Potentials & Fields

Potential formulation of EM fields

The Maxwell's equations

(i) $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$

(ii) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(iii) $\vec{\nabla}\cdot\vec{B} = 0$

(iv) $\vec{\nabla}\times\vec{B} = \mu_0(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t})$

generally deviate from the static case.

Hence one can't write $\vec{E} = -\nabla V$.

However, $\because \vec{\nabla}\cdot\vec{B} = 0$ is always true.

One can write $\vec{B} = \vec{\nabla}\times\vec{A}$ --- ①

Then the Faraday's law

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \text{ becomes}$$

$$\vec{\nabla}\times(\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0 \quad \text{--- ②}$$

$$\therefore \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \quad \text{--- ③}$$

As we have indicated, $\because \oint \nabla V \cdot d\vec{\ell} = \int \vec{\nabla}\times\vec{\nabla}V \cdot d\vec{a} = 0$

$\therefore$ Only $-\frac{\partial \vec{A}}{\partial t}$ contributes the emf $\oint \vec{E}\cdot d\vec{\ell}$!

$\therefore -\frac{\partial \vec{A}}{\partial t}$ is the induced electric field.

Eqs. ① & ③ are equivalent to (i) & (iii).

Using ① & ③, (i) becomes

$$\vec{\nabla}\cdot\left(-\nabla V - \frac{\partial \vec{A}}{\partial t}\right) = \rho/\varepsilon_0$$

$$\therefore \nabla^2 V + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = -\rho/\varepsilon_0 \quad \text{-- ④}$$

Which replaces the Poisson's equation $\nabla^2 V = -\rho/\varepsilon_0$ in the static case.

(ii) becomes

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} - \mu_0\epsilon_0\nabla\frac{\partial V}{\partial t} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}$$

$$\because \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

$\therefore$ We get
$$\underbrace{\left(\nabla^2 - \epsilon_0\mu_0\frac{\partial^2}{\partial t^2}\right)}_{\Box^2}\vec{A} - \nabla\left(\nabla\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J} \quad \text{-- ⑤}$$

Therefore, finding solutions to the Maxwell's equations reduces to find V & $\vec{A}$ that satisfy eqs. ④ & ⑤.

This is the potential formulation of Maxwell's equations.

Example. Find the charge & current distributions that give rise to the potential

$$U = 0, \quad \vec{A} = \frac{\mu_0 k}{4c}(ct-|x|)^2 \hat{z} \quad |x| < ct$$

$$\qquad\qquad\quad 0 \qquad\qquad |x| > ct$$

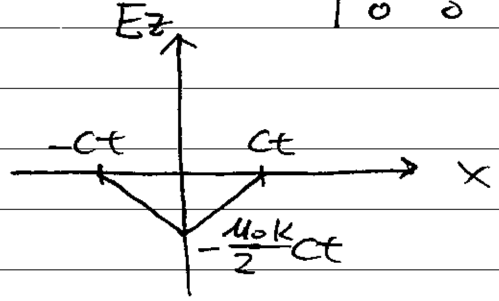
$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Solution:

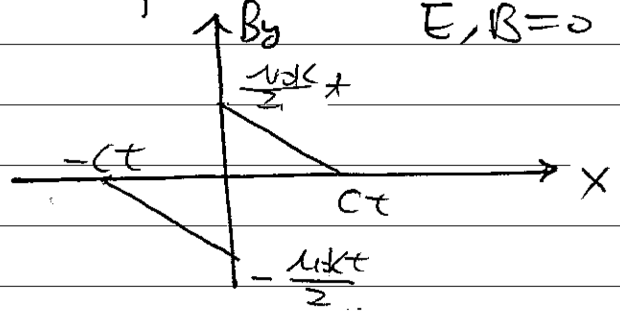
The corresponding $\vec{E}$ & $\vec{B}$ fields are

$$\left.\begin{aligned} \vec{E} &= -\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 k}{2}(ct-|x|)\hat{z} \\ \vec{B} &= \vec{\nabla}\times\vec{A} = -\frac{\mu_0 k}{4c}\frac{\partial}{\partial x}(ct-|x|)^2 \hat{y} \end{aligned}\right\} \; |x| < ct$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\mu_0 k}{4c}(ct-|x|)^2 \end{vmatrix} = \pm \frac{\mu_0 k}{2c}(ct-|x|)\hat{y}$$



$\vec{E}, \vec{B} = 0 \quad |x| > ct$



$$\therefore \vec{\nabla}\cdot\vec{E} = \frac{\partial E_z}{\partial z} = 0, \quad \rho = 0$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \left(\frac{\partial B_y}{\partial y} = 0\right)$$

$$\vec{\nabla}\times\vec{E} = \frac{\partial}{\partial x}\frac{\mu_0 k}{2}(ct-|x|)\hat{y} = \mp\frac{\mu_0 k}{2}\hat{y}$$

$$\vec{\nabla}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \pm\frac{\mu_0 k}{2c}(ct-|x|) & 0 \end{vmatrix}$$

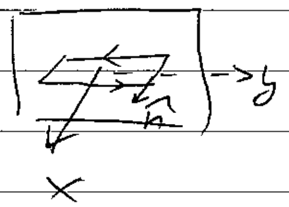
$$= \pm\frac{\mu_0 k}{2c}\frac{\partial}{\partial x}(ct-|x|)\,\hat{z} = -\frac{\mu_0 k}{2c}\hat{z}$$

$\because \frac{\partial \vec{E}}{\partial t} = -\frac{\mu_0 k c}{2}$ $\quad\therefore\ \epsilon_0\mu_0\frac{\partial \vec{E}}{\partial t} = -\frac{\mu_0 k}{2c}$

Hence from $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$, one gets $\vec{J} = 0$.

However, $\because B_y$ is discontinuous at $x=0$

& $\hat{y}\,[B_y(x=0^+) - B_y(x=0^-)] = \mu_0(\vec{K}\times\underbrace{\hat{x}}_{\hat{n}})$



$\therefore\ \mu_0 k t\,\hat{y} = \mu_0\vec{K}\times\hat{x}$

$\therefore\ \vec{K}\parallel\hat{z}$ with $\vec{K} = K\hat{z}$.

$\mu_0 K = \mu_0 k t \quad \therefore K = kt$

$\vec{K} = kt\,\hat{z}$

$\therefore$ There is a surface current at $x=0$ flowing in $\hat{z}$ direction: $\vec{K} = kt\,\hat{z}$

Gauge transformations & special gauges

In classical EM theory, $\vec{A}$ & V are potentials that are introduced mathematically to describe $\vec{E}$ & $\vec{B}$.

$\vec{A}$ & V have no absolute meaning and only their "relative values" yield $\vec{E}$ & $\vec{B}$ that are physical.

This reflects in the degree of freedom that one can choose $\vec{A}$ & V, called gauge freedom:

$\vec{A}'$ & $\vec{A}$ describes the same $\vec{B}$

as long as $\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A}$, i.e. $\vec{\nabla}\times(\vec{A}'-\vec{A}) = 0$

$$\therefore \vec{A}' = \vec{A} + \nabla\chi \quad \text{---} (6)$$

However, $\vec{E}$ is described by $\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t}$,

Changing $\vec{A}$ also changes $\vec{E}$.

To described the same $\vec{E}$, V also changes

$V' = V + \phi$, $\vec{E}' = -\nabla V' - \frac{\partial\vec{A}'}{\partial t} = -\nabla V - \nabla\phi - \frac{\partial\vec{A}}{\partial t} - \frac{\partial}{\partial t}\nabla\chi$

$= \vec{E} = -\nabla V - \nabla\phi$

$\therefore$ One requires $\nabla\phi + \frac{\partial}{\partial t}\nabla\chi = 0$

$\therefore$ One can choose $\phi = -\frac{\partial \chi}{\partial t}$

Therefore, $\vec{A}' = \vec{A} + \nabla\chi$

$V' = V - \frac{\partial \chi}{\partial t}$ --- ⑦

$(\vec{A}', V')$ & $(\vec{A}, V)$ describe the same $\vec{E}$ & $\vec{B}$

Eq. ⑦ is called the gauge transformation.

Under such transformation, $\vec{E}$ & $\vec{B}$ are the same, called gauge invariant.

$\vec{E}$ & $\vec{B}$ are

Since there are many possible χ,

there are many

ways to choose $\vec{A}$ & V, called gauge degree freedoms.

Depending on problems' nature, one can choose the most convenient $\vec{A}$ & V to work on. This is called choosing the gauge.

Coumlomb gauge $\vec{\nabla}\cdot\vec{A}=0$

If initially, $\vec{\nabla}\cdot\vec{A}^{\circ}\neq 0$, one can choose χ

$\ni$ $\vec{\nabla}\cdot(\vec{A}^{\circ}+\nabla\chi)=0$

$\therefore \nabla^2\chi = -\vec{\nabla}\cdot\vec{A}^{\circ}$ → viewed as $\frac{1}{\varepsilon_0}(\underbrace{-\varepsilon_0\vec{\nabla}\cdot\vec{A}^{\circ}}_{\rho})$

$\therefore$ One can choose $\chi(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{-\varepsilon_0\vec{\nabla}\cdot\vec{A}^{\circ}(r')}{|\vec{r}-\vec{r}'|}d\tau'$

So that $\nabla^2\chi = -\vec{\nabla}\cdot\vec{A}^{\circ}$

$\therefore$ One can always set $\vec{\nabla}\cdot\vec{A}=0$ --- (8)

This is the Coumlomb gauge. In this case, eq. (4) becomes

$$\nabla^2 V = -\rho/\varepsilon_0 \quad \text{--- (9)}$$

Which is the same as the Poisson's equation.

$\therefore$ $V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}',t')}{|\vec{r}-\vec{r}'|}d\tau'$ if V is taken to zero as $r\to\infty$ --- (10)

as the one given by the Coumlomb's law.

In this gauge, it does not mean that everything is the same as electrostatic as ρ may depend on t & V can also depend on time. Furthermore,

$\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t}$ still depends on $\vec{A}$.

through eq. ⑤ which becomes

$$\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \epsilon_0 \mu_0 \vec{\nabla} \frac{\partial V}{\partial t} \quad \text{--- (11)}$$

The advantage of the Coulomb gauge is that V is easily found. However, the disadvantage is that eq. ⑩ for $\vec{A}$ is difficult to solve.

Using eq. ⑩, one has ($\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$)

$$\frac{\partial V}{\partial t} = \frac{1}{4\pi\epsilon_0} \int d\tau' \frac{\frac{\partial \rho(r',t)}{\partial t}}{|\vec{r}-\vec{r}'|}$$

$$= -\frac{1}{4\pi\epsilon_0} \int d\tau' \frac{\vec{\nabla}' \cdot \vec{J}(r',t)}{|\vec{r}-\vec{r}'|}$$

One gets

$$\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \left[\vec{J} + \nabla \int \frac{d\tau'}{4\pi} \frac{\vec{\nabla}' \cdot \vec{J}(\vec{r}',t)}{|\vec{r}-\vec{r}'|} \right] \quad \text{- (12)}$$

Hence $\vec{A}$ is determined once $\vec{J}(\vec{r},t)$ is known.

Note that $\because \vec{\nabla} \cdot \vec{A} = 0$, RHS of eq. ⑫ needs to be divergenceless as well:

$$\nabla \cdot \text{RHS} = \vec{\nabla} \cdot \vec{J} + \nabla^2 \int \frac{d\tau'}{4\pi} \frac{1}{|\vec{r}-\vec{r}'|} \vec{\nabla}' \cdot \vec{J}(r',t) = 0 ?$$

Indeed, $\because \nabla^2 \frac{1}{|\vec{r}-\vec{r}'|} = -4\pi\, \delta(\vec{r}-\vec{r}')$

$$\nabla^2 \int \frac{dz'}{4\pi} \frac{1}{|\vec{r}-\vec{r}'|} \vec{\nabla}' \cdot \vec{J}(r',t)$$

$$= \int \frac{dz'}{4\pi} \nabla^2 \frac{1}{|\vec{r}-\vec{r}'|} \vec{\nabla}' \cdot \vec{J}(r',t) = -\int dz' \delta(\vec{r}-\vec{r}') \vec{\nabla}' \cdot \vec{J}(\vec{r}',t)$$

$$= -\vec{\nabla} \cdot \vec{J}(\vec{r},t)$$

$\therefore \vec{\nabla} \cdot \text{RHS} = 0 \quad \cdots (13)$

As we know; any vector $\vec{V}$ can be written as $\vec{V} = \vec{V}_\perp + \vec{V}_\parallel$

with $\vec{\nabla} \cdot \vec{V}_\perp = 0$ and $\vec{\nabla} \times \vec{V}_\parallel = 0$
(no divergence) (curless)

$\therefore$ Eq. (13) implies RHS of eq (12) $= \vec{J}_\perp$

(i.e. $\vec{\nabla} \cdot \vec{J}_\perp = 0$), i.e. divergenceless part of $\vec{J}$

$\therefore \vec{A}$ satisfies

$$\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}_\perp \quad \cdots (12)$$

<u>The Lorentz gauge</u> $\quad \vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} = 0$
$L \cdots (14)$

In this gauge, one chooses $\vec{A}$ & V to satisfy eq. (14), called the Lorentz gauge.

