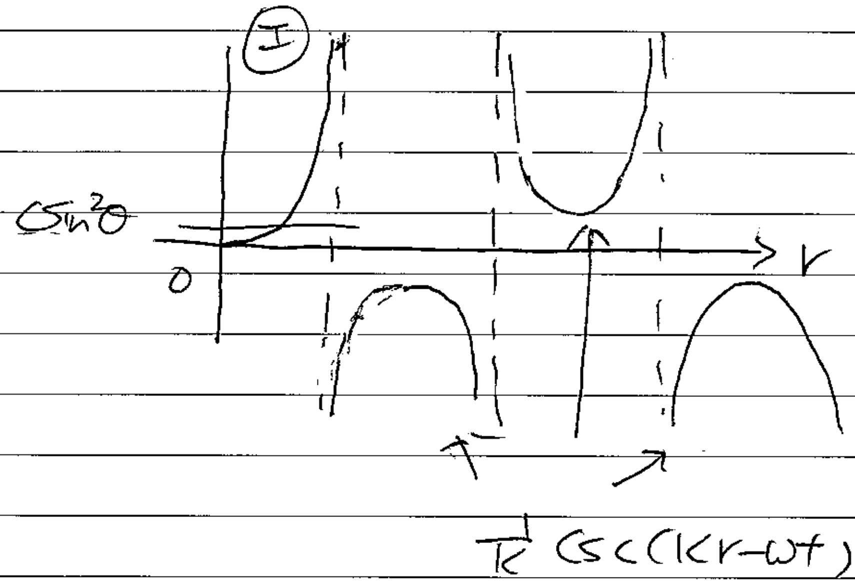
$$\therefore C \cdot \sin^2\theta = \underbrace{\frac{r}{kr\sin(kr - \omega t) + \cos(kr - \omega t)}}_{= f(r,t)}, \quad C = \text{const}$$

describes the $\vec{E}$ field lines

In eq. (25), for $r \to \infty$ ∵ kr dominates in the denominator.

∴ $C\sin^2\theta \to \frac{1}{k}\csc(kr-\omega t)$

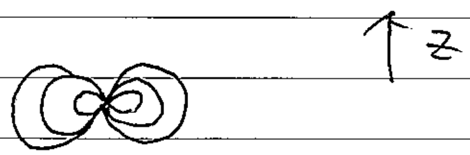
∴ $f(r,t)$ is sketched as



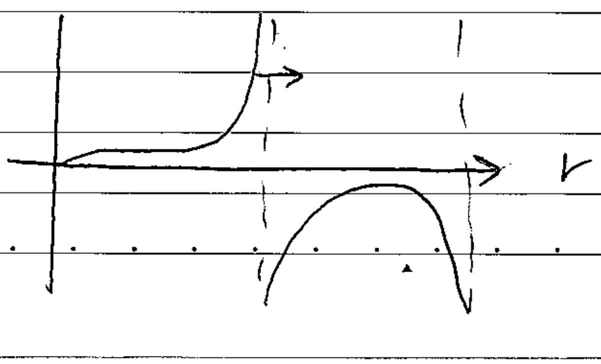
$r \to 0$, $r \simeq C\sin^2\theta\cos\omega t$

At $t=0$, $r \to 0$, $r \simeq C\sin^2\theta$ R is

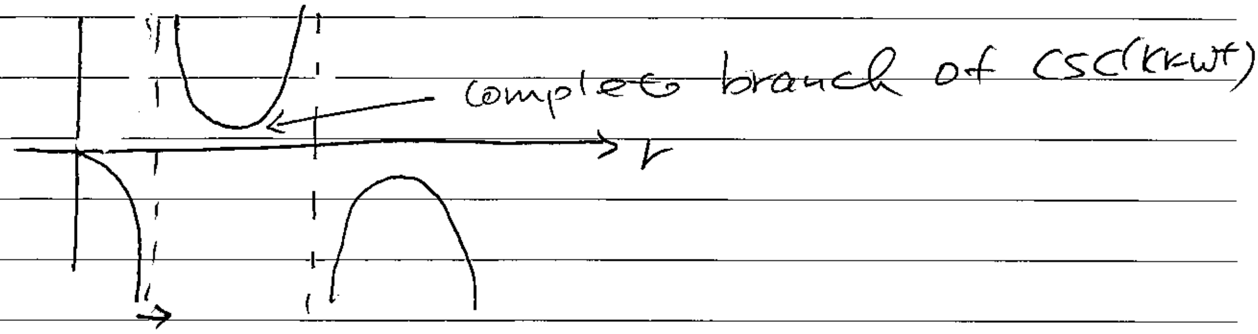
Characterized by, (I)



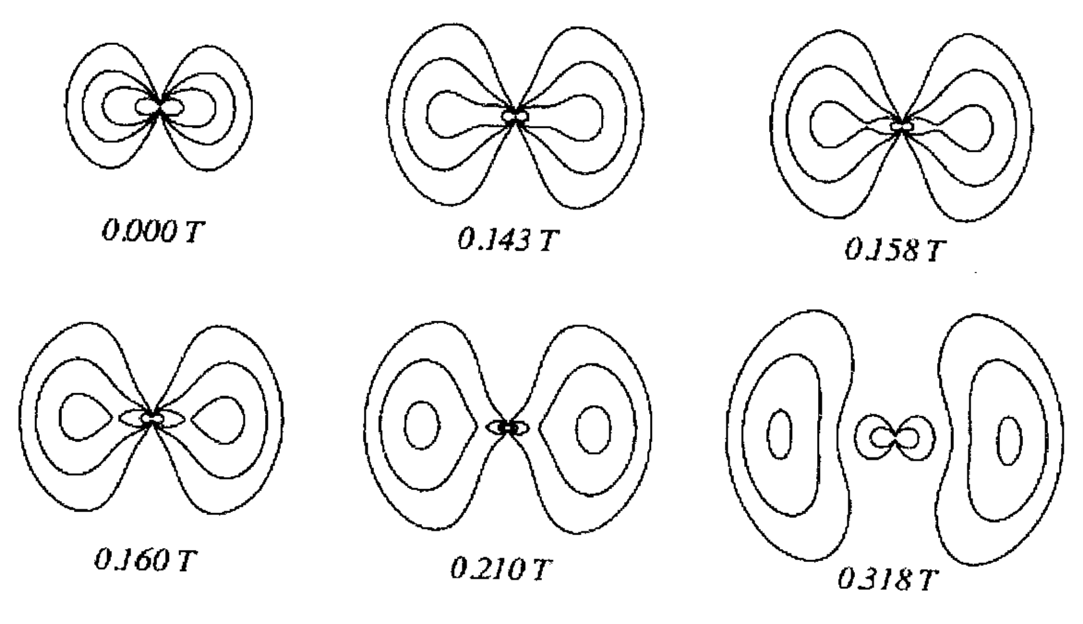
As $t\uparrow$, curves start to move toward large r.



until a complete $\overbrace{\csc(kr-\omega t)}^{\text{branch of}}$ forms



As a result of the above picture, one gets



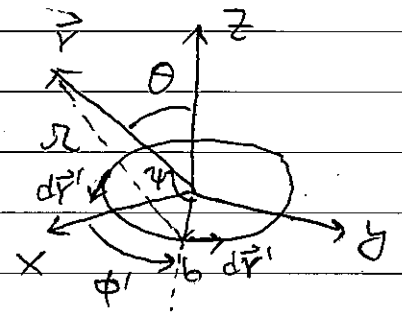
∴ We see that as t increases, reconnection of field lines occurs!

The field line loop closest to the dipole shrinks as t↑. The other loops expand and distort.

Eventually, other loops pinch. Closed loops of E lines form! These closed loops move away from the dipole while more loops are generated and ↳ carry away energy!

Magnetic dipole radiation

At the level of dipole, one can also have magnetic dipole that is oscillating and radiates.



Consider a loop of radius b on xy plane and its center is at (0,0,0) as shown in the left figure

Let the current on the loop be $I(t) = I_0 \cos\omega t$

$\therefore$ magnetic momentum $\vec{m}(t) = \pi b^2 I(t)\,\hat{z}$

$$= m_0 \cos\omega t\,\hat{z}$$

$$m_0 = I_0 \pi b^2$$

Since there is no charge density, $V=0$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{I_0 \cos[\omega(t-\mathscr{r}/c)]}{\mathscr{r}}\,d\vec{r}' \quad \text{--- (26)}$$

$$\mathscr{r} = |\vec{r}-\vec{r}'|$$

$\therefore$ If we choose coordinate system so that the point $\vec{r}$ is in XZ plane, Clearly $\mathscr{r} = |\vec{r}-\vec{r}'|$ is the same for two points $(x', y', 0)$ and $(x', -y', 0)$. The corresponding $d\vec{r}'$'s add to be along y direction.

Hence if $\vec{r}$ is in xz plane, $\vec{A} \parallel \hat{y}$

$$\vec{A}(\vec{r},t) = \frac{\mu_0 I_0 b}{4\pi}\hat{y}\int_0^{2\pi}\cos\phi' \frac{\cos[\omega(t-\frac{\mathcal{R}}{c})]}{|\vec{r}-\vec{r}'|}d\phi'$$

where $\mathcal{R} = |\vec{r}-\vec{r}'| = (r^2+b^2-2rb\cos\psi)^{1/2}$

Now $\vec{r} = r\sin\theta\,\hat{x} + r\cos\theta\,\hat{z}$

$\vec{b} = b\cos\phi'\,\hat{x} + b\sin\phi'\,\hat{y}$

$\vec{r}\cdot\vec{b} = rb\cos\psi = rb[\sin\theta\cos\phi']$

$\therefore \cos\psi = \sin\theta\cos\phi'$

$\mathcal{R} = [r^2+b^2-2rb\sin\theta\cos\phi']^{\frac{1}{2}}$

Point dipole approximation : $b \ll r$, $b \ll \frac{c}{\omega}$

$\mathcal{R} \simeq r(1-\frac{b}{r}\sin\theta\cos\phi')$

$\frac{1}{\mathcal{R}} \simeq \frac{1}{r}(1+\frac{b}{r}\sin\theta\cos\phi')$ ... (27)

$\cos[\omega(t-\frac{\mathcal{R}}{c})] \simeq \cos[\omega(t-r/c)+\frac{\omega b}{c}\sin\theta\cos\phi']$

$= \cos[\omega(t-r/c)]\cos[\frac{\omega b}{c}\sin\theta\cos\phi']$

$-\sin[\omega(t-r/c)]\sin[\frac{\omega b}{c}\sin\theta\cos\phi']$

for $b \ll \frac{c}{\omega}$, $\frac{\omega b}{c} \ll 1$ ... (28)

$\therefore \cos[\omega(t-\frac{\mathcal{R}}{c})] \simeq \cos[\omega(t-r/c)] - \frac{\omega b}{c}\sin\theta\cos\phi'\sin[\omega(t-r/c)]$

Combining eqs (27), (28) and (26), we obtain

$$\vec{A}(\vec{r},t) = \frac{\mu_0 I_0 b}{4\pi r}\hat{y}\int_0^{2\pi} d\phi' \cos\phi'$$

$$\times\left\{\cos[\omega(t-r/c)] + b\sin\theta\cos\phi'\left(\frac{\cos\omega(t-r/c)}{r} - \frac{\omega}{c}\sin\omega(t-r/c)\right)\right\}$$

$$\int_0^{2\pi}\cos\phi' d\phi' = 0 \qquad \int_0^{2\pi}\cos^2\phi' d\phi' = \pi$$

$$\therefore \vec{A}(\vec{r},t) = \frac{\mu_0 I_0 b^2}{4\pi}\frac{\sin\theta}{r}\left\{\frac{\cos[\omega(t-r/c)]}{r} - \frac{\omega}{c}\sin[\omega(t-r/c)]\right\}\hat{y}$$

This is the case when $\vec{r}$ is chosen to be in xz plane, in which $\vec{A} \parallel \hat{y}$, $\hat{y} \perp$ xz plane. In general, one replaces $\hat{y}$ by $\hat{\phi}$.

$$\therefore \vec{A}(\vec{r},t) = \frac{\mu_0 m_0}{4\pi}\left(\frac{\sin\theta}{r}\right)\left\{\frac{\cos[\omega(t-r/c)]}{r} - \frac{\omega}{c}\sin[\omega(t-r/c)]\right\}\hat{\phi} \quad \ldots (29)$$

The vector potential $\vec{A}$ can also be written as

$$\vec{A} = \vec{\nabla}\times\left[\frac{\mu_0}{4\pi}\frac{\vec{m}(t_{ret})}{r}\right] \quad \ldots (30)$$

where $\vec{m} = m(t)\hat{z} = m(t)(\cos\theta\hat{r} - \sin\theta\hat{\theta})$.

Check:

$$\vec{\nabla}\times\frac{\vec{m}}{r} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ \frac{m\cos\theta}{r} & -m\sin\theta & 0\end{vmatrix} = \frac{1}{r}\left[\frac{\partial}{\partial r}(-\sin\theta\, m) - \frac{\partial}{\partial\theta}\left(\frac{m\cos\theta}{r}\right)\right]\hat{\phi}$$

$\because \; m(t) = m_0 \cos[\omega(t - r/c)]$

$\therefore \; \frac{\partial}{\partial r} m = m_0 \frac{\omega}{c} \sin[\omega(t - r/c)]$

$\therefore \; \vec{\nabla} \times (\vec{m}/r) \Big|_{t = t_{ret}} = \frac{m_0 \sin\theta \hat{\phi}}{r} \left\{ \frac{\cos[\omega(t - r/c)]}{r} - \frac{\omega}{c} \sin[\omega(t - r/c)] \right\}$

Therefore, $\vec{A} = \vec{\nabla} \times \left[\frac{\mu_0 \vec{m}(t_{ret})}{4\pi r} \right]$

In the static limit, $\omega = 0$

Eq (29) becomes $\vec{A} = \frac{\mu_0}{4\pi} \frac{m_0 \sin\theta}{r^2} \hat{\phi}$

which reduces to the vector potential of a static current loop.

<u>Near zone & radiation zone</u>

Similar to the electric dipole radiation,

In the near zone, $r \ll \frac{c}{\omega}$

$$\vec{A}(\vec{r}, t) \approx \frac{\mu_0}{4\pi} \frac{\sin\theta}{r^2} m_0 \cos[\omega(t - r/c)]$$

$$= \frac{\mu_0}{4\pi} \frac{\vec{m}(t_{ret}) \times \vec{r}}{r^3} \quad \cdots (31)$$

In the radiation zone, $r \gg c/\omega$

$$\vec{A}(\vec{r}, t) = -\frac{\mu_0 m_0 \omega}{4\pi c} \frac{\sin\theta}{r} \sin[\omega(t - r/c)] \hat{\phi} \quad \cdots (32)$$

From $\vec{A}$, one can compute $\vec{E}$ & $\vec{B}$.