This is always possible. If $\vec{A}_0$ & V_0

do not satisfy eq. (14); one performs a gauge transformation

$$\vec{A} = \vec{A}_0 + \nabla\chi$$

$$V = V_0 - \frac{\partial \chi}{\partial t}$$

so that

$$\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}$$

$$= \vec{\nabla}\cdot\vec{A}_0 + \mu_0\epsilon_0\frac{\partial V_0}{\partial t} + \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\chi$$

with $\frac{1}{c^2} = \mu_0\epsilon_0$

Hence as long as one can find a χ satisfying

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\chi = -\vec{\nabla}\cdot\vec{A}_0 - \epsilon_0\mu_0\frac{\partial V_0}{\partial t}$$

$$\equiv f(\vec{r}, t) \quad \cdots \quad (15)$$

then $\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t} = 0$ is satisfied.

We shall see later that the solution to eq. (15) is

$$\chi(r,t) = -\frac{1}{4\pi}\int\frac{f(\vec{r}\,', t_r)}{|\vec{r}-\vec{r}\,'|}d\tau' . \quad \cdots \quad (16)$$

with $t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{c}$

Hence $\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t} = 0$ is achieved.

Using the Lorentz gauge, eq. (4) becomes

$$\left(\nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\right)V = -\rho/\epsilon_0 \quad \cdots \quad (17)$$

where we rewritten $\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = -\left(\frac{\partial^2}{\partial t^2}V\right)\mu_0\epsilon_0$.

Eq. ⑤ simplies to

$$\left(\nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0\vec{J} \quad \text{--- (18)}$$

Both $\vec{A}$ & V satisfies equations in a similar form

$$\Box^2\vec{A} = -\mu_0\vec{J} \quad \text{--- (19)}$$

$$\Box^2 V = -\rho/\epsilon_0 \quad \text{--- (20)}$$

With $\Box^2 = \nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ = d'Alembertian

$\Box^2 f = 0$ is the 4-dimensional version of Laplace equation. but $\vec{A}$ & V satisfy inhomogenous wave equations with sources $\vec{J}$ & ρ on RHS.

Since the Lorentz gauge yields more symmetric equations for $\vec{A}$ & V, we shall use the Lorentz gauge mostly in later part of this class.

<u>Lorentz force in potential form</u>

In the presence of $\vec{E}$ & $\vec{B}$, the force on a charge q is $q(\vec{E} + \vec{v} \times \vec{B})$. Hence

The equation of motion is

$$\frac{d\vec{\pi}(\vec{r})}{dt} = q\left[\vec{E}(\vec{r}(t), t) + \vec{v}(t) \times \vec{B}(\vec{r}(t), t)\right]$$

with $\vec{\pi} = m\vec{v}$ = momentum of particle L-(21)

Here we include the time-dependence of $\vec{r}(t)$ to emphasize that $\vec{E}$ & $\vec{B}$ are evaluated at the position $\vec{r}(t)$ of the particle.

It's more convenient to express eq. (21) in terms of potentials via

$$\vec{E} = -\nabla U - \frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\therefore \frac{d\vec{\pi}}{dt} = q\left[-\nabla U - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times (\vec{\nabla} \times \vec{A})\right]$$

Now $\vec{v} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{v} \cdot \vec{A}) - (\vec{v} \cdot \vec{\nabla})\vec{A}$

only act on $\vec{A}$

($\vec{v}$ has no position dependence)

$$\therefore \frac{d\vec{\pi}}{dt} = -q\left[\frac{\partial \vec{A}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{A} + \vec{\nabla}(U - \vec{v} \cdot \vec{A})\right] \cdots (22)$$

Here one first treat $\vec{r}$ & t in $\vec{A}(\vec{r}, t)$ as independent variables and perform $\vec{\nabla}$ & $\frac{\partial}{\partial t}$.

After differentiation, one then set $\vec{r} = \vec{r}(t)$.

The combination

$$\frac{\partial \vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A}$$

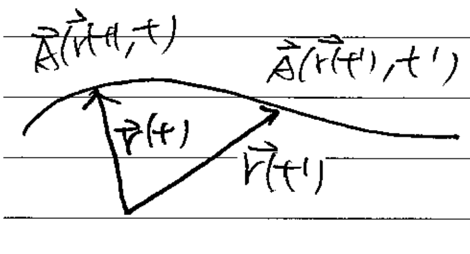
is called convective derivation of $\vec{A}$.

Similar to $\frac{d\vec{B}}{dt} = \frac{\partial \vec{B}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{B}$ as we discussed in Faraday's law for moving objects,

$$\frac{\partial \vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A} = \text{total time derivation of } \vec{A} \text{ that follows the particle}$$

Physically, as the particle moves, the vector potential it experiences is

$$\vec{a}(t) = \vec{A}(\vec{r}(t), t)$$



$$\therefore \frac{d\vec{a}}{dt} = \frac{d}{dt}\vec{A}(\vec{r}(t), t)$$

Using chain rule, $\frac{d\vec{A}(\vec{r}(t),t)}{dt} = \frac{d\vec{A}(x(t), y(t), z(t), t)}{dt}$

$$= \frac{dx(t)}{dt}\frac{\partial \vec{A}}{\partial x} + \frac{dy(t)}{dt}\frac{\partial \vec{A}}{\partial y} + \frac{dz(t)}{dt}\frac{\partial \vec{A}}{\partial z} + \frac{\partial \vec{A}}{\partial t}$$

$$\because \frac{d\vec{r}(t)}{dt} = \vec{v}(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$\therefore \frac{d\vec{A}(\vec{r}(t),t)}{dt} = v_x \frac{\partial}{\partial x}\vec{A} + v_y \frac{\partial}{\partial y}\vec{A} + v_z \frac{\partial \vec{A}}{\partial z} + \frac{\partial \vec{A}}{\partial t}$$

$$= \frac{\partial \vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A}$$

$$\therefore \frac{\partial \vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A} = \frac{d\vec{A}}{dt}$$

Eq. (22) can be written as

$$\frac{d}{dt}\vec{\pi} = -q\frac{d\vec{A}}{dt} - q\nabla(V - \vec{v}\cdot\vec{A})$$

$$\frac{d}{dt}(\vec{\pi} + q\vec{A}) = -\nabla U \quad \cdots (23)$$

with $U = V - \vec{v}\cdot\vec{A}$

Eq (23) is similar to the standard formula in mechanics : $\frac{d\vec{p}}{dt} = -\nabla U$ $\cdots (24)$ with $\vec{p}$ = canonical momentum.

Therefore, in the presence of $\vec{E}$ & $\vec{B}$, one replaces $\vec{p} = m\vec{v}$ by $\vec{p} = m\vec{v} + q\vec{A} = \vec{\pi} + q\vec{A}$ - (25) as the canonical momentum.

The usual eq. $\frac{d\vec{p}}{dt} = -\nabla U$

still works with $U = q(V - \vec{v}\cdot\vec{A})$ being a velocity-dependent potential.

The canonical momentum $\vec{p} = m\vec{v} + q\vec{A}$ plays a fundamental role in problems with particles in EM fields.

In the Hamiltonian formulation, one needs to express $\vec{v}$ in terms of $\vec{p}$ so that $\dot{p} = -\frac{\partial H}{\partial q}$ & $\dot{q} = \frac{\partial H}{\partial p}$ apply:

$$H = \frac{1}{2}mv^2 + qV = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + qV.$$

The energy is still given by $\frac{1}{2}mv^2 + qV$ but Hamiltonian is expressed in $\vec{p}$!

The appearance of $q\vec{A}$ in $\vec{p}$ and equation of motion enables one to view $\vec{A}$ as "potential momentum" per unit charge while V as potential energy per unit charge, which is also reflected

In the equation for change of energy :

$$\frac{d}{dt}\left(\frac{1}{2}mv^2 + qV\right) = \frac{\partial}{\partial t}\left[q(V - \vec{v}\cdot\vec{A})\right] \quad \text{(26)}$$

(problem 10.8)

General solutions for continuous charges and currents distribution

Retarded potentials

In the static case, eqs (18) & (20) reduce to

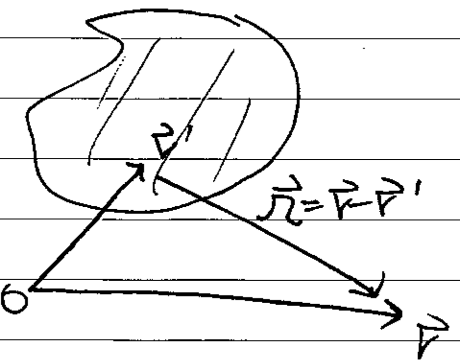
$$\nabla^2 V = -\frac{1}{\varepsilon_0}\rho$$

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

with the familiar solutions

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,d\tau' \quad \text{(27)}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,d\tau \quad \text{(28)}$$



To solve eq. (18) & (20), we first realize that changes of $\rho(\vec{r},t)$ & $\vec{J}(\vec{r},t)$ propagate in speed of light c.

Therefore, the potentials received at time t are actually containing information at older times

The older time is earlier.

Suppose the information at $\vec{r}'$ at some early time t_r propagates to $\vec{r}$ at time t.

t_r is called retarded time.
(延遲)

and satisfies $\quad t_r = t - \underbrace{\frac{|\vec{r}-\vec{r}'|}{c}}_{\text{propagation time}}$

Natural guessed solutions to Eqs (19) & (20) would be (in paralell to eqs. (27) & (28))

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}', t_r)}{|\vec{r}-\vec{r}'|} d\tau' \quad \cdots (29)$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}', t_r)}{|\vec{r}-\vec{r}'|} d\tau' \quad \cdots (30)$$

with t replaced by retarded time t_r.

These are called retarded potentials.

The guessed solutions of eqs (29) & (30) appear to be reasonable but they certainly need justification, as such guess would lead to wrong guess.:

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')\,d\tau'$$

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}',t_r)\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,d\tau'$$

Check: To check if (29) & (30) are indeed solutions, one needs to perform $\nabla^2 V$ & $\frac{\partial^2}{\partial t^2}V$.

Since $t_r = t - \mathscr{r}/c$, t_r depends on $\vec{r}$ as well.

$$\therefore \nabla V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\left[(\nabla\rho)\frac{1}{|\vec{r}-\vec{r}'|} + \rho\,\nabla\frac{1}{|\vec{r}-\vec{r}'|}\right]d\tau'$$

$$\nabla\frac{1}{|\vec{r}-\vec{r}'|} = -\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\nabla\rho = \nabla\rho(\vec{r}',t_r) = \underbrace{\frac{\partial}{\partial t_r}\rho(\vec{r}',t_r)}_{=\frac{\partial}{\partial t}\rho(\vec{r}',t_r)}\cdot\underbrace{\nabla t_r}_{-\frac{1}{c}\nabla|\vec{r}-\vec{r}'|}$$

$$= -\frac{\dot{\rho}}{c}\nabla\mathscr{r} = -\frac{\dot{\rho}}{c}\hat{\mathscr{r}}$$

$$\therefore \nabla V = \frac{1}{4\pi\epsilon_0}\int\left(-\frac{\dot{\rho}}{c}\frac{\hat{\mathscr{r}}}{\mathscr{r}} - \rho\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau' \quad \cdots (31)$$

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int\left\{-\frac{1}{c}\left[\frac{\hat{\mathscr{r}}}{\mathscr{r}}\cdot\nabla\dot{\rho} + \dot{\rho}\,\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right] - \left[\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\nabla\rho + \rho\,\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]\right\} \quad \cdots (32)$$

Now $\nabla\dot{\rho} = \frac{\partial\dot{\rho}}{\partial t_r}\cdot\nabla t_r = \ddot{\rho}\,\frac{-1}{c}\nabla\mathscr{r} = -\frac{\ddot{\rho}}{c}\hat{\mathscr{r}}$

$$\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}} = \nabla\cdot\frac{\vec{\mathscr{r}}}{\mathscr{r}^2} = (\vec{\nabla}\cdot\vec{\mathscr{r}})\frac{1}{\mathscr{r}^2} + \vec{\mathscr{r}}\cdot\nabla\frac{1}{\mathscr{r}^2}$$

$$= \frac{3}{\mathscr{r}^2} - \vec{\mathscr{r}}\cdot\hat{\mathscr{r}}\frac{\partial}{\partial\mathscr{r}}\frac{1}{\mathscr{r}^2} = \frac{1}{\mathscr{r}^2}$$

$$\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 4\pi\delta^3(\vec{\mathscr{r}})$$

$\therefore$ The integrand in eq. (31)

$$= \frac{-1}{c}\left[\frac{\hat{\mathscr{r}}}{\mathscr{r}}\cdot\frac{-\ddot{\rho}}{c}\hat{\mathscr{r}} + \cancel{\dot{\rho}\frac{1}{\mathscr{r}^2}}\right]$$

$$-\left[\cancel{\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\frac{-\dot{\rho}}{c}\hat{\mathscr{r}}} + \rho\,4\pi\delta^3(\vec{\mathscr{r}})\right]$$

$$= \frac{1}{c^2}\frac{\ddot{\rho}}{\mathscr{r}} - 4\pi\rho\,\delta^3(\vec{\mathscr{r}})$$

Eq. (31) becomes

$$\nabla^2 V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\left[\frac{1}{c^2}\frac{\ddot{\rho}}{\mathscr{r}} - 4\pi\rho\,\delta^3(\vec{\mathscr{r}})\right]d\tau' \quad \text{--- (33)}$$

$$\because \int\rho(\vec{r}',t_r)\,\delta^3(\vec{r}-\vec{r}')\,d^3\tau'$$

$$= \rho(\vec{r}', t_r|_{\vec{r}'=\vec{r}}) \underset{\uparrow}{=} \rho(\vec{r},t)$$

$$t_r = t - \frac{\mathscr{r}}{c}\Big|_{\mathscr{r}=0} = t$$

$$\ddot{\rho} = \frac{\partial^2}{\partial t^2}\rho(\vec{r},t_r)$$

$\therefore$ Eq. (33) implies $\nabla^2 V(\vec{r},t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underbrace{\int\frac{\rho}{4\pi\epsilon_0\mathscr{r}}d\tau'}_{V} - \frac{1}{\epsilon_0}\rho(\vec{r},t)$

$\therefore$ $\nabla^2 V(r,t) = -\rho/\epsilon_0$, V satisfies eq. (20).

eq. (29) is indeed a solution.

Similar computations for A_x & J_x, A_y & J_y and A_y & J_z show that indeed eq. (18) is a solution to eq. (19).

Advanced potentials

In the above derivation, $t_r = t - \frac{\mathscr{R}}{c}$, if one replaces t_r by $t_a = t + \mathscr{R}/c$ (advanced time),

$$V_a(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}', t_a)}{|\vec{r}-\vec{r}'|} d\tau'$$

$$\vec{A}_a(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}', t_a)}{|\vec{r}-\vec{r}'|} d\tau'$$

are called advanced potentials.

Clearly, to satisfy eq (18) & (20), t_r has be taken derivative twice. In this case, replacing t_r by t_a would not change eq. (33). Hence V_a & $\vec{A}_a$ satisfy eqs. (18) & (20) as well.

However, V_a & $\vec{A}_a$ represent solutions propagate from the future to t. They violate the principle of causality and hence are not considered!

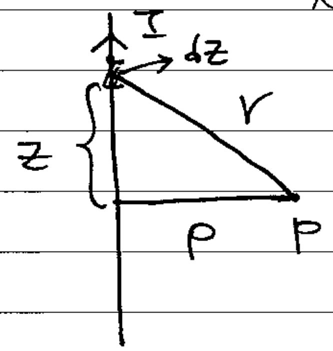
<u>Example</u>: An infinite straight wire carries the current

$$I(t) = \begin{cases} 0 & \text{for } t \le 0 \\ I_0 & \text{for } t > 0 \end{cases}$$

Find E & B.

<u>Solution</u>

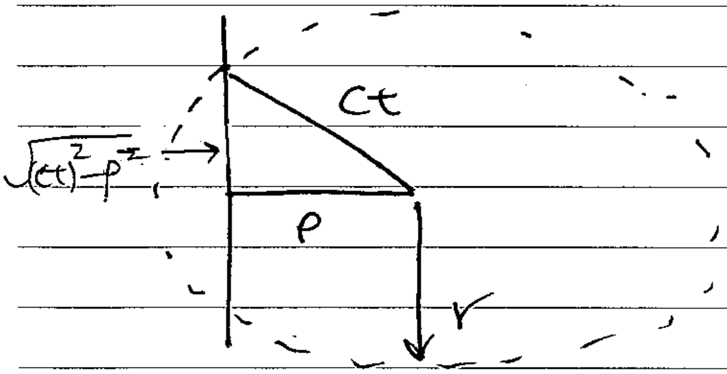
At the $\underbrace{t_r = t - r/c}_{\text{retarded time}}$,



$I(t_r) \neq 0$ only $t_r \geq 0$

$\therefore r \leq ct \quad I = I_0 \neq 0$

$r > ct \quad I = 0$



$\therefore$ At time t,

only $z \leq \sqrt{(ct)^2 - \rho^2}$

$I = I_0$

$z > \sqrt{(ct)^2 - \rho^2}$

$I = 0$

$$\therefore \vec{A}(\rho, t) = \frac{\mu_0}{4\pi} \hat{z} \int_{-\sqrt{(ct)^2-\rho^2}}^{\sqrt{(ct)^2-\rho^2}} \frac{dz\, I_0}{\sqrt{\rho^2+z^2}}$$

$$= \frac{2\mu_0 I_0}{4\pi} \hat{z} \int_0^{\sqrt{(ct)^2-\rho^2}} \frac{dz}{\sqrt{z^2+\rho^2}}$$

$z = \rho \sinh \theta$,

$$\frac{dz}{\sqrt{z^2+\rho^2}} = \frac{\rho\cosh\theta}{\rho\cosh\theta}\,d\theta = d\theta$$

$$\therefore \int \frac{dz}{\sqrt{z^2+\rho^2}} = \theta = \sinh^{-1}\frac{z}{\rho}$$

$$\sinh^{-1} y = x \qquad y = \sinh x = \frac{1}{2}(e^{x} - e^{-x})$$

$$\therefore (e^x)^2 - 2y(e^x) - 1 = 0$$

$$e^x = y + \sqrt{y^2+1} \qquad x = \ln\left(y + \sqrt{y^2+1}\right)$$

$$\therefore \int_0^{\sqrt{(ct)^2-\rho^2}} \frac{dz}{\sqrt{z^2+\rho^2}} = \ln \frac{z}{\rho} + \sqrt{\left(\frac{z}{\rho}\right)^2+1}\ \Bigg|_{z=0}^{z=\sqrt{(ct)^2-\rho^2}}$$

$$= \ln\left[\frac{\sqrt{(ct)^2-\rho^2}}{\rho} + \frac{ct}{\rho}\right]$$

$$\therefore \vec{A}(\rho,t) = \frac{\mu_0 I_0}{2\pi} \ln\left(\frac{ct + \sqrt{(ct)^2-\rho^2}}{\rho}\right)\hat{z}$$

$$\vec{E}(\rho,t) = -\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 I_0}{2\pi}\hat{z}$$

$$\times \frac{1}{\cancel{ct+\sqrt{(ct)^2-\rho^2}}} \underbrace{\left[c + \frac{1}{\cancel{2}}\frac{\cancel{2}c^2t}{\sqrt{(ct)^2-\rho^2}}\right]}_{\parallel \atop c \times \frac{\cancel{ct+\sqrt{(ct)^2-\rho^2}}}{\sqrt{(ct)^2-\rho^2}}}$$

$$= -\frac{\mu_0 I_0 c}{2\pi\sqrt{(ct)^2-\rho^2}}\hat{z}$$

$$\vec{B} = \nabla\times\vec{A} = -\frac{\partial A_z}{\partial \rho}\hat{\phi} = \frac{-\mu_0 I_0}{2\pi}\hat{\ } \left[\frac{\frac{-\rho}{\sqrt{(ct)^2-\rho^2}}}{ct+\sqrt{(ct)^2-\rho^2}} - \frac{1}{\rho}\right]$$

$$= \frac{\mu_0 I_0}{2\pi}\hat{\phi}\frac{1}{\rho[ct+\cancel{\sqrt{(ct)^2-\rho^2}}]}\Big[\underbrace{\frac{\rho^2}{\sqrt{(ct)^2-\rho^2}}+ct+\sqrt{(ct)^2-\rho^2}}_{\shortparallel}\Big]$$

$$\frac{1}{\sqrt{(ct)^2-\rho^2}}\left[ct\cancel{\sqrt{(ct)^2-\rho^2}}+(ct)^2\right]$$

$$= \frac{\mu_0 I_0}{2\pi}\frac{ct}{\sqrt{(ct)^2-\rho^2}}\hat{\phi}$$

$\underline{I(t)=q_0\delta(t)}$

$$\vec{A}(\rho,t)=\frac{\mu_0}{4\pi}\hat{z}\int_{-\infty}^{\infty}dz\,\frac{q_0\delta(\overset{t_r}{t-r/c})}{r}$$

$$r=\sqrt{\rho^2+z^2}$$

$$=\frac{\mu_0 q_0}{2\pi}\hat{z}\int_0^{\infty}dz\,\frac{\delta(t-r/c)}{r}$$

$z=\sqrt{r^2-\rho^2}$, $dz=\dfrac{r\,dr}{\sqrt{r^2-\rho^2}}$ $\quad z=0 \Leftrightarrow r=\rho$

$z=\infty$. $r=\infty$

$$\therefore \vec{A}(\rho,t)=\frac{\mu_0 q_0}{2\pi}\hat{z}\int_0^{\infty}\frac{1}{r}\delta(t-r/c)\frac{r\,dr}{\sqrt{r^2-\rho^2}}$$

Now, $\delta(t-r/c)=c\,\delta(r-ct)$

$$\therefore \vec{A}(\rho,t)=\frac{\mu_0 q_0}{2\pi}\hat{z}\,c\int_{\rho}^{\infty}\frac{\delta(r-tc)}{\sqrt{r^2-\rho^2}}dr$$

$$=\frac{\mu_0 q_0 c}{2\pi}\frac{1}{\sqrt{(ct)^2-\rho^2}}\hat{z} \qquad \rho<ct$$

$$=0 \qquad \rho>ct$$

$$\therefore \vec{E}(\rho,t) = -\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0 \dot{g}_0 c}{2\pi} \frac{c^2 t}{[(ct)^2-\rho^2]^{3/2}} \hat{z} \quad \text{for } \rho < ct$$

$$= 0 \quad \text{for } \rho > ct$$

$$\vec{B}(\rho,t) = -\frac{\partial A_z}{\partial \rho}\hat{\phi} = -\frac{\mu_0 g_0 c}{2\pi} \frac{\rho}{[(ct)^2-\rho^2]^{3/2}} \hat{z} \quad \text{for } \rho < ct$$

$$= 0 \quad \text{for } \rho > ct$$

$\vec{E}$ & $\vec{B}$ – Jefimenko's equation

Once $\vec{A}$ & U are known, one can deduce $\vec{E} = -\nabla U - \frac{\partial \vec{A}}{\partial t}$ & $\vec{B} = \nabla \times \vec{A}$.

∇U is given by eq. (31).

$$\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0}{4\pi} \int \frac{\frac{\partial}{\partial t}\vec{J}(\vec{r}', t_r)}{|\vec{r}-\vec{r}'|} d\tau' = \frac{1}{4\pi\epsilon_0}\frac{1}{c^2}\int \frac{\frac{\partial \vec{J}(\vec{r}', t_r)}{\partial t}}{|\vec{r}-\vec{r}'|} d\tau'$$

$$\therefore \vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \left[\frac{\rho(\vec{r}', t_r)}{\mathscr{R}^2}\hat{\mathscr{R}} + \frac{\dot{\rho}(\vec{r}', t_r)}{c\mathscr{R}}\hat{\mathscr{R}} - \frac{\dot{\vec{J}}(\vec{r}', t_r)}{c^2\mathscr{R}}\right] d\tau' \quad \text{L.. (34)}$$

$$\therefore \vec{E}(\vec{r},t) \neq \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}', t_r)}{\mathscr{R}^2}\hat{\mathscr{R}}\, d\tau' \quad (\text{Coulomb-like})$$

In addition to the Coulomb-like term, $\vec{E}$ contains $\dot{\rho}$ & $\dot{\vec{J}}$ terms due to ρ & $\vec{J}$'s time-dependence!

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To find $\vec{B}$, we have

$$\vec{\nabla}\times\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \nabla\times\frac{\vec{J}(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|}\,d\tau'$$

$\vec{J}(\vec{r}',t_r)$ now depends on r via t_r

$$t_r = t-\frac{|\vec{r}-\vec{r}'|}{c} = t-\frac{\mathcal{R}}{c}$$

$$\therefore \vec{\nabla}\times\frac{\vec{J}(\vec{r}',t_r)}{\mathcal{R}} = \frac{1}{\mathcal{R}}\vec{\nabla}\times\vec{J}(\vec{r}',t_r) + \nabla\frac{1}{\mathcal{R}}\times\vec{J}(\vec{r}',t_r)$$

$$\because \frac{\partial J_\alpha(\vec{r}',t_r)}{\partial x_\beta} = \frac{\partial J_\alpha}{\partial t_r}\frac{\partial t_r}{\partial x_\beta} = -\frac{1}{c}\frac{\partial\mathcal{R}}{\partial x_\beta}\frac{\partial J_\alpha}{\partial t}$$

$$= -\frac{1}{c}\dot{J}_\alpha\frac{\partial\mathcal{R}}{\partial x_\beta}$$

$$\therefore [\vec{\nabla}\times\vec{J}(\vec{r}',t_r)]_\alpha = \varepsilon_{\alpha\beta\gamma}\frac{\partial}{\partial x_\beta}J_\gamma$$

$$= -\frac{1}{c}\varepsilon_{\alpha\beta\gamma}\frac{\partial\mathcal{R}}{\partial x_\beta}\dot{J}_\gamma$$

$$= \frac{1}{c}(\dot{\vec{J}}\times\nabla\mathcal{R})_\alpha = \frac{1}{c}(\dot{\vec{J}}\times\hat{\mathcal{R}})_\alpha$$

$$\therefore \frac{1}{\mathcal{R}}\vec{\nabla}\times\vec{J} = \frac{1}{c}\frac{1}{\mathcal{R}}\dot{\vec{J}}\times\hat{\mathcal{R}}$$

$$\because \nabla\frac{1}{\mathcal{R}} = \frac{-1}{\mathcal{R}^2}\hat{\mathcal{R}}$$

$$\therefore \vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\int\left[\frac{\vec{J}(\vec{r}',t_r)\times\hat{\mathcal{R}}}{\mathcal{R}^2} + \frac{\dot{\vec{J}}(\vec{r}',t_r)\times\hat{\mathcal{R}}}{c\mathcal{R}}\right]d\tau'$$

$$= \frac{\mu_0}{4\pi}\int\left[\frac{\vec{J}(\vec{r}',t_r)}{\mathcal{R}^2} + \frac{\dot{\vec{J}}(\vec{r}',t_r)}{c\mathcal{R}}\right]\times\hat{\mathcal{R}}\,d\tau' \quad \text{--- (35)}$$

Eqs. (34) & (35) are explicit expressions for $\vec{E}$ & $\vec{B}$ and are called Jefimenko's equations.