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\frac{\sin\theta}{r}\cos[\omega(t - r/c)]\,\hat{\phi} \quad \text{--- (33)}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & \underbrace{r\sin\theta}A_\phi \end{vmatrix}$$

$$= \underbrace{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}[r\sin\theta A_\phi]\,\hat{r}}_{O(\frac{1}{r^2})\text{ neglected}} - \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}(r\sin\theta A_\phi)\,r\hat{\theta}$$

$$= \frac{1}{r}\frac{\mu_0 m_0 \omega}{4\pi}\frac{\partial}{\partial r}\sin[\omega(t - r/c)]\,\hat{\theta}$$

$$= -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\frac{\sin\theta}{r}\cos[\omega(t - r/c)]\,\hat{\theta} \quad \text{--- (34)}$$

Clearly, we see $E/B = c$.

and $\vec{B} = \frac{1}{c}\hat{r}\times\vec{E}$. --- (35)

The energy flux is given by

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{\mu_0}{c}\left[\frac{m_0\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\right]^2\hat{r}$$

$$\therefore \text{The intensity} = \langle\vec{S}\rangle = \frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3}\frac{\sin^2\theta}{r^2}\,\hat{r}$$

Total power radiated $\langle P\rangle = \oint_{S_{r=\infty}} \langle\vec{S}\rangle\cdot d\vec{a}$

$$= \frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3} \times \underbrace{2\pi}_{\int d\phi} \times \underbrace{\frac{4}{3}}_{\int \sin^2\theta \, \sin\theta d\theta}$$

$$= \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$$

$\therefore \langle P \rangle \propto \omega^4$ as well.

However, in comparison to the electric dipole, one finds

$$\frac{P_{magnetic}}{P_{electric}} = \left(\frac{m_0}{P_0 c}\right)^2$$

$$\because \frac{m_0}{P_0 c} = \frac{\pi b^2 I_0}{q_0 d c}, \quad I_0 = q_0 \omega, \quad b \sim d$$

for comparison. set $d = \pi b$

$$\frac{P_{magnetic}}{P_{electric}} = \left(\frac{\omega b}{c}\right)^2 \ll 1$$

$\therefore$ Electric dipole radiation dominates!

Radiation from an arbitrary source

For general current and charge distributions,

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d\tau' \frac{\vec{J}(\vec{r}', t_r)}{|\vec{r}-\vec{r}'|} \qquad t_r = t - \frac{|\vec{r}-\vec{r}'|}{c}$$

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d\tau' \frac{\rho(\vec{r}', t_r)}{|\vec{r}-\vec{r}'|}.$$

For currents & charges localized in space,

One can take $r \gg r'$

$$\because |\vec{r}-\vec{r}'| = (r^2 - 2\vec{r}\cdot\vec{r}' + r'^2)^{\frac{1}{2}}$$

$$= r\left[1 - 2\frac{\hat{r}\cdot\vec{r}'}{r} + \left(\frac{r'}{r}\right)^2\right]^{\frac{1}{2}}$$

$$= r\left\{1 - \left(\frac{r'}{r}\right)(\hat{r}\cdot\hat{r}') + O\left(\frac{r'}{r}\right)^2\right\}$$

$\therefore$ To order of $O\left(\frac{1}{r}\right)$

$$\frac{\vec{J}(\vec{r}', t_r)}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\vec{J}\left(\vec{r}', t - r/c + \frac{\hat{r}\cdot\vec{r}'}{c}\right)$$

hence in the radiation zone $\vec{A} \to \vec{A}_{rad}$, $V \to V_{rad}$

$$\vec{A}_{rad}(\vec{r},t) = \frac{\mu_0}{4\pi r}\int d\tau' \, \vec{J}\left(\vec{r}', t - r/c + \frac{\hat{r}\cdot\vec{r}'}{c}\right) \quad \cdots (36)$$

$$V_{rad}(\vec{r},t) = \frac{1}{4\pi\epsilon_0 r}\int d\tau' \quad \left(\vec{r}', t - r/c + \frac{\hat{r}\cdot\vec{r}'}{c}\right) \quad \cdots (37)$$

$\therefore$ In the radiation zone, the retarded time become $t_r^* = t - r/c + \hat{r}\cdot\vec{r}'/c$ --- (38)

and $\nabla t_r^* = -\frac{1}{c}\nabla r + \frac{1}{c}\nabla(\hat{r}\cdot\vec{r}')$

$$= -\frac{\hat{r}}{c} - \frac{\hat{r}}{c}\times(\hat{r}\times\frac{\vec{r}'}{r}) \approx -\frac{\hat{r}}{c} \quad \text{--- (39)}$$

$\therefore \vec{B}_{rad} = \vec{\nabla}\times\vec{A}_{rad}$

$$= \frac{\mu_0}{4\pi}\vec{\nabla}\times\left(\frac{1}{r}\int d\tau'\,\vec{J}\right)$$

$$= \underbrace{\cancel{\frac{\mu_0}{4\pi}(\nabla\frac{1}{r})\times\int d\tau'\,\vec{J}}}_{O(\frac{1}{r^2})} + \frac{\mu_0}{4\pi}\frac{1}{r}\int d\tau'\,\vec{\nabla}\times\vec{J}$$

$$\because \frac{\partial}{\partial x_i}J_j(r', \underbrace{t - r/c + \frac{\hat{r}\cdot\vec{r}'}{c}}_{t_r^*})$$

$$= \frac{\partial J_j}{\partial t_r^*}\frac{\partial t_r^*}{\partial x_i} = -\frac{(\hat{r})_i}{c}\frac{\partial J_j}{\partial t_r^*}$$

$$\therefore \vec{\nabla}\times\vec{J} = -\frac{1}{c}\hat{r}\times\frac{\partial\vec{J}}{\partial t_r^*}$$

$$\vec{B}_{rad}(\vec{r},t) = -\frac{\mu_0}{4\pi c}\frac{\hat{r}}{r}\times\int d\tau'\,\frac{\partial}{\partial t}\vec{J}(\vec{r}',t)\Big|_{t_r^*} \quad \text{--- (40)}$$

Similarly, $= -\frac{1}{c}\hat{r}\times\frac{\partial\vec{A}_{rad}}{\partial t}$

$$\because \vec{E}_{rad}(\vec{r},t) = -\nabla V_{rad} - \frac{\partial\vec{A}_{rad}}{\partial t}$$

$$\nabla V_{rad} = \underbrace{\cancel{\frac{1}{4\pi\epsilon_0}(\nabla\frac{1}{r})\int d\tau'\,\rho}}_{O(\frac{1}{r^2})} + \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int d\tau'\,\nabla\rho$$

$$\nabla \rho(\vec{r}', t_r^*) = \frac{\partial \rho(\vec{r}', t_r^*)}{\partial t_r^*} \nabla t_r^* = -\frac{\hat{r}}{c} \frac{\partial \rho(\vec{r}', t_r^*)}{\partial t_r^*}$$

$$\frac{\partial \vec{J}(\vec{r}', t_r^*)}{\partial t} = \frac{\partial \vec{J}(\vec{r}', t_r^*)}{\partial t_r^*}$$

$$\therefore \vec{E}_{rad}(\vec{r}, t) = \frac{\hat{r}}{4\pi\epsilon_0 c r} \int dz' \frac{\partial}{\partial t_r^*} \rho(\vec{r}', t_r^*)$$

$$- \frac{\mu_0}{4\pi} \int dz' \frac{\partial}{\partial t_r^*} \vec{J}(\vec{r}', t_r^*) \quad \text{-- (41)}$$

Now, from the charge conservation,

$$\frac{\partial \rho(\vec{r}', t_r^*)}{\partial t_r^*} = -\Big[\underbrace{\vec{\nabla}'}_{\text{acting only on } \vec{r}'} \cdot \vec{J}(\vec{r}', t')\Big]\Big|_{t' = t_r^*}$$

On the other hand, the total derivative on $\vec{r}'$

$$\vec{\nabla}' \cdot \vec{J}(\vec{r}', \underbrace{t_r^*}_{\substack{\text{depends on } \vec{r}' \\ \text{as well}}}) = \left[\vec{\nabla}' \cdot \vec{J}(\vec{r}', t')\right]\Big|_{t' = t_r^*}$$

$$+ \frac{\partial \vec{J}(\vec{r}', t_r^*)}{\partial t_r^*} \cdot \vec{\nabla}' t_r^*$$

$$= \left(\vec{\nabla}' \cdot \vec{J}(\vec{r}', t')\right)\Big|_{t' = t_r^*} + \frac{\hat{r}}{c} \cdot \frac{\partial \vec{J}(\vec{r}', t_r^*)}{\partial t_r^*}$$

$$\therefore \frac{\partial \rho(\vec{r}', t_r^*)}{\partial t_r^*} = -\vec{\nabla}' \cdot \vec{J}(\vec{r}', t_r^*) + \frac{\hat{r}}{c} \cdot \frac{\partial \vec{J}(\vec{r}', t_r^*)}{\partial t_r^*} \quad \text{-- (42)}$$

The divergent term $\vec{\nabla}' \cdot \vec{J}(\vec{r}', t_r^*)$

can be integrated to $S_{r=\infty}$:

$$\int d\tau' \, \vec{\nabla}' \cdot \vec{J}(\vec{r}', t_r^*) = \oint_{S_{r=\infty}} d\vec{a}' \cdot \vec{J}(\vec{r}', t_r^*) = 0$$

$\because \hat{r}(\hat{r} \cdot \vec{a}) - \vec{a} = \hat{r} \times (\hat{r} \times \vec{a})$ for any

vector $\vec{a}$, therefore, eq. (41) becomes

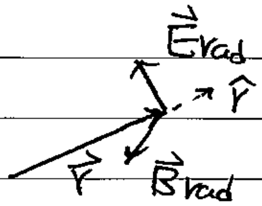
$$\vec{E}_{rad}(\vec{r}, t) = \hat{r} \times \left[\frac{\mu_0}{4\pi} \frac{\hat{r}}{r} \times \int d\tau' \frac{d}{dt} \vec{J}(\vec{r}', t - r/c + \frac{\hat{r} \cdot \vec{r}'}{c}) \right]$$

$\frac{1}{\epsilon_0 c^2} = \mu_0$ ... (43)

Comparing eqs (43) & (40), we get

$$\vec{B}_{rad} = \frac{1}{c} \hat{r} \times \vec{E}_{rad} \quad \cdots \quad (44)$$

is satisfied, and both $\vec{E}_{rad}$ & $\vec{B}_{rad} \perp \hat{r}$.



$$\because \vec{S} = \frac{1}{\mu_0} \vec{E}_{rad} \times \vec{B}_{rad}$$

$$= \frac{1}{\mu_0} E_{rad} B_{rad} \hat{r}$$

$$= \frac{c}{\mu_0} |\vec{B}_{rad}|^2 \hat{r}$$

$$\therefore \frac{dP}{d\Omega} = r^2 \frac{c}{\mu_0} |\vec{B}_{rad}|^2 = \frac{1}{\mu_0 c} \left| \hat{r} \times \frac{\partial \vec{A}_{rad}}{\partial t} \right|^2 \quad \cdots \quad (45)$$

Frequency domain

For a fixed angular frequency,

$$\vec{J}(\vec{r},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \vec{J}(\vec{r},\omega)\, e^{-i\omega t}$$

$$\vec{A}_{rad}(\vec{r},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \vec{A}_{rad}(\vec{r},\omega)\, e^{-i\omega t}$$

Eqs. 36 & 37 become

$$\therefore \vec{J}(\vec{r}', t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c})$$

$$= \int \frac{d\omega}{2\pi}\, \vec{J}(\vec{r}',\omega)\, e^{-i\omega(t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c})}$$

$$= \int \frac{d\omega}{2\pi} \left[\vec{J}(\vec{r}',\omega)\, e^{ikr}\, e^{-i\frac{\omega}{c}\hat{r}\cdot\vec{r}'}\right] \times e^{-i\omega t} \qquad k \equiv \frac{\omega}{c}$$

Let $\vec{k} \equiv \frac{\omega}{c}\hat{r} = k\hat{r}$

$$\therefore \vec{J}(\vec{r}',\omega) \rightarrow \vec{J}(\vec{r}',\omega)\, e^{ikr}\, e^{-i\vec{k}\cdot\vec{r}'}$$

$$\rho(\vec{r}',\omega) \rightarrow \rho(\vec{r}',\omega)\, e^{ikr}\, e^{-i\vec{k}\cdot\vec{r}'} \quad \text{--- (46)}$$

$$\therefore \text{Eq. (36)} \Rightarrow \vec{A}_{rad}(\vec{r},\omega) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r} \underbrace{\int d^3r'\, \vec{J}(\vec{r}',\omega)\, e^{-i\vec{k}\cdot\vec{r}'}}_{\vec{J}(\vec{k},\omega)}$$

$$V_{rad}(\vec{r},\omega) = \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r} \underbrace{\int d^3r'\, \rho(\vec{r}',\omega)\, e^{-i\vec{k}\cdot\vec{r}'}}_{\rho(\vec{k},\omega)}$$

$$\text{Eq. (40)} \Rightarrow \vec{B}_{rad}(\vec{r},\omega) = \frac{i\omega}{c}\, \hat{r} \times \vec{A}_{rad}(\vec{r},\omega) \quad \text{--- (47)}$$

$$\vec{E}_{rad} = -c\hat{r} \times \vec{B}_{rad}$$

$$= -i\omega\, \hat{r} \times [\hat{r} \times \vec{A}_{rad}(\vec{r},\omega)] \quad \text{--- (48)}$$

$$\therefore \left\langle \frac{dP}{d\Omega} \right\rangle_{(45)} = \frac{\omega^2}{2\mu_0 c} \left|\hat{r} \times \vec{A}_{rad}(r,\omega)\right|^2$$

(For a fixed ω, $\langle \cos^2\omega t\rangle = \frac{1}{2}$)

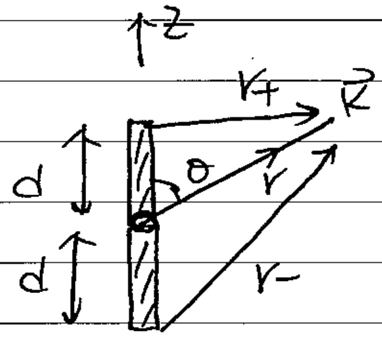
Using eq. (46)

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\omega^2}{2\mu_0 c}\left(\frac{\mu_0}{4\pi}\right)^2 |\hat{r}\times\vec{J}(\vec{k},\omega)|^2$$

$$= \frac{\mu_0\omega^2}{32\pi^2}|\vec{J}_\perp(\vec{k},\omega)|^2 \quad \text{-- (49)}$$

$\vec{J}_\perp \perp \hat{r}$

Example Dipole Antenna



For finite d of the case for the electric dipole, if the current is driven by an input voltage at the center with time dependence $e^{-i\omega t}$,

$I(z,t)$ in the wire

$$= \underbrace{I_0 \sin k(d-|z|)}_{I(z,\omega)} e^{-i\omega t} \quad -d\le z\le d$$

where $k = \omega/c$.