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General solutions for point charges

Liénard-Wiechert potentials

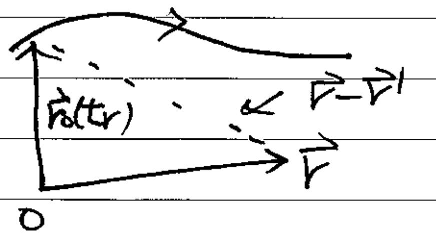
For a point charge q, it has a specific trajectory $\vec{r} = \vec{r}_0(t)$

= position of q at time t.

$$\therefore \quad \rho(\vec{r}, t) = q\,\delta(\vec{r} - \vec{r}_0(t))$$

$$\rho(\vec{r}\,', t_r) = q\,\delta(\vec{r}\,' - \vec{r}_0(t_r)) \quad \text{--- (3-6)}$$

$$t_r = t - \frac{|\vec{r} - \vec{r}\,'|}{c}, \quad \vec{r} = \text{position for observation}$$



It may seem naively that the retarded potential

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,', t_r)}{|\vec{r} - \vec{r}\,'|}\, d\tau'$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r} - \vec{r}_0(t_r)|} \quad ?$$

but it turns out that this is wrong!
The δ function in ρ indeed enables one to move $|\vec{r} - \vec{r}\,'|$ out and set $\vec{r}\,' = \vec{r}_0(t_r)$!

However, $\int \rho(\vec{r}', t_r)\, d\tau' \neq q$! ...... (37)

due to a subtle effect: a moving charge may appear as a different size with different q due to ~~that~~ one needs to collect ρ in different $t_r = t - \frac{|\vec{r} - \vec{r}_0(t_r)|}{c}$.

Mathematically, one can insert another $\delta(t' - t_r)$ into $V(\vec{r}, t)$:

replaced by t_r

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int d\tau' \int dt' \, \frac{\delta(t' - t_r)\; q\,\delta(\vec{r}' - \vec{r}_0(t'))}{|\vec{r} - \vec{r}'|} \quad \text{...... (38)}$$

Now one can perform $\int d\tau'$ in eq (38), one gets using $t_r = t - \frac{|\vec{r} - \vec{r}'|}{c}$

$$V(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \int dt' \, \frac{\delta\left(t' - t + \frac{|\vec{r} - \vec{r}_0(t')|}{c}\right)}{|\vec{r} - \vec{r}_0(t')|} \quad \text{...... (39)}$$

To integrate t', one performs a change of variable:

$$u = t' - t + \frac{|\vec{r} - \vec{r}_0(t')|}{c}$$

$$= t' - t + \frac{\sqrt{(\vec{r} - \vec{r}_0(t'))^2}}{c}$$

$$\therefore \quad \frac{du}{dt'} \underset{\substack{\uparrow \\ t=\text{fixed}}}{=} 1 + \frac{1}{c}\frac{d}{dt'}\sqrt{\underbrace{(x-x_0(t'))^2+(y-y_0(t'))^2+(z-z_0(t'))^2}_{\omega}}$$

$$= 1 + \frac{1}{c}\frac{d\sqrt{\omega}}{d\omega}\frac{d\omega}{dt'}$$

$$= 1 + \frac{1}{c}\frac{1}{2}\frac{1}{\sqrt{\omega}}\Big[2(x-x_0(t'))\Big(\underbrace{-\frac{dx_0(t')}{dt'}}_{-v_x(t')}\Big)$$

$$+2(y-y_0(t'))\Big(\underbrace{-\frac{dy_0(t')}{dt'}}_{-v_y(t')}\Big)$$

$$+2(z-z_0(t'))\Big(\underbrace{-\frac{dz_0(t')}{dt'}}_{-v_z(t')}\Big)\Big]$$

$$= 1 - \frac{1}{c}\frac{[\vec{r}-\vec{r}_0(t')]\cdot\vec{v}(t')}{|\vec{r}-\vec{r}_0(t')|} = 1 - \frac{\hat{\mathcal{R}}(t')\cdot\vec{v}(t')}{c} \qquad \text{L - (39) - 1}$$

where $\hat{\mathcal{R}}(t') = \dfrac{\vec{r}-\vec{r}_0(t')}{|\vec{r}-\vec{r}_0(t')|}$

$\therefore$ Eq. (39) becomes

$$V(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\int du\,\frac{dt'}{du}\,\frac{\delta(u)}{|\vec{r}-\vec{r}_0(t')|}$$

$$= \frac{q}{4\pi\epsilon_0}\int \frac{du}{1-\frac{\hat{\mathcal{R}}(t')\cdot\vec{v}(t')}{c}}\,\frac{\delta(u)}{|\vec{r}-\vec{r}_0(t')|} \quad \cdots (40)$$

where t' is considered as a function of u

such that $u = t'(u) - t + \frac{1}{c}|\vec{r}-\vec{r}_0(t'(u))|$.

and $\hat{\mathcal{R}}(t') = \dfrac{\vec{r}-\vec{r}_0(t')}{|\vec{r}-\vec{r}_0(t')|}$

$\delta(u)$ enforces $u=0$

$$\therefore \; t' = t - \frac{1}{c}\left|\vec{r} - \vec{r}_0(t')\right|$$

$t' \doteq t_r$ of t

$\therefore$ The integration of eq. (40) can be done now, one gets

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \frac{q}{\left(1 - \frac{\vec{\mathcal{R}}(t_r)\cdot\vec{v}(t_r)}{c}\right)\cdot \mathcal{R}(t_r)}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{cq}{c\mathcal{R}(t_r) - \vec{\mathcal{R}}(t_r)\cdot\vec{v}(t_r)} \quad \cdots \text{(41)}$$

With $t_r = t - \frac{1}{c}\left|\vec{r} - \vec{r}_0(t_r)\right|$

Similarly, for a point charge q,

$$\vec{J}(r,t) = \rho(\vec{r},t)\,\vec{v}(t),$$

Eq. (30) implies

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d\tau' \frac{\rho(\vec{r}',t_r)\,\vec{v}(t_r)}{|\vec{r}-\vec{r}'|}$$

Following the same treatment as eq. (38), one gets

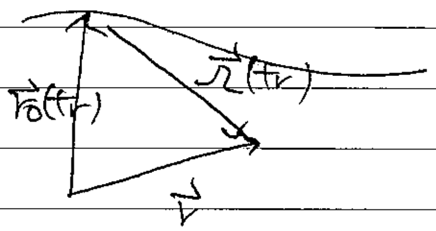
$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{cq\,\vec{v}(t_r)}{c\mathcal{R}(t_r) - \vec{\mathcal{R}}(t_r)\cdot\vec{v}(t_r)} \quad \cdots \text{(42)}$$

Eqs. (41) & (42) are the well-known Liénard-Wiechert

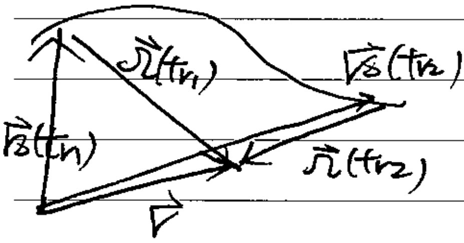
~~potentials with t_r being the recorded time~~ of t:

$$c(t-t_r) = \underbrace{|\vec{r}-\vec{r}_0(t_r)|}_{\vec{\mathcal{R}}(t_r)} \quad --- \quad (43)$$

It's important to note that for a given t, there is only one $\vec{r}_0(t_r)$ satisfying eq. (43): if there are two points on the trajectory $\vec{r}_0(t)$ that satisfy eq. (43), they must occur at two different t_{r1} & t_{r2}.

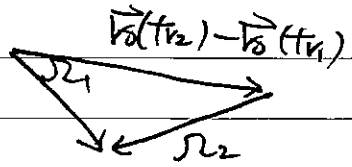


so that
$$c(t-t_{r1}) = |\vec{r}-\vec{r}_0(t_{r1})| \equiv \mathcal{R}_1$$
$$c(t-t_{r2}) = |\vec{r}-\vec{r}_0(t_{r2})| \equiv \mathcal{R}_2$$



Then $c(t_{r2}-t_{r1}) = \mathcal{R}_1 - \mathcal{R}_2$

For any triangle,



$$|\mathcal{R}_1-\mathcal{R}_2| < |\vec{r}_0(t_{r2})-\vec{r}_0(t_{r1})|$$

$$\therefore \ c < \frac{|\vec{r}_0(t_{r2})-\vec{r}_0(t_{r1})|}{|t_{r2}-t_{r1}|}$$

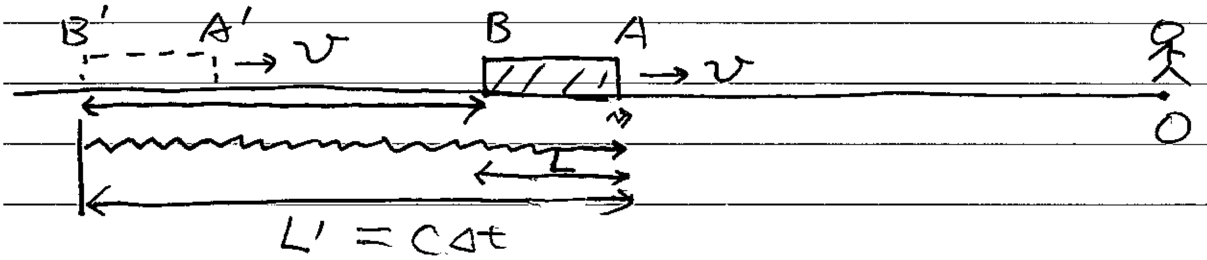
$\therefore$ The average speed of the particle $> c$ which is not possible. $\therefore$ There is only one retarded position for any t & $\vec{r}$!

Physically, total charge at retarded positions $\int \rho(\vec{r}\,', t_r)\, d\tau' = \dfrac{q}{1 - \frac{\hat{\mathscr{r}}(t_r)\cdot\vec{v}(t_r)}{c}}$ --- (44)

$\neq q$

This is a purely geometry effect and can be understood by a more vivid example:

a moving train looks longer when it comes towards you or shorter when it moves away from you!



As shown on the above, light from the back of train (B, B') needs to travel more distance to arrive at the same time to the observer O. The extra distance can be found by considering the moment when light just leaves A.

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Let the actual length of train be L.

The extra length B travels $= L'-L$

The duration for light travelling from B' to A

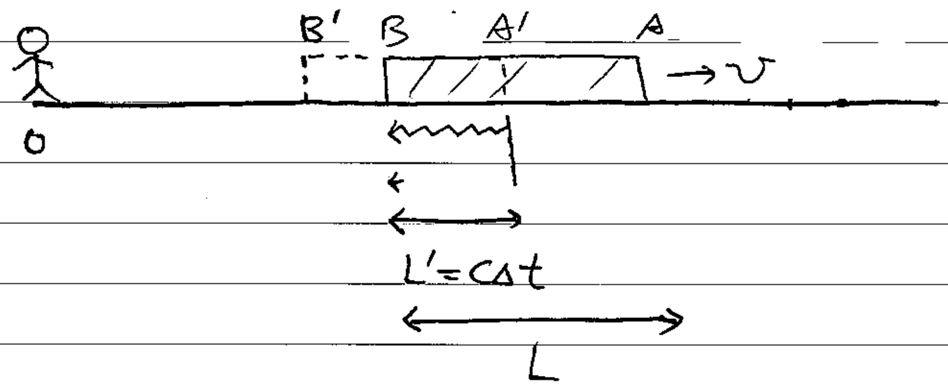
$= \Delta t$, $\therefore$ $L' = c\Delta t$

~~$L'-L = v\Delta t$~~

$$\frac{L'}{c} = \frac{L'-L}{v} \Rightarrow L' = \frac{L}{1-v/c}$$

L' is the apparent length of the tram that O found, which is longer than L!