$$\vec{J}(\vec{k},\omega) \Rightarrow I(\vec{k},\omega) = \int_{-d}^{d} dz'\, I(z',\omega)\underbrace{e^{-i\vec{k}\cdot\vec{r}'}}_{e^{-ikz'\cos\theta}}$$

$$= \int_{-d}^{d} dz'\, I_0 \sin k(d-|z|)\, e^{-ikz'\cos\theta}$$

Use the identity $\int dz\, e^{az}\sin(bz+c) = \frac{e^{az}}{a^2+b^2}[a\sin(bz+c) - b\cos(bz+c)]$

we find

$$I(\vec{R},\omega)=\frac{I_0 e^{-ikz'\cos\theta}}{-k^2\cos^2\theta+k^2}\Big[-ik\cos\theta\sin k(d-z') + k\cos k(d-z')\Big]\Big|_0^d$$

$$+\frac{I_0 e^{-ikz'\cos\theta}}{-k^2\cos^2\theta+k^2}\Big[-ik\cos\theta\sin k(d+z') - k\cos k(d+z')\Big]\Big|_{-d}^0$$

$$=\frac{I_0}{k\sin^2\theta}\Big[ke^{-ikd\cos\theta}+\cancel{i\cos\theta\sin kd}-\cos kd\Big]$$

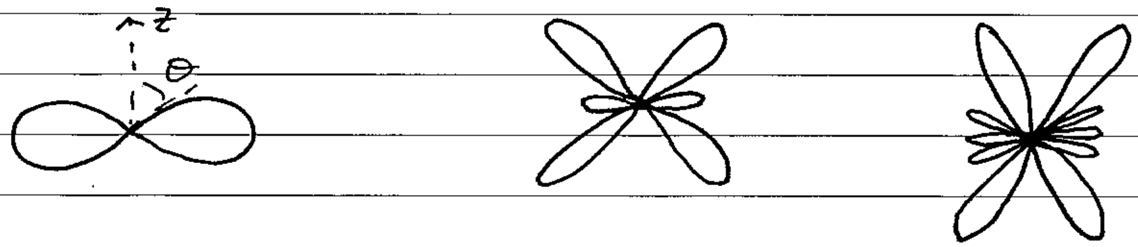
$$+\frac{I_0}{k\sin^2\theta}\Big[\cancel{-i\cos\theta\sin kd}-\cos kd+ke^{ikd\cos\theta}\Big]$$

$$=\frac{2I_0}{k\sin^2\theta}\Big[\cos(kd\cos\theta)-\cos kd\Big]$$

$\therefore$ Eq. (46) $\Rightarrow$ $$\vec{A}_{rad}(\vec{r},\omega)=\frac{\mu_0 I_0}{2\pi}\frac{e^{ikr}}{kr}\frac{\cos(kd\cos\theta)-\cos kd}{\sin^2\theta}$$

$$\left\langle\frac{dP}{d\Omega}\right\rangle=\frac{\mu_0 I_0^2 \underset{\uparrow\ \frac{\omega}{k}=c}{c}}{8\pi^2}\left(\frac{\cos(kd\cos\theta)-\cos kd}{\sin^2\theta}\right)^2$$

$kd=\pi$ ($d=\lambda/2$) $\quad\quad d=3\lambda/2 \quad\quad d=5\lambda/2$



Multipole radiation

From $\vec{A}_{rad}(\vec{r},t) = \frac{\mu_0}{4\pi r}\int d\tau' \, \vec{J}(\vec{r}', t - r/c + \frac{\hat{r}\cdot\vec{r}'}{c})$

and $V_{rad}(\vec{r},t) = \frac{1}{4\pi\epsilon_0 r}\int d\tau' \rho(\vec{r}', t - r/c + \frac{\hat{r}\cdot\vec{r}'}{c})$.

one obtains the multipole expansion for radiation fields by expanding

$$\rho(\vec{r}', t - r/c + \frac{\hat{r}\cdot\vec{r}'}{c}) = \rho(\vec{r}', \underbrace{t - r/c}_{t_0}) + \dot{\rho}(\vec{r}', t_0)\frac{\hat{r}\cdot\vec{r}'}{c}$$

$$+ \frac{1}{2}\ddot{\rho}\left(\frac{\hat{r}\cdot\vec{r}'}{c}\right)^2 + \frac{1}{3!}\dddot{\rho}\left(\frac{\vec{r}'\cdot\hat{r}}{c}\right)^3 + \cdots \quad (50)$$

$$\vec{J}(\vec{r}', t - r/c + \frac{\hat{r}\cdot\vec{r}'}{c}) = \vec{J}(\vec{r}', t - r/c) + \frac{\hat{r}\cdot\vec{r}'}{c}\frac{\partial}{\partial t}\vec{J}(\vec{r}', t - r/c)$$

$$+ \frac{1}{2}\left(\frac{\hat{r}\cdot\vec{r}'}{c}\right)^2 \frac{\partial^2}{\partial t^2}\vec{J}(\vec{r}', t - r/c) + \cdots \quad (51)$$

$$\therefore \vec{A}_{rad}(\vec{r},t) = \frac{\mu_0}{4\pi r}\int d\tau' \, \vec{J}(\vec{r}', t - r/c)$$

$$+ \frac{\mu_0}{4\pi r}\int d\tau' \, \frac{\hat{r}\cdot\vec{r}'}{c}\frac{\partial}{\partial t}\vec{J}(\vec{r}', t - r/c)$$

$$+ \cdots \quad (52)$$

$$V_{rad}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int d\tau' \, \rho(\vec{r}', t - r/c)$$

$$+ \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int d\tau' \, \frac{\hat{r}\cdot\vec{r}'}{c}\dot{\rho}(\vec{r}', t - r/c) + \cdots$$

$$(53)$$

Eqs. (52) & (53) are the multipole expansions for radiation fields.

In the expansion (50) & (51), if we set

$L =$ characteristic size of source

$T =$ time scale for changes of source,

we see eq. (53) implies that

the nth term $\sim \left(\frac{\vec{r}\cdot\vec{r}'}{c}\right)^n \frac{\partial^n}{\partial t^n}\rho$

is the order of $\left(\frac{L}{cT}\right)^n \rho$

Hence the expansion is according to $\left(\frac{L}{cT}\right)^n$

$= \left(\frac{\omega}{c}L\right)^n$ --- (54)

For $\vec{A}_{rad}$ in eq. (52), $\because \mu_0 = \frac{1}{\epsilon_0 c^2}$,

$[\vec{J}] \sim [\rho v] \sim \frac{L}{T}\rho$, $\therefore$ the nth term

in eq. (52) $\sim \frac{1}{c^2}\frac{L}{T}\cdot\left(\frac{L}{cT}\right)^n = \frac{1}{c}\left(\frac{L}{cT}\right)^{n+1}$

$= \frac{1}{c}\left(\frac{\omega}{c}L\right)^{n+1}$ --- (55)

Hence for $L \ll \frac{c}{\omega}$ (or $c\tau$, λ), one can neglect higher order terms!

General
Electric dipole radiation

From the above, one finds that the lowest order term is $O(\frac{\omega}{c}L)$

∴ One sets $n=1$ for eq. (53) & $n=0$ for eq. (54).

Hence

$$E \sim O(\tfrac{1}{r^2})$$

$$V_{rad}^{dipole} = \frac{1}{4\pi\epsilon_0 r}\cancel{\int \rho(\vec{r}', t_0)\, d\tau'}$$

$$+ \frac{1}{4\pi\epsilon_0 r}\frac{\hat{r}}{c}\cdot\frac{d}{dt}\int \vec{r}'\rho(\vec{r}', t_0)\, d\tau' \quad \text{--- (56)}$$

$t_0 = t - r/c$.

$$A_{rad}^{dipole} = \frac{\mu_0}{4\pi}\frac{1}{r}\int \vec{J}(\vec{r}', t_0)\, d\tau' \quad \text{--- (57)}$$

The first term in $V = \frac{1}{4\pi\epsilon_0}\frac{1}{r}Q$, Q = total charge = const (conserved)

∴ The corresponding $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$ is not a radiation field, ∴ we can drop it.

∵ $\int \vec{r}'\rho(\vec{r}', t_0)\, d\tau' = \vec{p}(t - r/c)$ = dipole moment of charge distribution at time $t - r/c$ --- (58)

$$\therefore\ U_{rad}^{dipole} = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\dot{\vec{p}}(t-r/c)}{rc} \quad \cdots (59)$$

Now, in general $\vec{J}(\vec{r}',t) = \sum_k q_k\, \delta(\vec{r}'-\vec{r}_k(t))\, \vec{v}_k$

$$\therefore \int d\tau'\, \vec{J}(\vec{r}',t) = \sum_k q_k \vec{v}_k = \frac{d}{dt}\sum_k q_k \vec{r}_k(t)$$

$$= \frac{d}{dt}\int d\tau'\, \underbrace{\sum_k q_k \delta(\vec{r}'-\vec{r}_k(t))}_{\rho(\vec{r}',t)}\, \vec{r}'$$

$$= \frac{d}{dt}\int d\tau'\, \vec{r}'\rho(\vec{r}',t) = \frac{d}{dt}\vec{p}(\vec{r}',t)$$

$\therefore$ Eq. (17) becomes

$$\vec{A}_{rad}^{dipole} = \frac{\mu_0}{4\pi}\frac{\dot{\vec{p}}(\vec{r}, t-r/c)}{r} \quad \cdots (60)$$

which is the same as eq (12) for an oscillating point dipole.

From eqs (59) & (60), one can compute $\vec{E}$ & $\vec{B}$ for general electric dipole radiation.

$$\because \nabla t_0 = -\frac{1}{c}\nabla r = \frac{-\hat{r}}{c}$$

$$\therefore \nabla U_{rad}^{dipole} = \frac{1}{4\pi\epsilon_0}\nabla\frac{\hat{r}\cdot\dot{\vec{p}}(t_0)}{rc} \underset{\text{to } O(\frac{1}{r})}{=} \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\ddot{\vec{p}}(t_0)}{rc}\nabla t_0$$

$$= \frac{1}{4\pi\epsilon_0 c^2} \frac{\hat{r}\cdot\ddot{\vec{p}}(t-r/c)}{r}\hat{r}$$

$$= \frac{-\mu_0}{4\pi} \frac{(\hat{r}\cdot\ddot{\vec{p}}(t-r/c))}{r}\hat{r}$$

$$\frac{\partial \vec{A}^{dip}_{rad}}{\partial t} = \frac{\mu_0}{4\pi} \frac{\ddot{\vec{p}}(t-r/c)}{r}$$

$$\therefore \vec{E}^{dipole}_{rad}(r,\theta,t) = -\nabla U^{dipole}_{rad} - \frac{\partial \vec{A}^{dipole}_{rad}}{\partial t}$$

$$= \frac{\mu_0}{4\pi r}\left[(\hat{r}\cdot\ddot{\vec{p}}(t-r/c))\hat{r} - \ddot{\vec{p}}(t-r/c)\right]$$

$$= \frac{\mu_0}{4\pi r}\left[\hat{r}\times(\hat{r}\times\ddot{\vec{p}})\right]_{t-r/c} \quad \text{--- (61)}$$

Similarly,

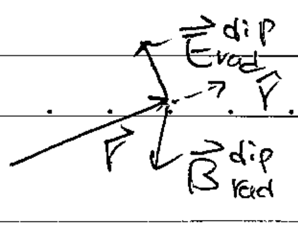
$$\vec{\nabla}\times\vec{A}^{dip}_{rad} \underset{\text{to } O(\frac{1}{r})}{=} \frac{\mu_0}{4\pi r}\vec{\nabla}\times\dot{\vec{p}}(t-r/c)$$

$$= \frac{\mu_0}{4\pi r}(\nabla t_0)\times\ddot{\vec{p}}(t_0)$$

$$= -\frac{\mu_0}{4\pi r c}\hat{r}\times\ddot{\vec{p}}(t-r/c)$$

$$\therefore \vec{B}^{dipole}_{rad}(\vec{r},t) = -\frac{\mu_0}{4\pi r c}\hat{r}\times\ddot{\vec{p}}(t-r/c) \quad \text{-- (62)}$$

$$\therefore \vec{E}^{dipole}_{rad} = -c(\hat{r}\times\vec{B}^{dipole}_{rad})$$



$\therefore$ Similar to eq. (45)

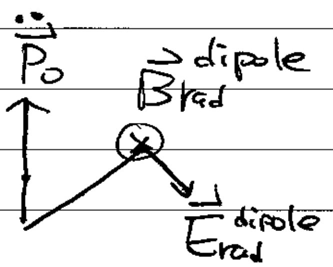
$$\vec{S} = \frac{c}{\mu_0}\left|B^{dipole}_{rad}\right|^2 \hat{r}$$

$$\frac{dP}{d\Omega} = r^2 \frac{c}{\mu_0}\left|B^{dipole}_{rad}\right|^2 \quad \cdots \quad (63)$$

If we choose $\ddot{\vec{p}}(t_0)$ aligned with z axis,

then $\vec{E}^{dipole}_{rad}(r,\theta,t) = \frac{\mu_0 \ddot{p}(t_0)}{4\pi}\frac{\sin\theta}{r}\hat{\theta}$

$\vec{B}^{dipole}_{rad}(r,\theta,t) = \frac{\mu_0 \ddot{p}(t_0)}{4\pi c}\frac{\sin\theta}{r}\hat{\phi}$.



The Poynting vector

$$\vec{S} = \frac{1}{\mu_0}\vec{E}^{dipole}_{rad} \times \vec{B}^{dipole}_{rad}$$

$$= \frac{\mu_0}{16\pi^2 c}\left(\ddot{p}(t_0)\right)^2 \frac{\sin^2\theta}{r^2}\hat{r}$$

$$\therefore \; P(r,t) = \oint \vec{S}\cdot d\vec{a} = \frac{\mu_0}{16\pi^2 c}\left(\ddot{p}(t_0)\right)^2 \times \int_0^{2\pi} d\phi \overbrace{\int_0^{\pi} \sin^2\theta \sin\theta \, d\theta}^{\frac{4}{3}}$$

$$= \frac{\mu_0}{6\pi c}\left[\ddot{p}(t - r/c)\right]^2$$

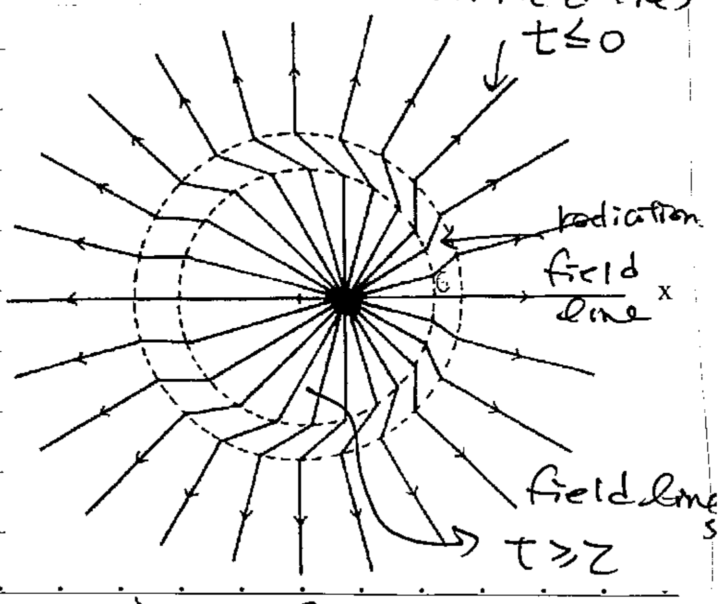
$$\therefore \; P_{rad}(t_0) = \frac{\mu_0}{6\pi c}\left[\ddot{p}(t_0)\right]^2$$

The above are lowest-order radiation — the electric dipole radiation. One can go to higher order $\left(\frac{\omega}{c}L\right)^2$ . . .

These are magnetic dipole radiation & electric quadrupole radiation, and magnetic quadrupole / electric octopole radiation . . . and so on. They become decreasingly small. We shall go into details for computing their contributions.

Radiation by point charges

Simple picture of the radiation for an accelerated charge.



To see how the radiation arises, consider a point charge q which is at rest at 0 for $t<0$.

For $t \ge 0$, the charge particle is suddenly accelerated for $0 \le t \le \tau$ with acceleration a. along x direction. After τ, it moves with constant velocity $\vec{v} = a\tau\hat{x}$. $a\tau \ll c$

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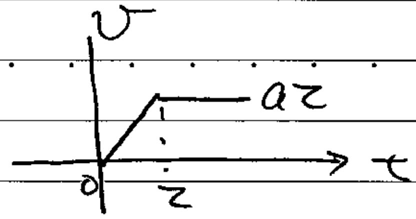


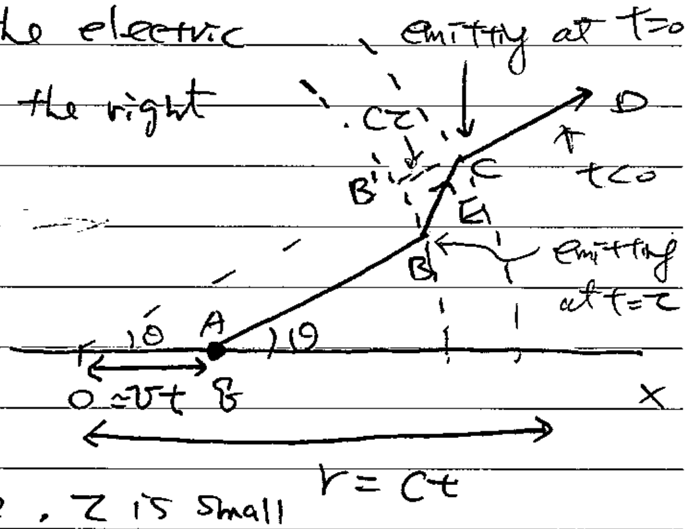
~~The electric field lines~~ are along radial ~~direction~~ for $t \leq 0$ emitting from $x=0$

and are along the radial direction emitting from the charge at $x = \frac{1}{2}a\tau^2$, as shown in the figure.

Because electric fields are continuous, one has to connect these two set of lines.

Clearly, one sees that there is a region that the field lines change direction. Thes are the radiation field lines.

Consider a time $t \gg \tau$, the electric ~~field line is shown in the right~~ figure. The



of the charge is at $x \simeq vt$.

$\because$ $r = ct$ is very large, τ is small

$\therefore$ $\overline{AB}$ is almost parallel to $\overline{CD}$!

$\overline{BB'} = vt\sin\theta$ is along the θ direction.

$\overline{B'C}$ is the distance propagate of the radiation field emitted at $t=0$

$\therefore \overline{B'C} = c\tau$

For $v \ll c$, the radial field can be approximated by static fields.

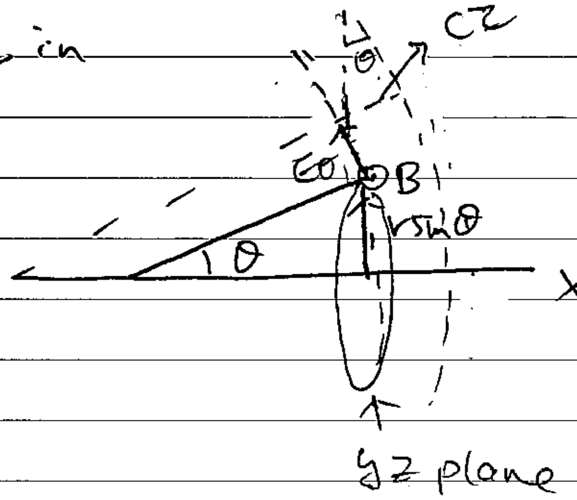
$\therefore E_r = \dfrac{q}{4\pi\epsilon_0 r^2}$

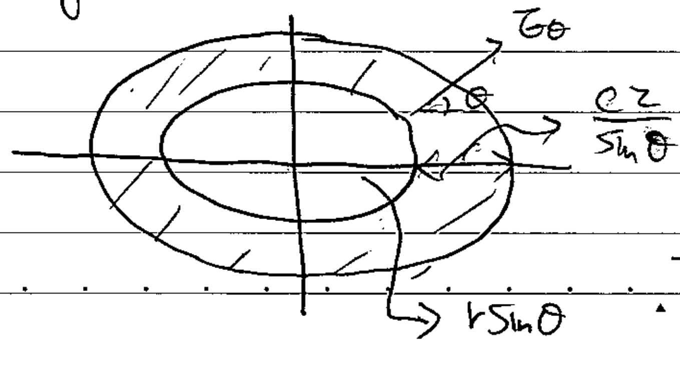
$$\because \frac{E_\theta}{E_r} = \frac{\overline{B'B}}{\overline{B'C}} = \frac{vt\sin\theta}{c\tau} = \frac{at\sin\theta}{c}$$

$$= \frac{a\sin\theta\, r}{c^2}$$

$$\therefore E_\theta = \frac{a\sin\theta}{c^2} r E_r = \frac{qa\sin\theta}{4\pi\epsilon_0 c^2 r} \quad \cdots (64)$$

As show in the left figure, in the yz plane, we have a region with E_θ.





Tho. electric flux.

$$\Phi_E = 2\pi r \sin\theta \cdot \underbrace{\frac{c\tau}{\sin\theta}}_{\text{width}} \cdot \underbrace{E_\theta \sin\theta}_{E_x}$$

$$= 2\pi r c\tau E_\theta \sin\theta$$

$$\therefore \frac{d\Phi_E}{dt} = \frac{\Phi_E}{\tau} = 2\pi r c E_\theta \sin\theta = \oint \vec{B}\cdot d\vec{\ell}$$

Maxwell - Ampere's law

$= 2\pi r \sin\theta\, B$. B is pointing in $\hat{r}\times E_\theta\hat{\theta}$ direction

$$\therefore B = c\mu_0\epsilon_0 E_\theta, \quad \vec{B}_{rad} = \frac{1}{c}\hat{r}\times\vec{E}_{rad}$$

with $\vec{E}_{rad} = E_\theta\hat{\theta}$. ... (65)

$$\therefore \frac{dP}{d\Omega} \underset{(63)}{=} r^2\frac{c}{\mu_0}B^2 = r^2\frac{c}{\mu_0}(c\mu_0\epsilon_0)^2 E_\theta^2$$

$$= r^2\epsilon_0 E_\theta^2 c$$

$$\therefore P = \int_0^\pi \epsilon_0 E_\theta^2 c\, 2\pi r^2 \sin\theta\, d\theta$$

$$= \left(\frac{qa}{4\pi\epsilon_0 c^2}\right)^2 2\pi c\,\epsilon_0 \underbrace{\int_0^\pi \sin^3\theta\, d\theta}_{4/3}$$

$$= \frac{q^2a^2}{6\pi\epsilon_0 c^3} = \frac{\mu_0 q^2 a^2}{6\pi c} \quad \text{--- (66)}$$

This is the Larmor formula, valid for $v \ll c$.

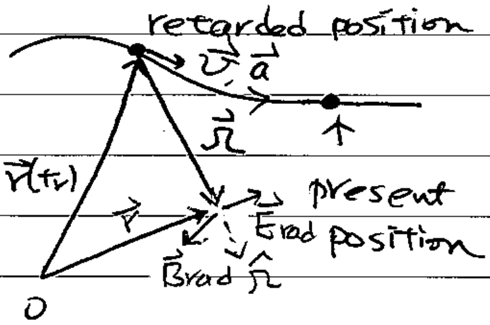
Radiation of an accelerated charge in general

For a generally accelerated charge,

the electric field is given by

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u}+\vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right] \quad \text{--- (66)}$$

with $\vec{B}(\vec{r},t) = \frac{1}{c}\hat{\mathscr{r}}\times\vec{E}(\vec{r},t)$.



Here $\vec{u} = c\hat{\mathscr{r}} - \vec{v}$

The first term (velocity field), as we discussed before, is the generalized Coulomb fied and goes as $O(\frac{1}{\mathscr{r}^2})$

The second term is the radiation field or the acceleration field

$$\therefore \vec{E}_{rad}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[\vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right] \quad \text{--- (67)}$$

where $\vec{\mathscr{r}} = \vec{r}-\vec{r}(t_r)$, $t_r = t - \frac{|\vec{r}-\vec{r}(t_r)|}{c}$

$\vec{a}$ & $\vec{v}$ both are evaluated at retarded time.

$$\vec{B}_{rad}(\vec{r},t) = \frac{1}{c}\hat{\mathscr{r}}\times\vec{E}_{rad}(\vec{r},t) \quad \text{--- (68)}$$

Using the radiation fields, one can find the Poynting vector

$$\vec{S} = \frac{1}{\mu_0}\vec{E}_{rad}\times\vec{B}_{rad}$$

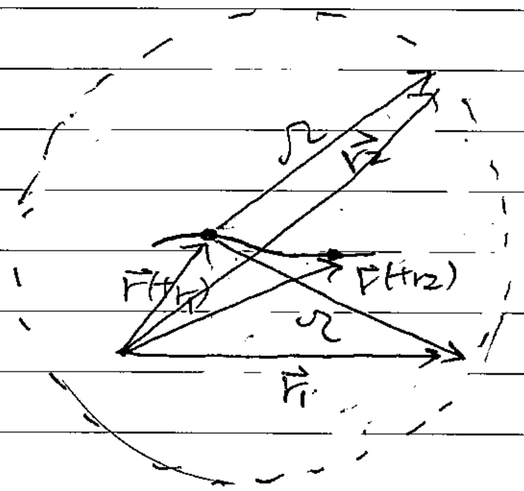
$$= \underbrace{\epsilon_0 c}_{\frac{1}{c\mu_0}}\, E_{rad}^2\,\hat{\mathscr{r}}$$

$$= \frac{q^2 c}{16\pi^2 \epsilon_0} \left| \frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3} (\vec{\mathscr{r}} \times (\vec{u}\times\vec{a})) \right|^2 \hat{\mathscr{r}} \Bigg|_{t=t_r} \quad \cdots (69)$$

where $\vec{\mathscr{r}} = \vec{r} - \vec{r}(t)$ and $\vec{u} = \vec{u}(t_r)$, $\vec{a} = \vec{a}(t_r)$.

For a moving charge, there is a subtley in going from $\vec{S}$ to find the power radiated:

(i) For a distant observer at surface $S_{\mathscr{r}}$ ($\mathscr{r} \to \infty$),



the measured radiation power = energy detected during dt at $\mathscr{r}$

$\because t_r = t - \dfrac{|\vec{r} - \vec{r}(t_r)|}{c}$

For a given t, different

$|\vec{r}_1 - \vec{r}(t_{r1})| = |\vec{r}_2 - \vec{r}(t_{r2})| = \mathscr{r}$

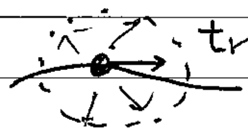
$\mathscr{r}$ corresponds to different retarded position $\vec{r}(t_r)$. (see the figure).

Hence the total $P(t) = \oint_{S_{\mathscr{r}}} \frac{dP}{d\Omega} d\Omega$ includes radiated energy of the same charge from different retarded time. Such computed radiated energy is determined by $\dfrac{dP(t)}{d\Omega} = \mathscr{r}^2 \vec{S}(t) \cdot \hat{\mathscr{r}} \Big|_{ret} \quad \cdots (70)$

with $\vec{S}(t)$ being given by eq. (68)

Clearly, the detected power at $S_{\mathcal{R}\to\infty}$ is not the total power radiated by a moving charge. It includes power radiated by the charge at different past times.

(ii) To compute the total power radiated by a moving charge, one evaluates the power nearby the charge at t_{ret}, i.e., the power left the charge.



In this case, the left the charge is detected during (t_r, t_r+dt_r) and is $\frac{dU_{rad}}{dt_r}$.

$$\because \frac{dU_{rad}}{dt_r} = \frac{dU_{rad}}{dt}\frac{dt}{dt_r} = \frac{\vec{\mathcal{R}}\cdot\vec{u}}{\mathcal{R}c}\frac{dU_{rad}}{dt}$$

eq. (10)-67

$$= \left(1-\frac{\hat{\mathcal{R}}\cdot\vec{v}}{c}\right)\frac{dU_{rad}}{dt} \quad \cdots (71)$$

$$\therefore \left.\frac{dP(t_r)}{d\Omega}\right|_{\text{power left charge at } t_r}$$

$$= \frac{dU_{rad}}{dt_r\,d\Omega} = \left(1-\frac{\hat{\mathcal{R}}\cdot\vec{v}}{c}\right)\left.\frac{dU_{rad}}{dt\,d\Omega}\right|_{\text{at } r,\, t_r}$$

$$= \left(1-\frac{\hat{\mathscr{R}}\cdot\vec{v}}{c}\right)\frac{dP(t)}{d\Omega}\bigg|_{\text{at } r,\,t} \quad \cdots (72)$$

That is, we can find the radiation power left the charge through the radiation power that goes in $r^2 d\Omega$ at time t.

Since $\frac{dP(t)}{d\Omega}$ is given in eq. (70). With all quantities in terms of t_r, the only time variable in the power left the charge is t_r.

We obtain that power radiated by the charge at t_r is

$$\frac{dP(t_r)}{d\Omega} = \frac{\vec{\mathscr{R}}\cdot\vec{u}}{\mathscr{R}c}\frac{1}{\mu_0 c}E_{rad}^2\,\mathscr{R}^2$$

$$= \frac{q^2}{16\pi^2\epsilon_0}\frac{\left\{\hat{\mathscr{R}}(t_r)\times\left[\vec{u}(t_r)\times\vec{a}(t_r)\right]\right\}^2}{\left[\hat{\mathscr{R}}(t_r)\cdot\vec{u}(t_r)\right]^5} \quad \cdots (73)$$

Note that since t_r is the only time variable in eq. (73), we can simply drop t_r and denote as t.

The total power radiated by the charge is obtained by integrating over $d\Omega$.

$$P = \int \frac{dP^{(tr)}}{d\Omega}\, d\Omega \quad \cdots (74)$$

This is a tedious work. We shall come back to evaluate it via the special relativity.