Similarly, if the tram is moving away, one has



$L' = c\Delta t$

$L - L' = v\Delta t$

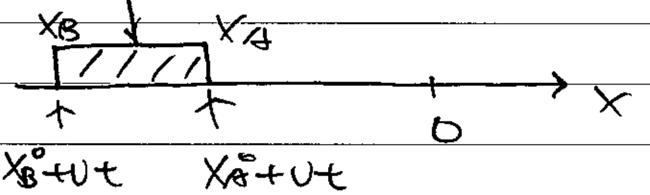
$$\frac{L'}{c} = \frac{L-L'}{v}$$

$\therefore L' = \frac{L}{1+v/c} < L$ is shorter!

The above derivation can be viewed more intuitively in the following way.

If one takes the moving direction as x-axis, the train can be described by

$X(t) = X_0 + vt$ with $X_B^0 \le X_0 \le X_B^0 + L = X_A^0 < 0$



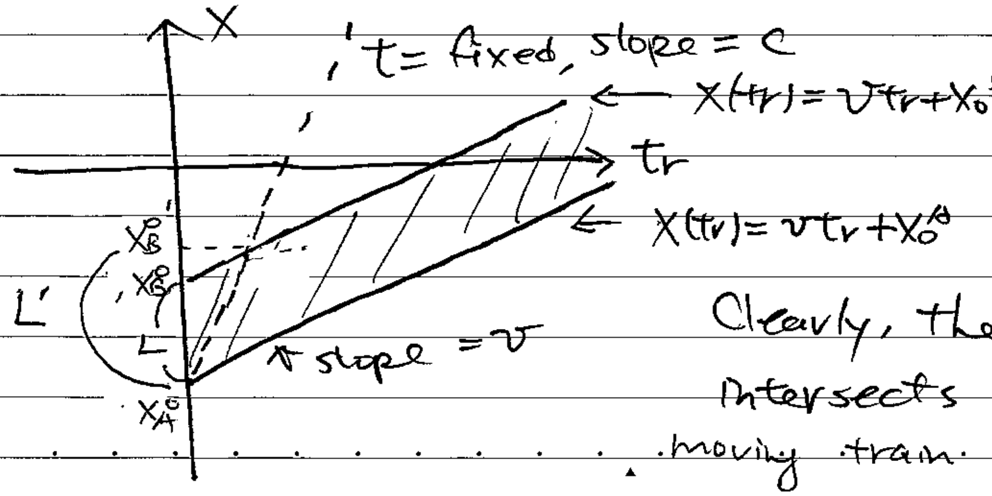
For a fixed observing time t, the points that be observed are (X, t_r) t_r = retorded time satisfying

$$C(t - t_r) = |X| \quad \text{--- (45)}$$

For $X < 0$, it becomes

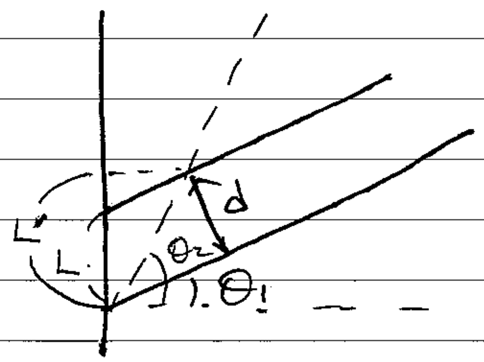
$$Ct = Ct_r - X \quad \text{--- (46)}$$

Which is a line of slope c shown in below.



Clearly, the t-fixed line intersects with the moving train at X_P & X_B^0!

Obviously, $X_B^{0\prime} - X_A^0 = L' > X_B^0 - X_A^0 = L$.



quantitatively,

$$d = L \sin(\pi/2 - \theta_1)$$
$$= \frac{L'}{\cos(\pi/2 - \theta_2)} \times \sin(\theta_2 - \theta_1)$$

$$\therefore \quad L \cos\theta_1 = \frac{L'}{\sin\theta_2}(\sin\theta_2\cos\theta_1 - \sin\theta_1\cos\theta_2)$$

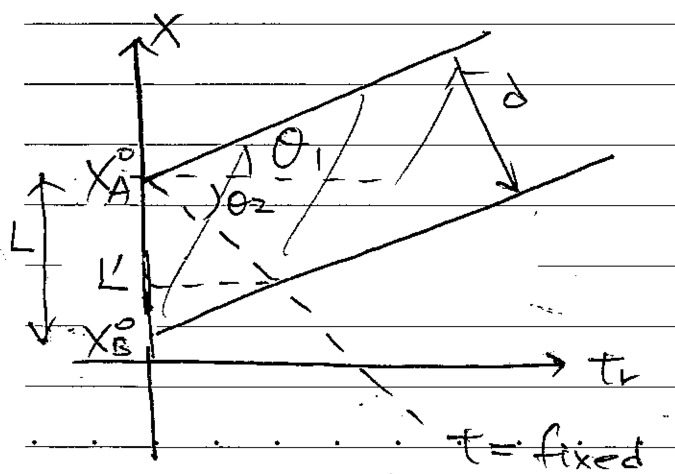
$$L' = \frac{L}{1 - \tan\theta_1 \cot\theta_2}$$

$\because \tan\theta_1 = v$, $\tan\theta_2 = c$ $\quad \therefore \tan\theta_1\cot\theta_2 = v/c$

$$\therefore \quad L' = \frac{L}{1 - v/c}$$

For $x > 0$, eq. (45) becomes $\quad ct = ct_r + x$ $\quad$ ← (47)

The slope of fixed-t line is negative.



For positive X_A^0 & X_B^0, the intersection is shown in the left figure.

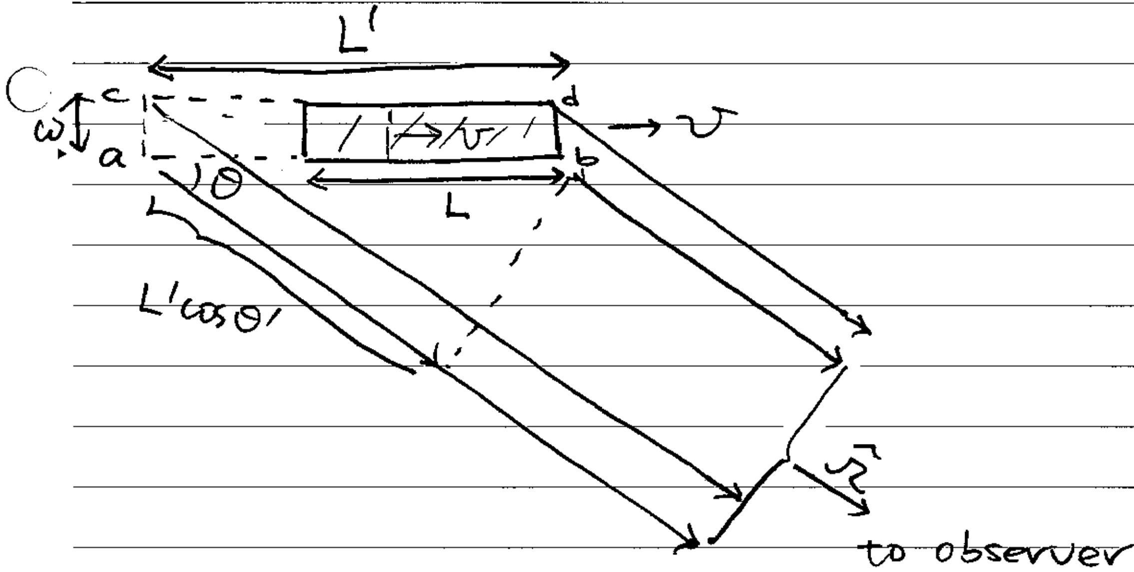
$$d = L \sin(\pi/2 - \theta_1)$$
$$= \frac{L'}{\cos(\pi/2 - \theta_2)} \sin(\theta_1 + \theta_2)$$

$$\therefore L\cos\theta_1 = \frac{L'}{\sin\theta_2}\left(\sin\theta_2\cos\theta_1 + \sin\theta_1\cos\theta_2\right)$$

$$L' = \frac{L}{1+\tan\theta_1\cot\theta_2} = \frac{L}{1+v/c}$$

In general, when $\vec{v}$ is not in the same direction as the connecting vector $\vec{r}$, one gets

$$L' = \frac{L}{1-\frac{1}{c}\vec{v}\cdot\hat{r}} \quad \text{---- (48)}$$



As show in the above figure, $\dfrac{L'\cos\theta}{c} = \dfrac{L'-L}{v}$

$$\therefore L' = \frac{L}{1-\frac{v}{c}\cos\theta} = \frac{L}{1-\frac{1}{c}\vec{v}\cdot\hat{r}}$$

This applies to both $\overline{ab}$ & $\overline{cd}$. Furthermore, lights from c&d are delayed relative to a&b. However, $\overline{ab}$ looks the same distance apart!

There is no moving along $\overline{ac}$ direction.

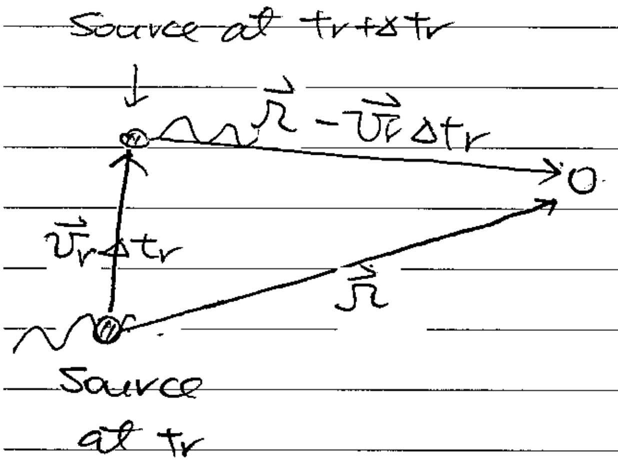
Hence $\overline{ac}$ will appear in the same length as w.

$\therefore$ The apparent volume τ' of the train is related to the actual volume τ by

$$\tau' = \frac{\tau}{1 - \frac{1}{c}\vec{v}\cdot\hat{\mathscr{r}}} \quad \text{--- (49)}$$

<u>Oberved pulse duration</u>

The effect of apparent volume can be also presented as changes of pulse duration.



Suppose that the charge is a source that emits a short pulse of radiation from time t_r to $t_r + \Delta t_r$.

The front of the pulse reaches the observer at $t = t_r + \frac{\mathscr{r}}{c}$ --- (50)

At the end of the pulse, the source is at the position $(-\vec{\mathscr{r}} + \vec{v}_r \Delta t_r)$ relative to O.

$\therefore$ The end of the pulse reaches the observer at $t' = t_r + \Delta t_r + \frac{1}{c}|\vec{\mathscr{r}} - \vec{v}_r \Delta t_r|$ --- (51)

For large $\vec{\mathscr{R}}$. $|\vec{\mathscr{R}} - \vec{v}_r \Delta t_r|$

$$= \sqrt{\mathscr{R}^2 - 2\vec{\mathscr{R}}\cdot\vec{v}_r \Delta t_r + v_r^2 \Delta t_r^2}$$

$$= \mathscr{R}\left[1 - 2\frac{\vec{v}_r\cdot\vec{\mathscr{R}}}{\mathscr{R}}\Delta t_r + \left(\frac{v_r}{\mathscr{R}}\right)^2 \Delta t_r^2\right]^{\frac{1}{2}}$$

$$\approx \mathscr{R}\left(1 - \frac{\vec{v}_r\cdot\vec{\mathscr{R}}}{\mathscr{R}}\Delta t_r\right) + O(\Delta t_r^2)$$

$$\therefore \quad t' = t_r + \Delta t_r + \frac{1}{c}(\mathscr{R} - \vec{v}_r\cdot\hat{\mathscr{R}}\Delta t_r) + O(\Delta t_r^2)$$

$$= t_r + \frac{\mathscr{R}}{c} + \Delta t_r\left(1 - \frac{1}{c}\vec{v}_r\cdot\hat{\mathscr{R}}\right) + O(\Delta t_r^2)$$

$\therefore$ The observed duration $t' - t = \Delta t \neq \Delta t_r$:

$$\Delta t = \Delta t_r\left(1 - \frac{1}{c}\vec{v}_r\cdot\hat{\mathscr{R}}\right) \quad \cdots (52)$$

For a point charge, it is a point

for a fixed t_r (not fixed t) $\quad \int dz'\, \delta(\vec{r}' - \vec{r}_0(t_r)) = 1$

This is why in eq. (38), one gets $\quad \int dz' = q$

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0}\int dz' \int \underbrace{dt'}_{\displaystyle \frac{du}{1 - \frac{1}{c}\vec{v}_r\cdot\hat{\mathscr{R}}}} \frac{\delta(t' - t_r)\, q\,\delta(\vec{r}' - \vec{r}_0(t'))}{|\vec{r} - \vec{r}'|}$$

$$\left(dt_r = \frac{du}{1 - \frac{1}{c}\vec{v}_r\cdot\hat{\mathscr{R}}}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{1 - \frac{1}{c}\vec{v}_r\cdot\hat{\mathscr{R}}} \quad \text{see eq. (38)} - 1$$

Relation to Doppler effect

Either the length change or the duration change

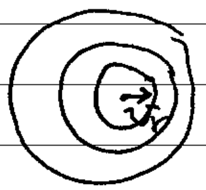
$$L' = \frac{L}{1 \mp v_r/c} \quad \cdots \text{(52)-1}$$

($-$ for approaching to observer, $+$ for leaving the observer)

$$\Delta t = \Delta t_r (1 \mp v_r/c) \quad \cdots \text{(52)-2}$$

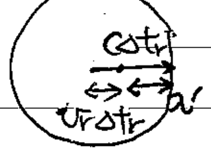
they are essentially non-relativistic Doppler effect:

Consider a point source of light moving in v_r



In the observer's frame, the source emits light $\underbrace{\ }_{\text{pulse}}$ every Δt_r period.