There are three special cases in which $\frac{dP^{(tr)}}{d\Omega}$ has a simple form.

<u>Non-relativistic motion</u> $\quad v \ll c$

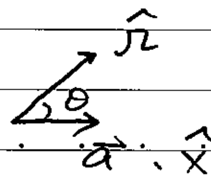
In this case, $v \ll c$ but $a = |\dot{\vec{v}}|$ can't be neglected. $\therefore$ We can set $\beta = v/c = 0$ as a good approximation.

$$\therefore \ \vec{u} = c(\hat{\mathscr{r}} - \vec{v}) \approx c\hat{\mathscr{r}}$$

$$\vec{E}_{rad} = \frac{q}{4\pi\epsilon_0}\frac{1}{c^2\mathscr{r}}\left[\hat{\mathscr{r}}\times(\hat{\mathscr{r}}\times\vec{a})\right]$$

eq. (67)

$$= \frac{\mu_0 q}{4\pi\mathscr{r}}\left[(\hat{\mathscr{r}}\cdot\vec{a})\hat{\mathscr{r}} - \vec{a}\right]$$



$$= \frac{\mu_0 q}{4\pi\mathscr{r}}\left[\cos\theta\,\hat{\mathscr{r}} - \hat{x}\right] a = \frac{q a \sin\theta}{4\pi\epsilon_0 c^2 \mathscr{r}}\,\hat{\theta} \quad \cdots (75)$$

$$\hat{x} = \cos\theta\,\hat{\mathscr{r}} + \sin\theta\,(-\hat{\theta})$$

∴ in agreement of eq. (64) obtained by consideration of field lines.

Hence P is given by eq. (66) ($1-\frac{\hat{r}\cdot\vec{v}}{c}=1$ in eq. (72))

$$P=\frac{\mu_0 q^2 a^2}{6\pi c}$$

Which is the Larmor formula.

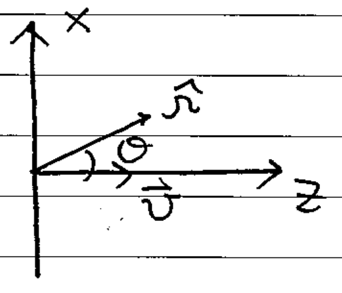
<u>Acceleration ∥ velocity</u>

$\because \vec{a}\parallel\vec{v}$, $\therefore$ $(\vec{u}\times\vec{a})=(c\hat{r}-\vec{v})\times\vec{a}=c\,\hat{r}\times\vec{a}$

$$\hat{r}\cdot\vec{u}=c-\vec{v}\cdot\hat{r}$$

Eq. (73) becomes

$$\frac{dP}{d\Omega}=\frac{q^2c^2}{16\pi^2\epsilon_0}\frac{\left[\hat{r}\times(\hat{r}\times\vec{a})\right]^2}{(c-\hat{r}\cdot\vec{v})^5}$$



If we align z axis along $\vec{v}$,

$$c-\hat{r}\cdot\vec{v}=c-v\cos\theta=c(1-\beta\cos\theta)$$

$$\hat{r}\times\vec{a}=-a\sin\theta\,\hat{y}$$

$$\hat{r}\times(\hat{r}\times\vec{a})=\hat{r}(\hat{r}\cdot\vec{a})-\vec{a}$$

$$|\hat{r}\times(\hat{r}\times\vec{a})|^2=(\hat{r}\cdot\vec{a})^2+a^2-2\hat{r}(\hat{r}\cdot\vec{a})\cdot\vec{a}=a^2-(\hat{r}\cdot\vec{a})^2$$

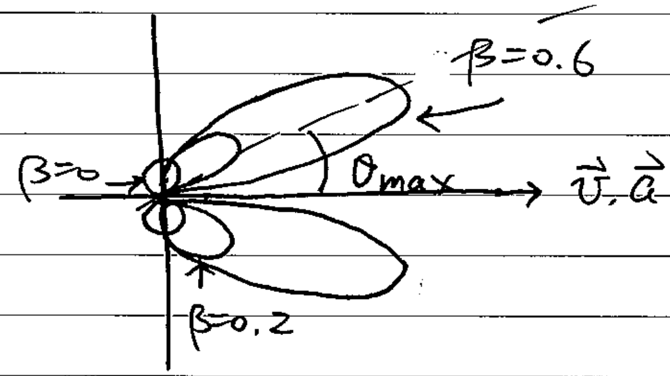
$$=a^2(1-\cos^2\theta)=a^2\sin^2\theta$$

$$\therefore\ \frac{dP}{d\Omega}=\frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\quad\text{--- (76)}$$

Which is consistent with the case $v=0$ in eq. (64).

However, for $v \to c$, $\beta \to 1$, the denominator contributes $(1-\beta\cos\theta)^{-5}$ which dominates at $\beta\cos\theta \sim 1$

Hence even though there is still no radiation precisely in forward direction $(\theta=0)$, most radiation is concentrated in a narrow cone in the forward direction for $v \to c$ as shown in the following figure.



$v \to c$, $\theta \sim 0$

$$1-\beta\cos\theta \simeq 1-\beta(1-\theta^2/2) = 1-\beta+\frac{\beta}{2}\theta^2 \approx 1-\beta+\frac{\theta^2}{2}$$

$$\because 1-\beta = \frac{1-\beta^2}{1+\beta} = \frac{1}{\gamma^2(1+\beta)} \approx \frac{1}{2\gamma^2} \qquad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$\therefore 1-\beta\cos\theta \simeq \frac{1+\gamma^2\theta^2}{2\gamma^2}$$

$$\frac{dP}{d\Omega} \approx \frac{2\mu_0 q^2 a^2}{\pi^2 c} \frac{\gamma^8(\gamma\theta)^2}{[1+(\gamma\theta)^2]^5} \qquad \gamma \gg 1$$

Maximizing eq. (77):

$$\frac{d}{dz}\frac{z^2}{(1+z^2)^5} = \frac{2z(1+z^2)^5 - z^2\, 5(1+z^2)^4 \cdot 2z}{(1+z^2)^{10}} = 0$$

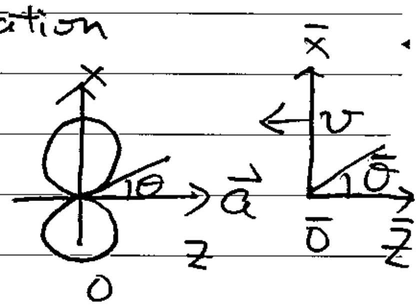
$$\therefore\ 4z^2 = 1 \quad \therefore\ \gamma\theta_{max} = \frac{1}{2}$$

$\therefore$ Radiation peaks at $\theta = \theta_{max} = \pm\frac{1}{2\gamma}$

With $\Delta\theta \sim 1/\gamma$ for $\gamma \gg 1$.

The physical mechanism for the concentration of radiation in a forward cone is due to the Doppler effect.

In the $\begin{matrix} v=0 \\ a\neq 0 \end{matrix}$ frame, the radiation is maximum at $\theta = \pi/2$



For a given radiation field with wave vector $\vec{k}$ & frequency ω, as shown in the above figure, the wave vector $\bar{\vec{k}}$ & $\bar{\omega}$ observed in the moving frame $\bar{O}$ are determined by $(\frac{\omega}{c}, \vec{k})$ nature:
the 4-vector

$$\bar{k}_z = \gamma\left(k_z + \beta\frac{\omega}{c}\right)$$

$$\bar{k}_{x/y} = k_{x/y} \quad \Rightarrow \quad \bar{k}_\perp = k_\perp$$

$$\therefore \cot\bar{\theta} = \frac{\bar{k}_z}{\bar{k}_\perp} = \frac{\gamma(k_z + \beta\,\omega/c)}{k_\perp}$$

$\because$ $k_z = k\cos\theta$, $k_\perp = k\sin\theta$, $\omega = ck$

$$\therefore \cot\bar{\theta} = \frac{\gamma(\cos\theta + \beta)}{\sin\theta} = \gamma\left(\cot\theta + \frac{\beta}{\sin\theta}\right)$$

$\therefore$ $\theta = \pi/2$ in the $v=0$ frame becomes

$$\cot\bar{\theta} = \frac{\gamma\beta}{\underset{\substack{\shortparallel \\ 1}}{\sin\theta}}, \quad \bar{\theta} = \tan^{-1}\frac{1}{\gamma\beta}$$

for $\beta \to 1$, $\bar{\theta} \sim 1/\gamma$ $\therefore$ radiations are
$\gamma \to \infty$ thus confined to $\Delta\bar{\theta} = \frac{1}{\gamma}$
due to the Doppler effect.

Total power radiated

Note that both $\frac{dP}{d\Omega}$ & P don't depends on sign of acceleration. $\therefore$ Both accelerating & decelerating radiate the same

Therefore, when a high speed electron hits a metal, it rapidly decelerates and gives off energy. This is known as bremsstrahlung "braking radiation", which is described by eq. (76).

The total emitted is

$$P = \int \frac{dP}{d\Omega} d\Omega$$

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int_0^{\pi} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \sin\theta \, d\theta \cdot 2\pi$$

$$= \frac{\mu_0 q^2 a^2}{8\pi c} \int_{-1}^{1} \frac{1-x^2}{(1-\beta x)^5} dx$$

$$\because \quad \frac{dx}{(1-\beta x)^5} = \frac{1}{4\beta} d(1-\beta x)^{-4}$$

$$\therefore \quad P = \frac{\mu_0 q^2 a^2}{8\pi c} \left[\frac{1-x^2}{4\beta} \frac{1}{(1-\beta x)^4} \Bigg|_{x=-1}^{x=1} + \frac{1}{2\beta} \int_{-1}^{1} \frac{x}{(1-\beta x)^4} dx \right]$$

$$\frac{dx}{(1-\beta x)^4} = \frac{1}{3\beta} d(1-\beta x)^{-3}$$

$$P = \frac{\mu_0 q^2 a^2}{8\pi c} \left[\cancel{\frac{1-x^2}{4\beta(1-\beta x)^4}}^{\,0} \Bigg|_{x=-1}^{x=1} + \frac{x}{6\beta^2(1-\beta x)^3} \Bigg|_{x=-1}^{x=1} - \frac{1}{6\beta^2} \underbrace{\int_{-1}^{1} \frac{dx}{(1-\beta x)^3}}_{\frac{1}{2\beta} \frac{1}{(1-\beta x)^2} \Big|_{x=-1}^{x=1}} \right]$$

$$= \frac{\mu_0 q^2 a^2}{8\pi c} \left[\frac{1}{6\beta^2(1-\beta)^3} + \frac{1}{6\beta^2(1+\beta)^3} - \frac{1}{12\beta^3(1-\beta)^2} + \frac{1}{12\beta^3(1+\beta)^2} \right]$$

$$= \frac{\mu_0 q^2 a^2}{8\pi c} \times \frac{4}{3} \frac{1}{(1-\beta^2)^3}$$

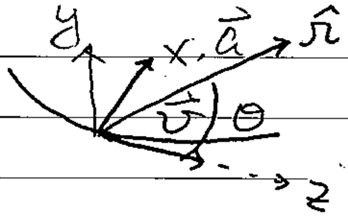
$$= \frac{\mu_0 q^2 a^2}{6\pi c} \times \gamma^6 \quad \cdots \quad (78)$$

which differs from the Larmor formula by the factor γ^6!

Acceleration $\perp$ Velocity :

When the charge particle moves on a circular arc, $\vec{a} \perp \vec{v}$. (Circular motion : Cyclotron radiation)

$\underbrace{\phantom{\vec{a}\perp\vec{v}}}_{\text{in this case,}}$



Let z axis be aligned with $\vec{v}$ and x axis be aligned with $\vec{a}$.

$\vec{u} = c\hat{r} - \vec{v} = c\hat{r} - v\hat{z}$. In spherical coordinates, $\hat{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$\hat{r}\cdot\vec{u} = c - v\cos\theta = c(1-\beta\cos\theta)$, $u^2 = c^2 + v^2 - 2cv\cos\theta$

$\because \hat{r}\times(\vec{u}\times\vec{a}) = (\hat{r}\cdot\vec{a})\,\vec{u} - (\hat{r}\cdot\vec{u})\,\vec{a}$ --- (79)

$$|\hat{r}\times(\vec{u}\times\vec{a})|^2 = (\hat{r}\cdot\vec{a})^2 u^2 + (\hat{r}\cdot\vec{u})^2 a^2 - 2(\vec{u}\cdot\vec{a})(\hat{r}\cdot\vec{a})(\hat{r}\cdot\vec{u})$$

$\hat{r}\cdot\vec{a} = a\sin\theta\cos\phi$, $\vec{u}\cdot\vec{a} = c(\hat{r}\cdot\vec{a}) = ca\sin\theta\cos\phi$

$$\therefore |\hat{r}\times(\vec{u}\times\vec{a})|^2 = (c^2+v^2-2cv\cos\theta)\,a^2\sin^2\theta\cos^2\phi - 2ca^2(\sin\theta\cos\phi)^2(c - v\cos\theta) + (c - v\cos\theta)^2 a^2$$

$$= a^2\left[c^2(1-\beta\cos\theta)^2 + \sin^2\theta\cos^2\phi\,(c^2+v^2\cancel{-2cv\cos\theta}-2c^2\cancel{+2cv\cos\theta})\right]$$

$$= a^2c^2\left[(1-\beta\cos\theta)^2 - (1-\beta^2)(\sin\theta\cos\phi)^2\right] \quad \text{--- (80)}$$

Eqs (73), (79) & (80) imply

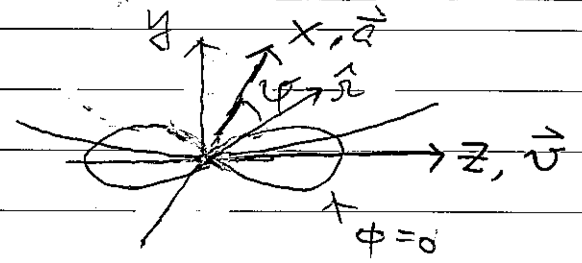
$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\,\frac{\left[(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\phi\right]}{(1-\beta\cos\theta)^5}$$

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{1}{(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right] \quad \text{--- (81)}$$

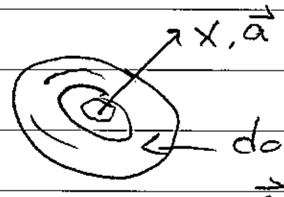
Clearly, in the non-relativistic limit $\beta \to 0$

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(1-\sin^2\theta\cos^2\phi\right)$$

This is the low velocity cyclotron radiation.



which is maximum when $\theta = 0$



donut shape surrounding $\vec{a}$

$\sin\theta\cos\phi$
$= \hat{r}\cdot\hat{a} = \cos\psi$ $\quad$ ψ = angle of $\hat{r}$ & $\vec{a}$

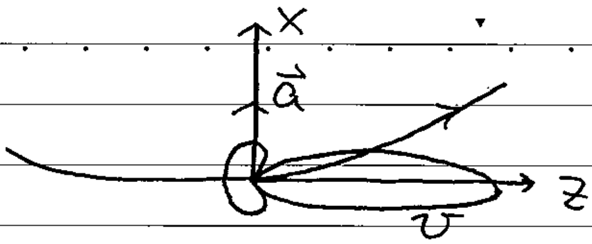
$\therefore \frac{dP}{d\Omega}$ = constant $\Rightarrow \psi$ = constant

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c}\sin^2\psi$$ which is exactly the same as $\frac{dP}{d\Omega}$ in eq. (66) when $\vec{v} \parallel \vec{a}$ ($v \ll c$).

For $\beta \to 1$, using eq. (77) $1-\beta\cos\theta \approx \frac{1+\gamma^2\theta^2}{2\gamma^2}$

$$\frac{dP}{d\Omega} \simeq \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{8\gamma^6}{(1+\gamma^2\theta^2)^3}\left[1-\frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta^2)^2}\right] \quad \gamma \gg 1$$

which is maximal when $\theta = 0$ and is confined to $\Delta\theta \sim 1/\gamma$ as shown in the following figure.



The same physical mechanism (Doppler effect) drives the radiation to focus on the v direction like the locomotive's headlights!

<u>Total power radiated</u>

$$P = \int \frac{dP}{d\Omega} d\Omega$$

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c} \iint d\phi \sin\theta d\theta \left[\frac{1}{(1-\beta\cos\theta)^3} - \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \frac{\cos^2\phi}{\gamma^2} \right]$$

From the evaluation of (f8),

$$\iint d\phi \sin\theta d\theta \frac{1}{(1-\beta\cos\theta)^3} = \frac{2\pi}{2\beta} \frac{1}{(1-\beta x)^2} \Big|_{x=-1}^{x=1} = \frac{4\pi}{(1-\beta^2)^2}$$

$$\int_0^{\pi} \sin\theta d\theta \frac{\sin^2\theta}{(1-\beta\cos\theta)^3} = \frac{4}{3} \frac{1}{(1-\beta^2)^3}$$

$$\because \int_0^{2\pi} \frac{1}{\gamma^2} \cos^2\phi \, d\phi = \frac{\pi}{\gamma^2}$$

$$\therefore P = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \left[\frac{4\pi}{(1-\beta^2)^2} - \frac{4\pi}{3(1-\beta^2)^2} \right] = \frac{\mu_0 q^2 a^2}{6\pi c} \gamma^4 \quad \cdots (f2)$$

Which differs from the Larmor formula by the factor γ^4

Liénard's generalization of the Larmor formula

As we have seen in eqs. (78) & (82), the total power radiated differs from the Larmor formula. They can be unified in the Liénard's formula:

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right) \quad \cdots (83)$$

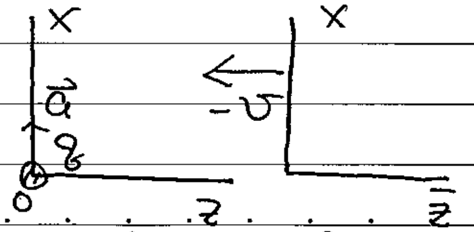
$$= \frac{1}{4\pi\epsilon_0}\frac{2q^2}{3c^3}\gamma^6\left[a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right]$$

Check: $\vec{v} \parallel \vec{a}$, $a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2 = a^2$

$\vec{v} \perp \vec{a}$ $\quad a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2 = \frac{1}{\gamma^2}a^2$

Eq. (83) can be obtained by straightforward (but tedious) integration of eq. (73). We shall not derive it differently by using the fact that $\left(\frac{U_{rad}}{c}, \vec{P}_{rad}\right)$ is a 4-vector.

Here $\vec{P}_{rad}$ & U_{rad} are total momentum and energy for the radiation field.



restframe of q

Consider the restframe O of the charge in which $v=0$, $a\neq 0$. The $\bar{O}$ frame moves $-v$ relative to O.

Therefore, in $\bar{O}$ frame, the charge moves with $\vec{v}$ & $\vec{\bar{a}}$ $(\neq \vec{a})$

Now, the total power radiated in O & $\bar{O}$ are $P = \frac{dU_{rad}}{dt}$, $\bar{P} = \frac{d\bar{U}_{rad}}{d\bar{t}}$

Here t = proper time of the charge $= \tau$

Consider a period dt. Since in O frame charge is at rest.

$\therefore$ the displacement of the charge $d\vec{r} = 0$.

Now $U_{rad} = \gamma(\bar{U}_{rad} + \vec{v}\cdot\vec{\bar{P}}_{rad})$ - - - (84)

where $\vec{\bar{P}}_{rad}$ is the total field moment in $\bar{O}$ frame. $\vec{v}\cdot\vec{\bar{P}}_{rad} = v\bar{P}_{rad\,z}$ if z-axis is aligned with $\vec{v}$.

$$t = \gamma\left(\bar{t} + \frac{1}{c^2}\vec{v}\cdot\vec{r}\right)$$

$$\therefore P = \frac{dU_{rad}}{dt} = \frac{\gamma(d\bar{U}_{rad} + \vec{v}\cdot d\vec{\bar{P}}_{rad})}{\gamma(d\bar{t} + \frac{1}{c^2}\vec{v}\cdot d\vec{r})} \quad - - - (85)$$

Now, as indicated before, $d\vec{r} = 0$.

In the rest frame of the charge, the field distribution is donut-like and symmetric.

$\therefore \vec{\bar{P}}_{rad} = 0$, $d\vec{\bar{P}}_{rad} = 0$

Eq. (85) implies

$$P = \frac{dU_{rad}}{dt} = \frac{d\bar{U}_{rad}}{d\bar{t}} = \bar{P} \quad \text{--- (86)}$$

∴ Total power radiated is the same in both inertial frames. This is reasonable as the relative motion is a constant velocity and it <u>does not change the acceleration of the charge!</u>

Since in the rest frame,

$$P = \frac{1}{4\pi\epsilon_0}\frac{2q^2}{3c^3}|\vec{a}|^2$$

$$= \frac{1}{4\pi\epsilon_0}\frac{2q^2}{3c^3}\left(\sum_{i=1}^{3} a^i a_i\right).$$

The invariance of $P = \bar{P}$ implies that one can replace $\sum_{i=1}^{3} a^i a_i$ by the scalar product of two 4-vectors:

$$\sum_{i=1}^{3} a_i a_i \to \sum_{\mu=0}^{4} a^\mu a_\mu \quad \left(= -a_0^2 + \sum_{i=1}^{3}(a_i)^2\right)$$

$$= \sum_{\mu} \bar{a}^\mu \bar{a}_\mu$$

Hence

$$\bar{P} = \frac{1}{4\pi\epsilon_0}\frac{2q^2}{3c^3}\sum_{\mu}\bar{a}^\mu \bar{a}_\mu \quad \text{--- (87)}$$

Where $a^\mu = (a^0, a^1, a^2, a^3)$ is the proper acceleration

$$\bar{a} = \frac{d\bar{\eta}}{d\tau}, \quad \bar{\eta} = \frac{d\bar{x}}{d\tau} = \text{4-velocity}$$

$$m_0 \bar{a} = \frac{d\bar{P}}{d\tau}, \quad \bar{P} = \left(\frac{E}{c}, \vec{p}\right), \quad \vec{p} = m\gamma\vec{v}$$

m_0 rest mass

$$E = m\gamma c^2$$

($\vec{v}$ has included ~~effect of~~ $\vec{a}$, $\therefore \neq \vec{v}$ $\vec{a}$)

$$\therefore \quad P = \frac{1}{4\pi\epsilon_0}\frac{2q^2}{3m_0^2c^3}\frac{d\bar{P}^\mu}{d\tau}\frac{d\bar{P}_\mu}{d\tau} \quad \cdots (8f)$$

$$\because \text{(i)} \quad \frac{d\bar{P}^\mu}{d\tau}\frac{d\bar{P}_\mu}{d\tau} = \frac{d\vec{p}}{d\tau}\cdot\frac{d\vec{p}}{d\tau} - \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2$$

$$= (m_0c\gamma)^2\left[\frac{d\gamma\vec{\beta}}{dt}\cdot\frac{d\gamma\vec{\beta}}{dt} - \left(\frac{d\gamma}{dt}\right)^2\right], \quad \vec{\beta} = \frac{\vec{v}}{c}$$

$$\frac{dt}{d\tau} = \gamma$$

$$\text{(ii)} \quad \frac{d\gamma}{dt} = \frac{d}{dt}\frac{1}{\sqrt{1-\beta^2}} = \left(\frac{-1}{2}\right)\frac{(-2\beta)\frac{d\beta}{dt}}{(1-\beta^2)^{3/2}} = \gamma^3\beta\frac{d\beta}{dt}$$

$$= \gamma^3\vec{\beta}\cdot\frac{d\vec{\beta}}{dt} = \gamma^3\vec{\beta}\cdot\dot{\vec{\beta}}$$

$$\text{(iii)} \quad \frac{d}{dt}(\gamma\vec{\beta}) = \frac{d\gamma}{dt}\vec{\beta} + \gamma\dot{\vec{\beta}} = \gamma^3(\vec{\beta}\cdot\dot{\vec{\beta}})\vec{\beta} + \gamma\dot{\vec{\beta}}$$

$$\therefore \quad P = \frac{1}{4\pi\epsilon_0}\frac{2q^2}{3c}\gamma^2\left[\gamma^6(\vec{\beta}\cdot\dot{\vec{\beta}})^2\vec{\beta}^2 + \gamma^2(\dot{\vec{\beta}})^2 - \gamma^6(\vec{\beta}\cdot\dot{\vec{\beta}})^2 + 2\gamma^4(\vec{\beta}\cdot\dot{\vec{\beta}})^2\right] \quad \cdots (8g)$$

Eq (8f) becomes

$$\gamma^2(\vec{\beta}^2 - 1) = \frac{1}{1-\beta^2}(\vec{\beta}^2 - 1) = -1$$

$$\therefore \quad \gamma^6(\vec{\beta}\cdot\dot{\vec{\beta}})^2\vec{\beta}^2 - \gamma^6(\vec{\beta}\cdot\dot{\vec{\beta}})^2 = -\gamma^4(\vec{\beta}\cdot\dot{\vec{\beta}})^2$$

$$\therefore [\cdots] \text{ in } (8g) = \gamma^2(\dot{\vec{\beta}})^2 + \gamma^4(\vec{\beta}\cdot\dot{\vec{\beta}})^2 = \gamma^4\left[(1-\beta^2)(\dot{\vec{\beta}})^2 + (\vec{\beta}\cdot\dot{\vec{\beta}})^2\right]$$

$$= \bar{\gamma}^4 \left\{ (\dot{\vec{\beta}})^2 - \underbrace{(\bar{\beta}^2 \dot{\vec{\beta}}^2 - (\vec{\beta}\cdot\dot{\vec{\beta}})^2)} \right\}$$

$$(\vec{\beta}\times\dot{\vec{\beta}})\cdot(\vec{\beta}\times\dot{\vec{\beta}}) = \frac{1}{c^4}|\vec{v}\times\dot{\vec{v}}|^2$$

$$\therefore\ \bar{P} = \frac{1}{4\pi\epsilon_0}\frac{2 e^2}{3c^3}\bar{\gamma}^6\left\{(\dot{\vec{v}})^2 - \frac{1}{c^2}|\vec{v}\times\dot{\vec{v}}|^2\right.$$

$$= \frac{1}{4\pi\epsilon_0}\frac{2e^2}{3c^3}\gamma^6\left\{ a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right\}$$

Where inside $\vec{a} = \frac{d\vec{v}}{dt}$ with $\vec{v}$, since $\vec{v}_\perp = 0$, $\therefore \vec{\bar{v}} = \vec{v}$, we has replaced $\vec{\bar{v}}$ by $\vec{v}$, $\bar{\gamma}$ by γ. (but $\hat{\vec{a}} \neq \vec{a}$)

Hence Liénard's formula is proved!

The γ^6 factor in Liénard's formula implies that for a given acceleration, the radiated power increases enormously as $v \Rightarrow c$!

Example Compare the power needed to supply for the linear acceleration and the circular accelerator of radius R. for a given $\frac{d\vec{p}}{dt}$. $\vec{p}$ = momentum of the particle being accelerated

Solution:

For $\hat{a}$ linear accelerator, eq. (88) implies

$$\frac{dp^\mu dp_\mu}{d\tau\, d\tau} = \gamma^2\left[\left(\frac{dp}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{dt}\right)^2\right]$$

$$\frac{d\vec{p}}{dt}\cdot\frac{d\vec{p}}{dt}$$

$\because E^2 = c^2p^2 + m_0^2c^4$

$\therefore 2E\,dE = 2c^2 p\,dp \qquad dp = \frac{E}{c^2p}dE = \frac{dE}{v}$

$\therefore \left(\frac{dp}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{dt}\right)^2 = (1 - v^2/c^2)\left(\frac{dp}{dt}\right)^2 = \frac{1}{\gamma^2}\left(\frac{dp}{dt}\right)^2$

$\therefore \frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau} = \left(\frac{dp}{dt}\right)^2$

$$\therefore P_{\parallel} = \frac{q^2 c}{6\pi\epsilon_0}\frac{1}{(m_0c^2)^2}\left(\frac{d\vec{p}}{dt}\right)^2 \quad \text{--- (90)}$$

For a circular accelerator of radius R, the magnitude of $\vec{p} = \gamma m_0 \vec{v}$ is fixed. $\therefore v = \text{constant}$

$$\therefore \frac{d\vec{p}}{dt} = \gamma m_0 \frac{d\vec{v}}{dt} = \gamma m_0 \vec{a}, \quad \left(\frac{d\vec{p}}{dt}\right)^2 = \gamma^2 m_0^2 a^2$$

The Liénord's formula implies

$$P_{\perp} = \frac{q^2}{6\pi\epsilon_0 c^3}\underbrace{\gamma^6 a^2(1-\beta^2)}_{\gamma^4 a^2 \,=\, \gamma^2 \frac{1}{m_0^2}\left(\frac{d\vec{p}}{dt}\right)^2}$$

$$\therefore P_{\perp} = \frac{q^2 c}{6\pi\epsilon_0}\frac{\gamma^2}{(m_0c^2)^2}\left(\frac{d\vec{p}}{dt}\right)^2 \quad \text{--- (91)}$$

Comparing eqs (90) & (91), we see that for a given $\frac{d\vec{p}}{dt}$, $\frac{P_\perp}{P_\parallel} = \gamma^2$ (independent of R)

For $v \ll c$, $P_\perp \gg P_\parallel$. The circular accelerator needs much more machine power to accelerate the charge particle due to the radiation loss.

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The circular accelerator, however, is popular because particles can pass each accelerator element many times and can have more opportunities to collide.