During $\overset{\text{a period}}{\Delta t_r}$, light propagates $c\Delta t_r$ while the source moves $v_r \Delta t_r$



$\therefore \lambda' = (c \mp v_r)\Delta t_r$, λ' = observed wavelength

Now, $\lambda' = c\Delta t$, Δt = observed period.

$\therefore \Delta t = \left(1 \mp \frac{v_r}{c}\right)\Delta t_r$, which is precisely eq (52)-2.

Now, according to special relativity, if the period in the rest frame of the source is Δt_0, then

$$\Delta t_r = \frac{\Delta t_0}{\sqrt{1 - v_r^2/c^2}} \quad \cdots \text{(52)-3} \qquad \therefore \Delta t = \left(1 \mp \frac{v_r}{c}\right)\frac{\Delta t_0}{\sqrt{1-(v_r/c)^2}}$$

$$\Delta t = \sqrt{\frac{1 \mp v_r/c}{1 \pm v_r/c}}\ \Delta t_0$$

In terms of frequency $\nu = \frac{1}{\Delta t}$, $\nu_0 = \frac{1}{\Delta t_0}$,

One gets

$$\nu = \sqrt{\frac{1+\beta_r}{1-\beta_r}}\ \nu_0 \quad (\beta_r = v_r/c, \text{ approaching})$$

--- (5-2)-4

or

$$\nu = \sqrt{\frac{1+\beta_r}{1-\beta_r}}\ \nu_0 \quad (\text{leaving})$$

Which are precisely the Doppler effect in terms of frequency!

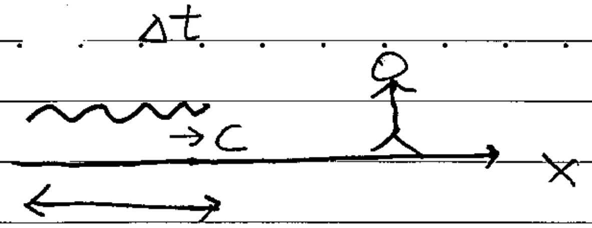
Note that for non-relativistic Doppler effect, one thinks that there is a medium for light to propagate and hence one stops at eq. (5-2)-2 and treat $\Delta t_r = \Delta t_0$ (t is universal in Galilean transformation). One does not have eq. (5-2)-3

Hence $$\nu = \frac{\nu_0}{1 \mp v_r/c}$$ (Galilean Doppler effect)

--- (5-2)-5

Relation (5-2)-1 is also the Doppler effect.

From eq. (5-2)-2, one can deduce eq. (5-2)-1 by considering a wave pulse during Δt
that moves in $+x$ direction of

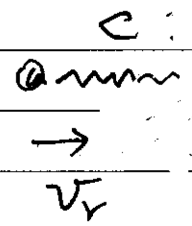


For a fixed observer,

$c\Delta t$ = length of pulse which plays the role of L (for the train), L is the real length of the pulse.

$\therefore c\Delta t = L$

If the pulse is emitted by a moving emitter,



the emitting time $\Delta t_r \neq \Delta t$

but satisfies $\Delta t = \Delta t_r(1 \mp v_r/c)$

$c\Delta t_r$ is the apparent length ("emitting length") seen by the emitter, $\therefore c\Delta t_r$ plays the role of L'. ($c\Delta t_r$ = traveling length for head of pulse $\cong \overline{B'A}$ in 10-31)

$\Delta t = \Delta t_r(1 \mp v_r/c)$ then implies

$c\Delta t = c\Delta t_r(1 \mp v_r/c)$, $L = L'(1 \mp v_r/c)$

$\therefore L' = \dfrac{L}{1 \mp v_r/c}$ which shows eq. (52) -1 is also due to Galilean Doppler effect.

Note that if L_0 = proper length of the train in the rest frame of the train, due to Lorentz contraction, $L = L_0\sqrt{1 - v_r^2/c^2}$. One gets

$$L' = \sqrt{\frac{1-\beta_r}{1+\beta_r}}\, L_0 \text{ (approaching)}, \quad \sqrt{\frac{1+\beta_r}{1-\beta_r}}\, L_0 \text{ (leaving)}$$

which are precisely the Doppler effect in terms of wavelength by setting $\nu\lambda = \nu_0\lambda_0 = c$ in eq. (52) -4.

Example Find the potentials for a point charge moving with constant velocity

Solution: Let the velocity be $\vec{v}$.

$$\vec{r}_0(t) = \vec{v}t$$

The retarded time t_r is determined by eq. (43)

$$c(t-t_r) = |\vec{r} - \vec{r}_0(t_r)| = |\vec{r} - \vec{v}t_r|$$

$$\therefore c^2(t-t_r)^2 = r^2 - 2\vec{r}\cdot\vec{v}t_r + v^2 t_r^2$$

$$[c^2 - v^2]t_r^2 + 2(\vec{r}\cdot\vec{v} - c^2 t)t_r + c^2t^2 - r^2 = 0$$

$$t_r = \frac{1}{c^2 - v^2}\left[(c^2t - \vec{r}\cdot\vec{v}) \pm \sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}\right]$$

$v \to 0$ $t_r \to t \pm r/c$. In this limit, charge is fixed at r $\therefore t_r = t - r/c$ $\therefore$ "$-$" is correct

$$\therefore t_r = \frac{1}{c^2 - v^2}\left[(c^2t - \vec{r}\cdot\vec{v}) - \sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}\right] \cdots (53)$$

$$\mathscr{R} = |\vec{r} - \vec{v}t_r| = c(t-t_r), \quad \hat{\mathscr{R}} = \frac{\vec{r} - \vec{r}_0(t_r)}{c(t-t_r)} = \frac{\vec{r} - \vec{v}t_r}{c(t-t_r)}$$

$$\therefore \mathscr{R}(1 - \hat{\mathscr{R}}\cdot\vec{v}/c)$$
$$= c(t-t_r)\left[1 - \frac{\vec{v}}{c}\cdot\frac{\vec{r} - \vec{v}t_r}{c(t-t_r)}\right] = c(t-t_r) - \frac{\vec{v}\cdot\vec{r}}{c} + \frac{v^2}{c}t_r$$

$$= \frac{1}{c}\left[(c^2 t - \vec{r}\cdot\vec{v}) - (c^2 - v^2)\, t_r\right]$$

Now eq. (13) implies

$$(c^2 - v^2)\, t_r = c^2 t - \vec{r}\cdot\vec{v} - \sqrt{(c^2 t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}$$

$\therefore\ \mathscr{R}(1 - \hat{\mathscr{R}}\cdot\vec{v}/c)$

$$= \frac{1}{c}\sqrt{(c^2 t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)} \quad \text{-- (13) -1}$$

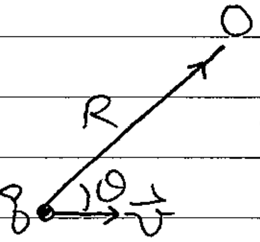
$$\therefore\ V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0}\,\frac{q}{\mathscr{R}\left(1 - \frac{\hat{\mathscr{R}}}{c}\cdot\vec{v}\right)}$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{qc}{\sqrt{(c^2 t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}$$

$$\vec{A}(\vec{r}, t) = \frac{\vec{v}}{c^2}\, V(\vec{r}, t)$$

It's convenient to define $\vec{R} = \vec{r} - \vec{v}t$ and

θ = angle between $\vec{R}$ and $\vec{v}$.



$$I \equiv (c^2 t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)$$

$$= \cancel{c^4 t^2} - 2c^2 t(\vec{r}\cdot\vec{v}) + (\vec{r}\cdot\vec{v})^2 - r^2 v^2$$

$$+ c^2 r^2 - \cancel{c^4 t^2} + c^2 t^2 v^2$$

$$= c^2\left[\vec{r} - \vec{v}t\right]^2 + (\vec{r}\cdot\vec{v})^2 - r^2 v^2$$

$$= c^2 R^2 + \left[(\vec{R} + \vec{v}t)\cdot\vec{v}\right]^2 - (\vec{R} + \vec{v}t)^2 v^2$$

$\vec{R}\cdot\vec{v} = Rv\cos\theta$

$$\therefore I = c^2R^2 + [Rv\cos\theta + v^2t]^2 - (R^2 + 2Rv\cos\theta t + v^2t^2)v^2$$

$$= c^2R^2 + R^2v^2(\cos^2\theta - 1)$$

$$= c^2R^2 - R^2v^2\sin^2\theta \quad \cdots (53)-2$$

$$\therefore V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{R\sqrt{1-v^2/c^2\sin^2\theta}}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{q\vec{v}}{R\sqrt{1-v^2/c^2\sin^2\theta}} \quad \cdots (54)$$

Which are consistent with results of special relativity that we derived before.

Eq. (54) shows the difference between the static potentials $V = \frac{1}{4\pi\epsilon_0}\frac{q}{R}$ & potentials when the charge moves with a constant velocity $\vec{v}$. The extra factor due to $\frac{1}{\sqrt{1-v^2/c^2\sin^2\theta}}$ is due to the Lorentz contraction.

$\vec{E}$ and $\vec{B}$ fields due to a moving point charge

Once $\vec{A}$ & U are known, one can calcultate $\vec{E} = -\nabla U - \frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla}\times\vec{A}$.

To perform ∇ & $\frac{\partial}{\partial t}$, one first needs to know a few results.

For a functor of t_r, $\vec{f}(t_r)$, one has

(i) $(\vec{d}\cdot\vec{\nabla})\vec{f}(t_r) = d_x\frac{\partial}{\partial x}\vec{f} + d_y\frac{\partial}{\partial y}\vec{f} + d_z\frac{\partial \vec{f}}{\partial z}$

$$= d_x\frac{d\vec{f}}{dt_r}\frac{\partial t_r}{\partial x} + d_y\frac{d\vec{f}}{dt_r}\frac{\partial t_r}{\partial y} + d_z\frac{d\vec{f}}{dt_r}\frac{\partial t_r}{\partial z}$$

$$= \frac{d\vec{f}(t_r)}{dt_r}(\vec{d}\cdot\vec{\nabla})t_r \quad \text{for any } \vec{d} \text{ vector} \quad \text{--- (55)}$$

(ii) $\vec{\nabla}\times\vec{f}(t_r) = \left(\vec{\nabla}t_r\frac{d}{dt_r}\right)\times\vec{f}(t_r) = -\frac{d\vec{f}}{dt_r}\times\vec{\nabla}t_r$ --- (56)

$\vec{\nabla} = \frac{d}{dt_r}\vec{\nabla}t_r$

(iii) $(\vec{u}\cdot\vec{\nabla})\vec{r} = u_x\frac{\partial}{\partial x}\vec{r} + u_y\frac{\partial}{\partial y}\vec{r} + u_z\frac{\partial \vec{r}}{\partial z} = (u_x, u_y, u_z) = \vec{u}$ --- (57)

(iv) $\because\ \vec{\mathscr{r}} = \vec{r} - \vec{r}_0(t_r) \qquad c(t-t_r) = \mathscr{r} = |\vec{r} - \vec{r}_0(t_r)|$

$\therefore\ -c\nabla t_r = \nabla\mathscr{r} = \nabla\sqrt{\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}} = \frac{d\sqrt{u}}{du}\nabla u$

$u = \mathscr{r}^2$

$$= \frac{1}{2\sqrt{\mathscr{r}^2}}\nabla\vec{\mathscr{r}}\cdot\vec{\mathscr{r}} \quad \text{--- (58)}$$

Using $\vec{\nabla}(\vec{a}\cdot\vec{b}) = (\vec{a}\cdot\vec{\nabla})\vec{b} + (\vec{b}\cdot\vec{\nabla})\vec{a} + \vec{a}\times(\vec{\nabla}\times\vec{b}) + \vec{b}\times(\vec{\nabla}\times\vec{a})$, one gets

$$\nabla(\vec{\mathscr{R}}\cdot\vec{\mathscr{R}}) = 2(\vec{\mathscr{R}}\cdot\vec{\nabla})\vec{\mathscr{R}} + 2\vec{\mathscr{R}}\times(\vec{\nabla}\times\vec{\mathscr{R}})$$

$$\therefore\ -c\nabla t_r = \frac{1}{\mathscr{R}}\left[(\vec{\mathscr{R}}\cdot\vec{\nabla})\vec{\mathscr{R}} + \vec{\mathscr{R}}\times(\vec{\nabla}\times\vec{\mathscr{R}})\right]$$

$$(\vec{\mathscr{R}}\cdot\vec{\nabla})\vec{\mathscr{R}} = (\vec{\mathscr{R}}\cdot\vec{\nabla})\vec{r} - (\vec{\mathscr{R}}\cdot\vec{\nabla})\vec{r}_0(t_r)$$

$$\underset{(i)}{=} (\vec{\mathscr{R}}\cdot\vec{\nabla})\vec{r} - \frac{d\vec{r}_0}{dt_r}(\vec{\mathscr{R}}\cdot\vec{\nabla})t_r$$

$$\underset{(iii)}{=} \vec{\mathscr{R}} - \vec{v}(\vec{\mathscr{R}}\cdot\vec{\nabla})t_r$$

$$\vec{\nabla}\times\vec{\mathscr{R}} = \underbrace{\cancel{\vec{\nabla}\times\vec{r}}}_{0} - \vec{\nabla}\times\vec{r}_0(t_r) \underset{(ii)}{=} \frac{d\vec{r}_0}{dt_r}\times\nabla t_r$$

$$= \vec{v}\times(\vec{\nabla}t_r)$$

$$\therefore\ -c\nabla t_r = \frac{1}{\mathscr{R}}\Big[\vec{\mathscr{R}} - \cancel{\vec{v}(\vec{\mathscr{R}}\cdot\vec{\nabla})t_r} + \underbrace{\vec{\mathscr{R}}\times(\vec{v}\times\nabla t_r)}_{=\ \cancel{\vec{v}(\vec{\mathscr{R}}\cdot\nabla t_r)} - (\vec{\mathscr{R}}\cdot\vec{v})\nabla t_r}\Big]$$

$$= \frac{1}{\mathscr{R}}\left[\vec{\mathscr{R}} - (\vec{\mathscr{R}}\cdot\vec{v})\nabla t_r\right]$$

$$\therefore\ \nabla t_r = \frac{-\vec{\mathscr{R}}}{\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v}} \quad \text{---} \quad (59)$$

Now, $\nabla V(\vec{r},t) = \dfrac{qc}{4\pi\epsilon_0}\,\dfrac{-1}{(\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v})^2}\,\nabla(\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v})$

$\because c(t-t_r) = \mathscr{r} \quad \therefore c\nabla \mathscr{r} = -c^2 \nabla t_r$ --- (60)

$$\nabla(\vec{\mathscr{r}}\cdot\vec{v}) \underset{\uparrow}{=} (\vec{\mathscr{r}}\cdot\vec{\nabla})\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{\mathscr{r}} + \vec{\mathscr{r}}\times(\vec{\nabla}\times\vec{v})$$

eq.(1-8) $\quad + \vec{v}\times(\vec{\nabla}\times\vec{\mathscr{r}})$

Using (i), $\vec{v} = \vec{v}(t_r)$

$$\therefore (\vec{\mathscr{r}}\cdot\vec{\nabla})\vec{v} = \frac{d\vec{v}}{dt_r}(\vec{\mathscr{r}}\cdot\vec{\nabla})t_r$$

$$= \vec{a}(\vec{\mathscr{r}}\cdot\vec{\nabla}t_r) \quad \text{--- (61)}$$

$$(\vec{v}\cdot\vec{\nabla})\vec{\mathscr{r}} = (\vec{v}\cdot\vec{\nabla})\vec{r} - (\vec{v}\cdot\vec{\nabla})\vec{r_0}(t_r)$$

$$= \vec{v} - \underbrace{\frac{d\vec{r_0}}{dt_r}}_{\vec{v}}(\vec{v}\cdot\vec{\nabla}t_r) = \vec{v}\left[1-\vec{v}\cdot\vec{\nabla}t_r\right]$$

(iii) & (ii) --- (62)

$$\vec{\nabla}\times\vec{v} = -\frac{d\vec{v}}{dt_r}\times\nabla t_r = -\vec{a}\times\nabla t_r \quad \text{--- (62)-1}$$

$$\therefore \vec{\mathscr{r}}\times(\vec{\nabla}\times\vec{v}) = -\vec{\mathscr{r}}\times(\vec{a}\times\nabla t_r) \quad \text{--- (63)}$$

$$\vec{\nabla}\times\vec{\mathscr{r}} = \vec{\nabla}\times\vec{r} - \vec{\nabla}\times\vec{r_0}(t_r) = +\frac{d\vec{r_0}}{dt_r}\times\nabla t_r$$

(0)

$$= \vec{v}\times\nabla t_r$$

$$\therefore \vec{v}\times(\vec{\nabla}\times\vec{\mathscr{r}}) = \vec{v}\times(\vec{v}\times\nabla t_r) \quad \text{--- (64)}$$

Combing (61) – (64), one gets

$$\vec{\nabla}(\vec{\mathscr{r}}\cdot\vec{v}) = \vec{a}(\vec{\mathscr{r}}\cdot\nabla t_r) + \vec{v} - \vec{v}(\vec{v}\cdot\nabla t_r)$$
$$-\vec{\mathscr{r}}\times(\vec{a}\times\vec{\nabla}t_r) + \vec{v}\times(\vec{v}\times\nabla t_r)$$

$$= \cancel{\vec{a}(\vec{\mathscr{r}}\cdot\nabla t_r)} + \vec{v} - \cancel{\vec{v}(\vec{v}\cdot\nabla t_r)} - \cancel{\vec{a}(\vec{\mathscr{r}}\cdot\nabla t_r)} + (\vec{\mathscr{r}}\cdot\vec{a})\nabla t_r$$
$$+ \cancel{\vec{v}(\vec{v}\cdot\nabla t_r)} - v^2\nabla t_r$$

$$= \vec{v} + (\vec{\mathscr{r}}\cdot\vec{a} - v^2)\nabla t_r \quad --- (65)$$

Combining eqs. (60) & (65), we get

$$\nabla V(\vec{r},t) = \frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^2}\left[\vec{v} + (c^2 - v^2 + \vec{\mathscr{r}}\cdot\vec{a})\nabla t_r\right]$$

$$\underset{(59)}{=} \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^2}\frac{1}{\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v}}$$

$$\times\left[\vec{v}(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v}) + (c^2 - v^2 + \vec{\mathscr{r}}\cdot\vec{a})(-\vec{\mathscr{r}})\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^3}\left[(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})\vec{v} - (c^2 - v^2 + \vec{\mathscr{r}}\cdot\vec{a})\vec{\mathscr{r}}\right]$$

$\quad$ L - (66)

Similarly, to find $-\frac{\partial \vec{A}}{\partial t}$, one needs to know $\frac{\partial t_r}{\partial t}$.

(v) From $c(t - t_r) = \mathscr{r}$, one gets

$$c^2(t - t_r)^2 = \mathscr{r}^2 = \vec{\mathscr{r}}\cdot\vec{\mathscr{r}}$$

$$\frac{\partial}{\partial t} \Rightarrow 2c^2(t - t_r)\left(1 - \frac{\partial t_r}{\partial t}\right) = 2\vec{\mathscr{r}}\cdot\frac{\partial\vec{\mathscr{r}}}{\partial t}$$

$$\therefore c\mathscr{r}\left(1 - \frac{\partial t_r}{\partial t}\right) = \vec{\mathscr{r}}\cdot\frac{\partial\vec{\mathscr{r}}}{\partial t} = \vec{\mathscr{r}}\cdot\left(-\frac{\partial \vec{r}_0(t_r)}{\partial t}\right)$$

$$= \vec{\mathscr{r}}\cdot\left(-\frac{d\vec{r}_0}{dt_r}\frac{\partial t_r}{\partial t}\right) = -\vec{\mathscr{r}}\cdot\vec{v}\,\frac{\partial t_r}{\partial t}$$

$$\therefore \frac{dt_r}{dt}(c\mathcal{R} - \vec{\mathcal{R}}\cdot\vec{v}) = c\mathcal{R}$$

$$\frac{dt_r}{dt} = \frac{c\mathcal{R}}{\mathcal{R}(c - \hat{\mathcal{R}}\cdot\vec{v})} = \frac{c\mathcal{R}}{\vec{\mathcal{R}}\cdot\underbrace{(c\hat{\mathcal{R}} - \vec{v})}_{\vec{u}}}$$

$$= \frac{c\mathcal{R}}{\vec{\mathcal{R}}\cdot\vec{u}} \quad \cdots \text{(67)}$$

$$\vec{u} = c\hat{\mathcal{R}} - \vec{v}$$

Now, $\vec{A}(\vec{r},t) = \frac{\vec{v}}{c^2}V(\vec{r},t)$

$$\therefore \frac{\partial\vec{A}}{\partial t} = \frac{1}{c^2}\left(\frac{\partial\vec{v}}{\partial t}V(\vec{r},t) + \vec{v}\frac{\partial V}{\partial t}\right) \cdots \text{(68)}$$

$$\frac{\partial\vec{v}}{\partial t} = \frac{d\vec{v}}{dt_r}\frac{\partial t_r}{\partial t} = \vec{a}\frac{\partial t_r}{\partial t} \quad \cdots \text{(69)}$$

$$\because V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\mathcal{R}(c - \hat{\mathcal{R}}\cdot\vec{v})} = \frac{1}{4\pi\epsilon_0}\frac{qc}{\vec{\mathcal{R}}\cdot(c\hat{\mathcal{R}} - \vec{v})}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{qc}{\vec{\mathcal{R}}\cdot\vec{u}}$$

$$\therefore \frac{\partial V}{\partial t} = \frac{1}{4\pi\epsilon_0}\frac{-qc}{(\vec{\mathcal{R}}\cdot\vec{u})^2}\frac{\partial}{\partial t}(\mathcal{R}c - \vec{\mathcal{R}}\cdot\vec{v})$$

$$= \frac{1}{4\pi\epsilon_0}\frac{-qc}{(\vec{\mathcal{R}}\cdot\vec{u})^2}\left[c\frac{\partial\mathcal{R}}{\partial t} - \frac{\partial\vec{\mathcal{R}}}{\partial t}\cdot\vec{v} - \vec{\mathcal{R}}\cdot\frac{\partial\vec{v}}{\partial t}\right]$$

$$\because \mathcal{R} = c(t - t_r) \therefore \frac{\partial\mathcal{R}}{\partial t} = c - c\frac{\partial t_r}{\partial t} = c\left(1 - \frac{\partial t_r}{\partial t}\right)$$

$$\vec{\mathcal{R}} = \vec{r} - \vec{r}_0(t_r), \quad \frac{\partial\vec{\mathcal{R}}}{\partial t} = -\frac{d\vec{r}_0}{dt_r}\frac{\partial t_r}{\partial t} = -\vec{v}\frac{\partial t_r}{\partial t}$$

$$\frac{\partial\vec{v}}{\partial t} = \frac{d\vec{v}}{dt_r}\frac{\partial t_r}{\partial t} = \vec{a}\frac{\partial t_r}{\partial t}$$

$$\therefore$$

$$\frac{\partial V(\vec{r},t)}{\partial t} = \frac{1}{4\pi\epsilon_0}\frac{-qc}{(\vec{\mathscr{r}}\cdot\vec{u})^2}$$

$$\times\left[c^2\left(1-\frac{\partial t_r}{\partial t}\right)+v^2\frac{\partial t_r}{\partial t}-\vec{\mathscr{r}}\cdot\vec{a}\frac{\partial t_r}{\partial t}\right] \quad \cdots (70)$$

Combing eqs. (68), (69) & (70), one gets

$$\frac{\partial \vec{A}}{\partial t} = \frac{1}{c^2}\frac{qc}{4\pi\epsilon_0}\left[\frac{\vec{a}}{\vec{\mathscr{r}}\cdot\vec{u}}\frac{\partial t_r}{\partial t}-\frac{\vec{v}}{(\vec{\mathscr{r}}\cdot\vec{u})^2}\left(c^2\left(1-\frac{\partial t_r}{\partial t}\right)+v^2\frac{\partial t_r}{\partial t}-\vec{\mathscr{r}}\cdot\vec{a}\frac{\partial t_r}{\partial t}\right)\right]$$

$$= \frac{q}{4\pi\epsilon_0 c(\vec{\mathscr{r}}\cdot\vec{u})^2}\left\{\vec{a}(\vec{\mathscr{r}}\cdot\vec{u})\frac{\partial t_r}{\partial t}-\vec{v}\left[c^2\left(1-\frac{\partial t_r}{\partial t}\right)+v^2\frac{\partial t_r}{\partial t}-\vec{\mathscr{r}}\cdot\vec{a}\frac{\partial t_r}{\partial t}\right]\right\}$$

$$= \frac{q}{4\pi\epsilon_0 c(\vec{\mathscr{r}}\cdot\vec{u})^2}\left\{-c^2\vec{v}+\left[(\vec{\mathscr{r}}\cdot\vec{u})\vec{a}+(c^2-v^2+\vec{\mathscr{r}}\cdot\vec{a})\vec{v}\right]\frac{\partial t_r}{\partial t}\right\}$$

$$= \frac{q}{4\pi\epsilon_0 c(\vec{\mathscr{r}}\cdot\vec{u})^2}\left\{-c^2\vec{v}+\left[(\vec{\mathscr{r}}\cdot\vec{u})\vec{a}+(c^2-v^2+\vec{\mathscr{r}}\cdot\vec{a})\vec{v}\right]\frac{c\mathscr{r}}{\vec{\mathscr{r}}\cdot\vec{u}}\right\}$$