$\because v =$ constant, $\therefore E = \gamma m_0 c^2$ is a constant. Using $a = v^2/R$, we can rewrite

$$P_\perp = \frac{q^2}{6\pi\epsilon_0 c^3}\gamma^6 a^2(1-\beta^2)$$

$$= \frac{q^2}{6\pi\epsilon_0 c^3}\gamma^4 a^2 = \frac{q^2}{6\pi\epsilon_0 c^3}\frac{\gamma^4 v^4}{R^2}$$

$$= \frac{q^2 c}{6\pi\epsilon_0}\frac{(\gamma\beta)^4}{R^2} = \frac{q^2 c}{6\pi\epsilon_0}\frac{\beta^4}{(m_0c^2)^2}\frac{E^4}{R^2}$$

Radiation loss can be reduced by increasing the radius of accelerator but as $E\uparrow$, R also needs to be further increased. The amount of resources for constructing the circular accelerator becomes prohibitive!

(Field)

Radiation reaction on a slowly moving charge

The Abraham-Lorentz formula

Energy consideration:

An accelerating charge radiates.

Hence its (kinetic) energy decreases, There must be a force $\vec{F}_{rad}$ acting back on the charge to decelerate it!

For $v \ll c$, the total power radiates

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

it's tempting to conclude $\quad \bullet \rightarrow v$

$$\vec{F}_{rad} \cdot \vec{v} = -P = -\frac{\mu_0 q^2 a^2}{6\pi c} \quad \text{--- (P2)}$$

where $\vec{v}$ = velocity of particle at t_r.

Eq. (P2) is not correct because P is not the total energy that is pumped into fields by the charge. P is only the power that can be carried to $r=\infty$. What is actually important is the "field reaction".

that acts on the particle charge.

This is the total force including the power that goes into the velocity field in eq. (66) $\vec{E}_u = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}[(c^2-u^2)\vec{u}]$!

Hence eq. (82) is not correct.

However, for $\underbrace{\text{periodic motion}}_{\text{in } (t_1, t_2)}$ of the charge, the energy in $\vec{E}_u$ (E_u) is the same at t_1 & t_2

$$E_u(t_1) = E_u(t_2)$$

$$\therefore \int_{t_1}^{t_2} \vec{F}_{rad}\cdot\vec{v}\,dt = \int_{t_1}^{t_2}\left(\frac{dE_u}{dt} + \frac{dE_a}{dt}\right)dt$$

due to velocity field

due to radiation field

$$= \cancel{E_u\Big|_{t_1}^{t_2}}_{\,0} + \int_{t_1}^{t_2}\frac{dE_a}{dt}dt$$

$$= -\frac{\mu_0 q^2}{6\pi c}\int_{t_1}^{t_2} a^2\,dt$$

$$= \frac{-\mu_0 q^2}{6\pi c}\int_{t_1}^{t_2}\frac{d\vec{v}}{dt}\cdot\frac{d\vec{v}}{dt}\,dt$$

$$= \frac{-\mu_0 q^2}{6\pi c}\left[\underbrace{\vec{v}\cdot\frac{d\vec{v}}{dt}\Big|_{t_1}^{t_2}}_{\text{periodic}} - \int_{t_1}^{t_2}\frac{d^2\vec{v}}{dt^2}\cdot\vec{v}\,dt\right]$$

$$\therefore \int_{t_1}^{t_2} \left(\vec{F}_{rad} - \frac{\mu_0 q^2}{6\pi c}\dot{\vec{a}}\right)\cdot\vec{v}\,dt = 0$$

For any $\vec{v}(t)$, $\therefore$ it's satisfied if

$$\vec{F}_{rad} = \frac{\mu_0 q^2}{6\pi c}\dot{\vec{a}} \quad \cdots \quad (93)$$

Which is the Abraham-Lorentz formula!

Dirac's arguement

Eq. (93) is derived in the circumstance when the motion of the particle is periodic.

Eq. (93) is actually generally true.

To see it, we follow the arguement made by Dirac: fields that propagate out at speed c are responsible for $\vec{F}_{rad}$ (field reaction force)

For this purpose, we expand $t_r = t - \frac{|\vec{r}-\vec{r}'|}{c}$ in terms of $\frac{|\vec{r}-\vec{r}'|}{c}$ as follows:

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d\tau' \frac{\vec{J}(\vec{r}', t-\frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|} = \frac{\mu_0}{4\pi}\int d\tau' \frac{\vec{J}(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$$

$$- \frac{\mu_0}{4\pi} \int d\tau' \frac{1}{|\vec{r}-\vec{r}'|} \frac{|\vec{r}-\vec{r}'|}{c} \frac{d}{dt} \vec{J}(\vec{r}',t) + \cdots$$

$$= \frac{\mu_0}{4\pi} \int d\tau' \frac{\vec{J}(\vec{r}',t)}{|\vec{r}-\vec{r}'|} - \frac{\mu_0}{4\pi c} \frac{d}{dt} \int d\tau' \vec{J}(\vec{r}',t) + \cdots$$

$$\text{--- (94)}$$

Similarly,

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \int d\tau' \frac{\rho(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{4\pi\epsilon_0} \Big[\int d\tau' \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|} \pm \underbrace{\frac{1}{c} \frac{d}{dt} \int d^3r' \rho(\vec{r}',t)}_{=0 \,=\, \text{Const}}$$

$$+ \frac{1}{2c^2} \frac{d^2}{dt^2} \int d\tau' \rho(\vec{r}',t) |\vec{r}-\vec{r}'|$$

$$- \frac{1}{3!c^3} \frac{d^3}{dt^3} \int d\tau' \rho(\vec{r}',t) |\vec{r}-\vec{r}'|^2 \Big] + \cdots$$

$$\text{--- (95)}$$

Clearly, to identify fields that do propagate out, one switch $c \to -c$. Terms that do not change are not propagating out!

Hence we identify $\vec{A}_{FR}$ & V_{FR} responsible for fields reaction as

$$\vec{A}_{FR}(\vec{r},t) = - \frac{\mu_0}{4\pi c} \frac{d}{dt} \int d\tau' \vec{J}(\vec{r}',t) \quad \text{--- (96)}$$

$$V_{FR}(\vec r', t) = \underbrace{\frac{-1}{4\pi\epsilon_0}\frac{1}{6c^3}}_{= -\frac{\mu_0}{24\pi c}} \frac{d^3}{dt^3}\int d\tau'\, \rho(\vec r', t)\,|\vec r - \vec r'|^2 \quad \text{--- (97)}$$

Now, ∵ $\vec A_{FR}$ doesn't depend on $\vec r$

$$\therefore\ \nabla\times\vec A_{FR} = \vec B_{FR} = 0 \quad \text{--- (98)}$$

For a point particle, $\vec J(\vec r', t) = q\,\delta(\vec r' - \vec r(t))\,\vec v(t)$

$$\therefore\ \int d\tau'\, \vec J(\vec r', t) = q\,\vec v(\vec r(t))$$

$$\frac{d}{dt}\int d\tau'\, \vec J(\vec r', t) = q\,\vec a(t)$$

$$\therefore\ \vec A_{FR}(\vec r, t) = -\frac{\mu_0 q}{4\pi c}\vec a(t) \quad \text{--- (99)}$$

To find $\vec E_{FR}$, one needs ~~to~~ calculate

∇V_{FR}: ∵ $\nabla|\vec r - \vec r'|^2 = 2(\vec r - \vec r')$

$$\therefore\ \nabla V_{FR} = -\frac{\mu_0}{12\pi c}\frac{d^3}{dt^3}\int d\tau'\, \rho(\vec r - \vec r')(\vec r - \vec r')$$

For a point charge, $\rho(\vec r', t) = \delta(\vec r' - \vec r(t))\,q$

$$\therefore\ \int d\tau'\, \rho(\vec r - \vec r')(\vec r - \vec r') = q(\vec r - \vec r(t))$$

$$\frac{d^3}{dt^3} q(\vec r - \vec r(t)) = -q\frac{d\vec a}{dt}$$

Hence $\nabla U_{FR} = \frac{\mu_0 q}{12\pi c}\frac{d\vec{a}}{dt}$ --- (100)

Combining (99) & (100), we get

$$\vec{E}_{FR} = -\nabla U_{FR} - \frac{\partial \vec{A}_{FR}}{\partial t} = \frac{\mu_0}{6\pi c} q \frac{d\vec{a}}{dt} \quad \text{--- (101)}$$

Hence $\vec{F}_{FR} = q\vec{E}_F = \frac{\mu_0}{6\pi c} q^2 \dot{\vec{a}}$

Runaway solution and preacceleration

Even though one can formally derive the Abraham-Lorentz formula, it yields unphysical runaway solutions when it is combined with Newton's 2nd law:

$$m\vec{a} = \vec{F}_{ext} + m\tau_0 \dot{\vec{a}} \quad \text{--- (102)}$$

where $m\tau_0 \equiv \frac{\mu_0}{6\pi c} q^2$. For electrons, $q=e$

$\tau_0 = \frac{\mu_0 e^2}{6\pi m c} = \frac{2}{3}\frac{r_e}{c}$, with r_e be the classical electron radius : $\frac{e^2}{4\pi\epsilon_0 r_e} = mc^2$.

$\therefore$ $\tau_0 \doteq$ time for light to travel r_e.

The problem with eq (102) is that it involves $\frac{d^3}{dt^3}\vec{r}(t)$ and $\vec{r}(t)$ is not determined by initial position $\vec{r}(0)$ and velocity $\dot{\vec{r}}(0) = \vec{v}(0)$!

Take $\vec{F}_{ext} = F\hat{x}$ as an example.

Eq. (102) becomes $ma = F + m\tau_0 \frac{da}{dt}$

$$\therefore \tau_0 \frac{da}{dt} = a - \frac{F(t)}{m}$$

If $F(t) = F$ for $t > 0$,

$$a = \frac{F}{m} + Be^{t/\tau_0} \quad \text{with } B = \text{some constant}$$

--- (103)

We see that if $B \neq 0$, $t \to \infty$, $a(t) \to \infty$.

The solution runs away and is unphysical.

To eliminate the run-away solution, one considers $F(t) = F$ for $0 \le t \le T$

$F(t) = 0$ otherwise.

For $t < 0$, $\tau_0 \frac{da}{dt} = a$, $a(t) = Ae^{t/\tau_0}$ $A = \text{const}$,

$0 \le t \le T$, $\tau_0 \frac{da}{dt} = a - \frac{F}{m}$, $a(t) = \frac{F}{m} + Be^{t/\tau_0}$

$t > T$ $\tau_0 \frac{da}{dt} = a$, $a(t) = Ce^{t/\tau_0}$ $C = \text{const}$.

Clearly, $t = 0$ $a(0^-) = a(0^+)$ $\therefore A = \frac{F}{m} + B$ --- (104)

$$t = T, \quad \frac{F}{m} + Be^{T/\tau_0} = Ce^{T/\tau_0}$$

$$\therefore C = B + \frac{F}{m}e^{-T/\tau_0} \quad \text{--- (105)}$$

By choosing $C = 0$, $B = -\frac{F}{m}e^{-T/\tau_0}$, $a(t) = 0$ for $t > T$.

$\therefore$ The run-away solution is eliminated.

However, eq. (104) implies

$$A = F/m + B = F/m\,(1 - e^{-T/\tau_0}) \neq 0$$

$\therefore$ $a(t) = A e^{t/\tau_0}$ for $t <$ $\neq 0$

The particle is accelerated before the force applies! This is also not physical!

We see that if one wants to eliminate the run-away solution, it results in pre-acceleration. If we eliminate the pre-acceleration, $A=0$, $B = -F/m$,

$$C = B + F/m\, e^{-T/\tau_0} = F/m\,(e^{-T/\tau_0} - 1) \neq 0$$

$a(t) \neq 0$ for $t > T$ and it runs away!

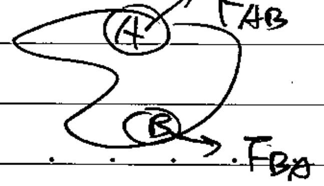
Hence the solution is always unphysical, either run-away or pre-accelerated!

Finite object — mechanism of radiation reaction

The problems that occur in the Abraham-Lorentz formula can be traced back to the treatment of charge particles as points. For point charges, $E \propto \frac{1}{r}$ diverges at the particle. The force due to acceleration is obscure due to $F = \infty$ since ∞ − finite is still ∞. One can pull out finite terms from ∞ arbitrarily.

We shall illustrate the mechanism of radiation reaction by considering a finite charge distribution and take size → 0 at the end.

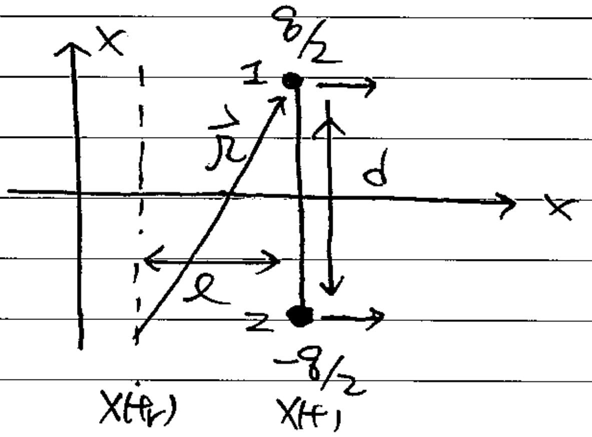
For a finite charge distribution, the failure of the 3rd Newton's law enables the existence of reaction force:



$$\vec{F}_{AB} + \vec{F}_{BA} \neq 0$$

This is the mechanism for the reaction force.

To illustrate the reaction force, we consider two point charges $\pm q/2$, connecting by a massless rod of length d. We shall take the limit $d \to 0$ at the end.



Suppose that this dumbbell moves in +X direction and is at rest at recorded time. The electric field on 1 due to 2 is (eq. 66)

$$\vec{E}_1 = \frac{q/2}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[(c^2-\underbrace{v^2}_{0})\vec{u}+\vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]$$

Where $\vec{u} = c\hat{\mathscr{r}} - \vec{v} = c\hat{\mathscr{r}}$ $(\vec{v}=0)$. $\begin{cases} \vec{\mathscr{r}} = l\hat{x} + d\hat{y} \\ l = X(t) - X(t_r) \\ t_r = \text{recorded time} \end{cases}$

$\because \vec{\mathscr{r}}\times(\vec{u}\times\vec{a}) = \vec{u}(\vec{\mathscr{r}}\cdot\vec{a}) - (\vec{\mathscr{r}}\cdot\vec{u})\vec{a}$

$\vec{\mathscr{r}}\cdot\vec{u} = c\mathscr{r}$, $\vec{\mathscr{r}}\cdot\vec{a} = la$

$\mathscr{r} = \sqrt{l^2+d^2}$, $u_x = c\dfrac{l}{\mathscr{r}}$

$$\therefore E_{1x} = \frac{q}{8\pi\epsilon_0}\frac{1}{c^3\mathscr{r}^2}\left[c^3\frac{l}{\mathscr{r}} + la\cdot c\frac{l}{\mathscr{r}} - c\mathscr{r}a\right]$$

$$= \frac{q}{8\pi\epsilon_0 c^2}\frac{1}{\mathscr{r}^3}\left[lc^2 + a(l^2-\mathscr{r}^2)\right]$$

$$= \frac{q}{8\pi\epsilon_0 c^2} \frac{[\ell c^2 - a d^2]}{(\ell^2+d^2)^{3/2}} \quad \text{---} (106)$$

Where $\ell = X(t) - X(t_r)$.

The force on 1 due to 2 is $\frac{q}{2}\vec{E}_1 \equiv \vec{F}_{12}$

Similarly, if the electric field on 2 due to 1 is $\vec{E}_2$, $\vec{F}_{21} = \frac{q}{2}\vec{E}_2$

The net force $\vec{F}^{int}_{self} = \frac{q}{2}(\vec{E}_1 + \vec{E}_2)$.

By symmetry, $E_{1y} + E_{2y} = 0$, $E_{1x} = E_{2x}$

$\therefore \vec{F}^{int}_{self} = \frac{q}{2}(E_{1x} + E_{2x})$

$$= q E_{1x} = \frac{q^2}{8\pi\epsilon_0 c^2} \frac{\ell c^2 - a d^2}{(\ell^2+d^2)^{3/2}} \hat{x} \quad \text{--} (107)$$

eq. (106)

$\vec{F}^{int}_{self}$ doesn't include the self-force of each $q/2$. To include those self-forces, one needs to consider $\pm q/2$ as extended charge distribution and find the self-force by integration. This would be more complicated. In the limit $d \to 0$, however, those self-forces can be included by assuming $F(q) =$ total self force on q in the limit $d \to 0$.

Then $$F(q) = 2F(q/2) + \lim_{d\to 0} F_{self}^{int} \quad \text{--- (108)}$$

Clearly, $\because F(q) \propto q^2$.

$$\therefore F(q/2) = \frac{1}{4} F(q)$$

$\therefore$ Eq. (108) implies

$$\vec{F}(q) = 2 \lim_{d\to 0} \vec{F}_{self}^{int}$$

$$= \lim_{d\to 0} \frac{q^2}{4\pi\epsilon_0 c^2} \frac{lc^2 - ad^2}{(l^2+d^2)^{3/2}} \hat{x} \quad \text{--- (109)}$$

$$= \frac{q^2}{4\pi\epsilon_0} \frac{1}{l^3} \hat{x} = \frac{q^2}{4\pi\epsilon_0} \frac{1}{|x(t) - x(t_r)|^3} \hat{x} \quad \text{--- (110)}$$

Eq. (110) is the exact field reaction of a point dumbbell charge. There is nothing unphysical!

For finite but small d,

$$\vec{F}_{rad} \underset{\text{eq (108)}}{=} \frac{q^2}{4\pi\epsilon_0 c^2} \frac{lc^2 - ad^2}{(l^2+d^2)^{3/2}} \hat{x} \quad \text{--- (111)}$$

If we expand $\vec{F}_{rad}$ in powers of d, we shall find the Abraham-Lorentz formula.

First,

$$X(t) = X(t_r) + \overset{u=0}{\dot{X}(t_r)}(t-t_r) + \frac{1}{2}\ddot{X}(t_r)(t-t_r)^2$$

$$+ \frac{1}{3!}\dddot{X}(t_r)(t-t_r)^3 + \cdots$$

Let $T = t - t_r$, one gets

$$l = X(t) - X(t_r) = \frac{1}{2}aT^2 + \frac{1}{6}\dot{a}T^3 + o(\ddot{a}T^4) \quad \text{— (112)}$$

$$a = \dot{x}(t_r)$$

$$\because (cT)^2 = r^2 = l^2 + d^2$$

$$\therefore d = \sqrt{(cT)^2 - l^2} = \left[(cT)^2 - \left(\frac{1}{2}aT^2 + \frac{1}{6}\dot{a}T^2 + \cdots\right)^2\right]^{\frac{1}{2}}$$

$$= cT\left[1 - \left(\frac{aT}{2c} + \frac{\dot{a}T^2}{6c} + \cdots\right)^2\right]^{1/2}$$

$$= cT\left[1 - \frac{1}{2}\left(\frac{aT}{2c}\right)^2 + (\ )T^3 + \cdots\right]$$

$$= cT - \frac{a^2}{8c}T^3 + (\ )T^4 + \cdots \quad (113)$$

Combing eqs (112) & (113) allows one to expand l in terms d. For this purpose we expand T in terms of d:

To $o(d)$, $d \simeq cT$, $T = \frac{d}{c} + o(d^3)$

$$\therefore (113) \Rightarrow d = cT - \frac{a^2}{8c}\left(\frac{d}{c}\right)^3 + o(d^4)$$

$$\therefore T = \frac{d}{c} + \frac{a^2 d^3}{8c^5} + o(d^4) \quad \text{--- (114)}$$

Substituting eq. (114) into (112), we get

$$l = \frac{a}{2c^2}d^2 + \frac{\dot{a}}{6c^3}d^3 + O(\ddot{a}d^4) + \cdots \quad (115)$$

Substituting eq. (115) into eq. (111), we find

$$\vec{F}_{rad} = \frac{q^2}{4\pi\epsilon_0 c^2}\, \frac{\frac{a}{2c^2}d^2c^2 - ad^2 + \frac{\dot{a}}{6c^3}d^3c^2 + \cdots}{\left[\left(\frac{a}{2c^2}d^2 + \frac{\dot{a}}{6c^3}d^3 + \cdots\right)^2 + d^2\right]^{3/2}}\,\hat{x}$$

$$= \frac{q^2}{4\pi\epsilon_0 c^2}\,\frac{1}{d}\,\frac{\frac{a}{2} - a + \frac{\dot{a}}{6c}d + O(\ddot{a}d^2)\cdots}{\left[1 + \underbrace{\left(\frac{a}{2c^2} + \frac{\dot{a}}{6c^3}d + \cdots\right)^2}_{O(a^2,\ \dot{a}a,\ \dot{a}^2)}\right]^{3/2}}\,\hat{x}$$

higher order

$$= \frac{q^2}{4\pi\epsilon_0}\left[-\frac{a}{2c^2 d} + \frac{\dot{a}}{6c^3} + O(\ddot{a})\right]\hat{x}$$

where $a = a(t_r)$ and $\dot{a} = \dot{a}(t_r)$

$$\because\ a(t_r) = a(t) + \dot{a}(t)(t_r - t) + \cdots$$

$$= a(t) - T\dot{a}(t) + \cdots$$

$$\underset{(114)}{=} a(t) - \dot{a}(t)\frac{d}{c} + \cdots$$

$$\therefore\ \vec{F}_{rad} = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{a(t)}{2c^2 d} + \frac{2\dot{a}(t)}{3c^3} + O(\ddot{a}d)\right]\hat{x} \quad (116)$$

Clearly, ($\because \vec{F}_{rad} = m\vec{a}$) the first term correct the mass of the charge. If m_0 is the mass of each $q/2$, one gets

$$m = 2m_0 + \frac{1}{4\pi\epsilon_0}\frac{q^2}{2c^2 d} \quad \text{--- (117)}$$

The correction $\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d}\frac{1}{c^2}$ ~ potential energy$/c^2$ ($E/c^2 = \Delta m$).

The 2nd term in (116) yields the Abraham–Lorentz formula:

$$F_{rad} = \frac{q^2 \dot{a}}{6\pi\epsilon_0 c^3}$$

$$= \frac{\mu_0 q^2}{6\pi c^2}\dot{a} \quad \text{--- (118)}$$

One sees that the Abraham–Lorentz term results from the expansion of the exact field reaction force in the size of the charge (d)

The mechanism for radiation reaction is clarified: it's due to the force of the charge on itself, i.e., force generated by different parts of charge distribution acting on one another.

The Landau-Lifshitz equation.

From the above dumbbell example, it is seen that the unphysical solution resulted from $\dot{a}$ is in fact due to the expansion in size of the charge.

$\therefore$ Eq. (102) is actually an expansion:

$$\vec{a} = \frac{\vec{F}_{ext}}{m} + \tau_0 \dot{\vec{a}} + \underbrace{\cdots}_{O(\ddot{\vec{a}} d)} \qquad (119)$$

If one can resum the series, one recovers eq. (110) without any unphysical results due to $\dot{a}$!

The problem with the Abraham-Lorentz equation is thus due to truncation of eq. (119)!

If we denote the characteristic time scale for change of a by T_a, $\frac{a}{\dot{a}} \sim T_a$

$\therefore \tau_0 \dot{a} \sim \frac{\tau_0}{T_a} a$. $\frac{\ddot{a}}{a} \sim \left(\frac{\tau_0}{T_a}\right)^2$

Eq. (119) should be:

$$\vec{a} = \frac{\vec{F}_{ext}}{m} + \tau_0 \dot{\vec{a}} + O\left(\frac{\tau_0^2}{T_a^2}\right) \quad \cdots (120)$$

Therefore, the expansion is valid only when $\tau_0 \dot{a} < \left(\frac{\tau_0}{T_a}\right)^2 a$. Hence the unphysical run-away solution can not grow beyond $\left(\frac{\tau_0}{T_a}\right)^2 a$. That is, if $a(t) = c\, e^{t/\tau_0}$ grows too large, the Abraham-Lorentz equation is no longer valid and its solution is no longer correct!

To get more sensible expansion, one first notices that Eq. (120) implies $\vec{a} = \frac{\vec{F}_{ext}}{m} + O\left(\frac{\tau_0}{T_a}\right)$

$\therefore$ By replace $\dot{\vec{a}}$ by $\frac{\dot{\vec{F}}_{ext}}{m}$, one gets $\vec{a} = \frac{\vec{F}_{ext}}{m} + \tau_0 \frac{\dot{\vec{F}}_{ext}}{m} + O\left(\frac{\tau_0^2}{T_a^2}\right)$ as well.

Hence by neglecting $O(\frac{\tau_0^2}{T_a^2})$, one obtains the Landau-Lifshitz equation:

$$m\vec{a} = \vec{F}_{ext} + \tau_0 \frac{d}{dt}\vec{F}_{ext} \quad \text{---} (120)$$

$$= \vec{F}_{ext} + \tau_0 \frac{\partial \vec{F}_{ext}}{\partial t} + \tau_0 (\vec{v}\cdot\vec{\nabla})\vec{F}_{ext}$$

L-- (121)

where $\vec{F}_{ext} = \vec{F}_{ext}(\underset{\uparrow}{\vec{r}(t)}, t)$

particle's position

$$\frac{d}{dt}\vec{F}_{ext}(\vec{r}(t), t) = \frac{\partial \vec{F}_{ext}}{\partial t} + (\underset{\uparrow}{\vec{v}}\cdot\vec{\nabla})\vec{F}_{ext}$$

$\dot{\vec{r}}(t)$

The Landau-Lifshitz equation is free from unphysical run-away solutions and can describe slowly moving charge particle more appropriately!

Example Radiation damping

In the Lorentz model of frequency dependent permittivity $\varepsilon(\omega)$, we model the electron as a harmonic oscillator. Let $\vec{r}(t)$ be the displacement of the electron from equilibrium. The external force $\vec{F}_{ext} = -m\omega_0^2 \vec{r}$ where ω_0 is the natural frequency.

The Landau-Lifshitz equation becomes

$$m\ddot{\vec{r}} = -m\omega_0^2 \vec{r} + \tau_0 \frac{d}{dt}(-m\omega_0^2 \vec{r})$$

$$= -m\omega_0^2 \vec{r} - m\tau_0\omega_0^2 \dot{\vec{r}} \quad \cdots (122)$$

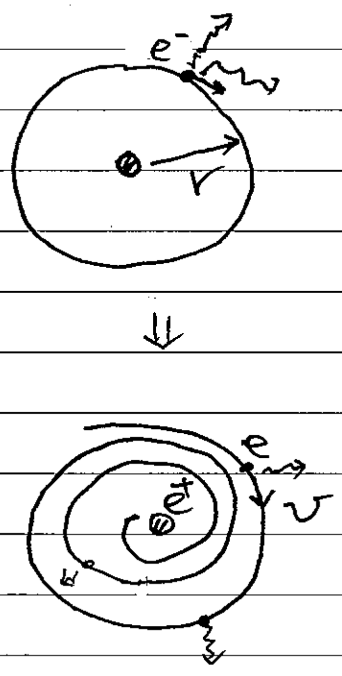
Therefore, the radiation introduces a damping force $-m\gamma\vec{v}$, with $\gamma = \tau_0\omega_0^2$

Note that in the Abraham-Lorentz formula, one obtains $\gamma = \tau_0\omega^2$ with ω being the frequency of oscillation, including correction of damping.

The damping is known as radiation damping. Without driving force, electrons will eventually be at rest at its equilibrium positions with $\vec{r} = 0$.

Example : Rutherford model of atom and its collapse.

Rutherford in 1911 proposed that atoms are similar to the system of the sun with planets (replaced by electrons) circulating around the sun.
The simplest atom is the hydrogen atom in which the electron makes circular motion around the neucleus (proton).



However, it's soon realized that atoms this model are unstable due to that electrons are accelerating and they will radiate energy and collapse to neucleus.

We can estimate the time of the electron to collapse by approximating intermediate orbits as circular orbits.

$$\therefore\ m\frac{v^2}{r} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2}, \quad mv^2 = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

$$\therefore\ E = \frac{1}{2}mv^2 - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r} = -\frac{1}{8\pi\epsilon_0}\frac{e^2}{r}$$

$$a = \frac{v^2}{r} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{mr^2}$$

Now. the power being radiated is

$$\frac{dE}{dt} = -\frac{\mu_0 q^2 a^2}{6\pi c}$$

$$\underset{\uparrow}{=} -\frac{1}{4\pi\epsilon_0}\frac{2e^2a^2}{3c^3}$$

$$\mu_0 = \frac{1}{\epsilon_0 c^2}$$

$$= -\left(\frac{1}{4\pi\epsilon_0}\right)^3 \frac{2e^6}{3c^3m^2r^4} \quad \cdots (123)$$

$$\frac{dE}{dt} = \frac{dE}{dr}\frac{dr}{dt} \simeq \frac{d}{dr}\left(-\frac{1}{8\pi\epsilon_0}\frac{e^2}{r}\right)\frac{dr}{dt}$$

$$= \frac{1}{8\pi\epsilon_0}\frac{e^2}{r^2}\frac{dr}{dt}$$

$\therefore$ Eq. (123) becomes $\displaystyle \frac{dr}{dt} \approx -\left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{4e^4}{3m^2c^3}\frac{1}{r^2}$

Integrating the above equation yields

$$r^3 \approx r_0^3 - \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{4e^4}{3m^2c^3} t$$

$$\equiv r_0^3 (1 - t/t_0) \qquad t_0^{-1} \equiv \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{4e^4}{m^2c^3r_0^3}$$

Neucleus $r = 0.5\times10^{-15}$ m, H-atom $r_0 = 0.5\times10^{-10}$ m

$$t \approx t_0 \ (\because r \ll r_0) = \left[(9\times10^9)^2 \frac{4\times(1.6\times10^{-19})^4}{(9.1\times10^{-31})^2(3\times10^8)^3(0.5\times10^{-10})^3}\right]^{-1}$$

$$= 1.3\times10^{-11}\,\text{sec}$$

$\therefore$ The atom collapses in 10^{-11} sec !