(67)

$$= \frac{q}{4\pi\epsilon_0 c(\vec{\mathscr{r}}\cdot\vec{u})^3}\left\{-c^2\vec{v}(\vec{\mathscr{r}}\cdot\vec{u})+c\mathscr{r}(\vec{\mathscr{r}}\cdot\vec{u})\vec{a}+c\mathscr{r}(c^2-v^2+\vec{\mathscr{r}}\cdot\vec{a})\vec{v}\right\}$$

$$= \frac{qc}{4\pi\epsilon_0(\mathscr{r}c-\vec{\mathscr{r}}\cdot\vec{v})^3}\left\{-\vec{v}(\vec{\mathscr{r}}\cdot\vec{u})+\frac{\mathscr{r}}{c}\vec{a}(\vec{\mathscr{r}}\cdot\vec{u})+\frac{\mathscr{r}}{c}(c^2-v^2+\vec{\mathscr{r}}\cdot\vec{a})\vec{v}\right\}$$

$$= \frac{qc}{4\pi\epsilon_0 (\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v})^3}\Big\{(\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v})\left(-\vec{v} + \frac{\mathscr{R}}{c}\vec{a}\right) + \frac{\mathscr{R}}{c}(c^2 - v^2 + \vec{\mathscr{R}}\cdot\vec{a})\vec{v}\Big\} \text{ --- } (71)$$

$\vec{\mathscr{R}}\cdot\vec{u} = \mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v}$

Combing eqs. (66) & (71), we obtain

$$\vec{E}(\vec{r},t) = \frac{qc}{4\pi\epsilon_0}\frac{-1}{(\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v})^3}$$

$$\times\Big\{\overbrace{(\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v})}^{\vec{\mathscr{R}}\cdot\vec{u}}\left(\frac{\mathscr{R}}{c}\vec{a}\right) - (c^2 - v^2 + \vec{\mathscr{R}}\cdot\vec{a})\underbrace{\left(\vec{\mathscr{R}} - \frac{\mathscr{R}}{c}\vec{v}\right)}_{\frac{\mathscr{R}}{c}[c\hat{\mathscr{R}} - \vec{v}] = \frac{\mathscr{R}}{c}\vec{u}}\Big\}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{\mathscr{R}}{(\vec{\mathscr{R}}\cdot\vec{u})^3}\Big\{-\vec{a}(\vec{\mathscr{R}}\cdot\vec{u}) + (c^2 - v^2)\vec{u} + (\vec{\mathscr{R}}\cdot\vec{a})\vec{u}\Big\}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{\mathscr{R}}{(\vec{\mathscr{R}}\cdot\vec{u})^3}\Big\{(c^2 - v^2)\vec{u} + \vec{\mathscr{R}}\times(\vec{u}\times\vec{a})\Big\} \text{ --- } (72)$$

Once $\vec{E}$ is found, one can also compute $\vec{B}$ by noting that $\vec{A} = \frac{\vec{v}}{c^2}V$

$$\vec{\nabla}\times\vec{A} = \frac{1}{c^2}\vec{\nabla}\times(\vec{v}V) = \frac{1}{c^2}\left[V\vec{\nabla}\times\vec{v} - \vec{v}\times\nabla V\right]$$

∇V is given by eq. (66), while $\vec{\nabla}\times\vec{v} \underset{(62)\text{-}1}{=} -\vec{a}\times\nabla t_r$

$$= \frac{\vec{a}\times\vec{\mathscr{r}}}{\mathscr{r}c-\vec{\mathscr{r}}\cdot\vec{v}} = \frac{\vec{a}\times\vec{\mathscr{r}}}{\vec{\mathscr{r}}\cdot\vec{u}}, \quad V = \frac{qc}{4\pi\epsilon_0}\frac{1}{\vec{\mathscr{r}}\cdot\vec{u}}$$

All together, we get

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{qc}{4\pi\epsilon_0}\frac{1}{c^2}\left\{\frac{\vec{a}\times\vec{\mathscr{r}}}{(\vec{\mathscr{r}}\cdot\vec{u})^2} - \vec{v}\times\frac{1}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[\cancelto{0}{(\vec{\mathscr{r}}\cdot\vec{u})\vec{v}} - (c^2-v^2+\vec{a}\cdot\vec{\mathscr{r}})\vec{\mathscr{r}}\right.\right.$$

$$= -\frac{1}{c}\frac{8}{4\pi\epsilon_0}\frac{1}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\vec{\mathscr{r}}\times\left\{+\vec{a}(\vec{\mathscr{r}}\cdot\vec{u}) + \vec{v}(c^2-v^2+\vec{a}\cdot\vec{\mathscr{r}})\right\}$$

$$= -\frac{1}{c}\frac{8}{4\pi\epsilon_0}\frac{1}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\vec{\mathscr{r}}\times\left[(c^2-v^2)\vec{v} + (\vec{\mathscr{r}}\cdot\vec{a})\vec{v} + (\vec{\mathscr{r}}\cdot\vec{u})\vec{a}\right]$$

--- (73)

$\therefore \vec{B}$ has a similar form to $\vec{E}$ in eq. (72)

for [...] terms

Now, $\because \vec{u} = c\hat{\mathscr{r}} - \vec{v}$

$$\therefore \vec{\mathscr{r}}\times\vec{u} = -\vec{\mathscr{r}}\times\vec{v}$$

bracket in (72)

$$\therefore \vec{\mathscr{r}}\times\left[(c^2-v^2)\vec{u} + \vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]$$

$$= \vec{\mathscr{r}}\times\left[(c^2-v^2)\vec{u} + \vec{u}(\vec{\mathscr{r}}\cdot\vec{a}) - (\vec{\mathscr{r}}\cdot\vec{u})\vec{a}\right]$$

$$= -\vec{\mathscr{r}}\times\left[(c^2-v^2)\vec{v} + \vec{v}(\vec{\mathscr{r}}\cdot\vec{a}) + (\vec{\mathscr{r}}\cdot\vec{u})\vec{a}\right]$$ bracket in (73)

$\therefore$ We conclude

$$\vec{B}(\vec{r},t) = \frac{1}{c}\hat{\mathscr{r}}\times\vec{E}(\vec{r},t) \quad \text{--- (74)}$$

$\therefore$ $\vec{B}$ field of a moving point charge

$\perp \vec{E}$ " " " "

and is always in perpendicular to $\hat{\mathscr{r}}$

(the vector from the retarded position to the observer!)

Now, $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left\{(c^2-v^2)\vec{u} + \vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right\}$ ← (75)

$\vec{u} = c\hat{\mathscr{r}} - \vec{v}$

In the limit $\vec{v}=0$, $\vec{a}=0$

$\vec{E}$ reduces to $\frac{q}{4\pi\epsilon_0}\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$

which comes from the term

$$\frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}(c^2-v^2)\vec{u} \sim O\left(\frac{1}{\mathscr{r}^2}\right) \text{ as } \mathscr{r}\to\infty$$

so it's called <u>generalized Coulomb field</u>,

while the term that is in proportional to $\vec{a}$ is responsible for radiation

$$\frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\vec{\mathscr{r}}\times(\vec{u}\times\vec{a}) \sim O\left(\frac{1}{\mathscr{r}}\right) \text{ as } \mathscr{r}\to\infty$$

so it's called <u>radiation field</u> (or <u>acceleration field</u>)

Given eqs. (74) & (75), one can deduce the force that acts other charge Q

$$\vec{F} = Q[\vec{E} + \vec{V}\times\vec{B})$$

with $\vec{E}$ & $\vec{B}$ being given in (74) & (75).

<u>Example</u> Find the electric and magnetic fields of a point charge q moving with a constant velocity.

<u>Solution</u>: For a constant velocity $\vec{v}$,

$\vec{a}=0$

$\therefore$ Eq. (75) reduces to

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\{(c^2-v^2)\vec{u}\}$$

where $\vec{u} = c\hat{\mathscr{r}} - \vec{v}$. Now, $\vec{r}_0(t) = \vec{v}t$, $\therefore \vec{\mathscr{r}} = \vec{r} - \vec{r}_0(t_r)$
$= \vec{r} - \vec{v}t_r$

$$\therefore \mathscr{r}\vec{u} = c\vec{\mathscr{r}} - \mathscr{r}\vec{v} = \underset{\substack{\uparrow \\ \mathscr{r}=c(t-t_r)}}{=} c(\vec{r}-\vec{r}_0(t_r)) - c(t-t_r)\vec{v}$$

$$= c(\vec{r}-\vec{v}t_r) - c(t-t_r)\vec{v} = c(\vec{r}-\vec{v}t) \quad \cdots (76)$$

Now $\vec{\mathscr{r}}\cdot\vec{u} = \mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v} \underset{\substack{\uparrow \\ \text{eq. (53)-1}}}{=} \sqrt{(c^2t-\vec{r}\cdot\vec{v})^2+(c^2-v^2)(r^2-c^2t^2)}$

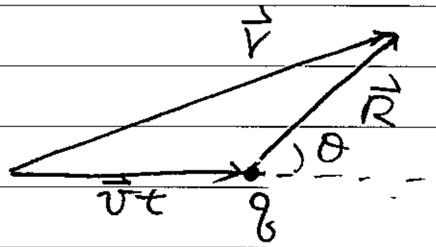
$$\underset{\substack{\uparrow \\ (53)-2}}{=} \sqrt{R^2c^2 - R^2v^2\sin^2\theta} = Rc\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{\frac{1}{2}} \quad (77)$$

with $\vec{R} = \vec{r} - \vec{v}t$. Combing eqs (76) & (77), we get

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{c(\vec{r}-\vec{v}t)(c^2-v^2)}{(Rc)^3\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1-v^2/c^2}{\left[1-\frac{v^2}{c^2}\sin^2\theta\right]^{3/2}}\frac{\hat{R}}{R^2}$$

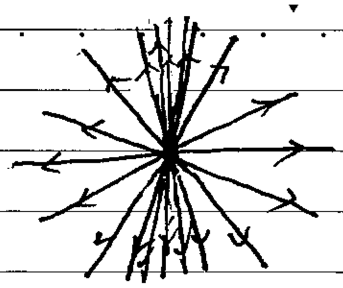
$\vec{R} = \vec{r} - vt$, in consistant with the formula obtain by special relativity.



Due to the factor $\sin^2\theta$ in the denominator, the forward and backward directions of $\vec{E}$ are reduced by a factor $(1-v^2/c^2)$ relative to the static field $\frac{q}{4\pi\epsilon_0}\frac{\hat{R}}{R^2}$

While for $\theta = \pi/2$, $\frac{1-v^2/c^2}{(1-\frac{v^2}{c^2}\sin^2\theta)^{3/2}} = \frac{1}{\sqrt{1-v^2/c^2}}$ is the enhanced factor.

$\therefore \vec{E}$ field-lines are still in radial direction but are more dense in $\theta = \pi/2$ direction as shown in below:



For $\vec{B}$ field, $\vec{B} = \frac{1}{c}\hat{\mathscr{R}} \times \vec{E}(r,t)$.

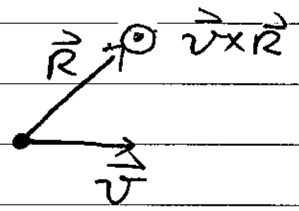
$$\hat{\mathscr{R}} = \frac{\vec{r} - \vec{v}t_r}{\mathscr{R}} = \frac{\vec{r} - \vec{v}t + (t - t_r)\vec{v}}{\mathscr{R}}$$

$$= \frac{\vec{R}}{\mathscr{R}} + \vec{v}$$

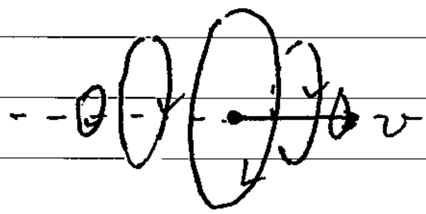
$\mathscr{R} = c(t - t_r)$

$\vec{E} \parallel \hat{R} \quad \therefore \frac{\vec{R}}{\mathscr{R}} \times \vec{E} = 0$

$$\vec{B} = \frac{1}{c}\vec{v} \times \vec{E}(r,t) \quad \propto \quad \frac{\vec{v} \times \vec{R}}{R^3}$$



$\therefore$ B field-lines are circles around the charge.



When $v/c \ll 1$, one may neglect v^2/c^2.

$\therefore \vec{E}(\vec{r},t) \approx \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\hat{R} \quad \vec{B}(\vec{r},t) \approx \frac{\mu_0}{4\pi}\frac{q}{R^2}\vec{v} \times \hat{R}$ — (18)

Which are the Coulomb law & the Biot-Savart law for a point charge, valid in quasi-static limit that we mentioned!

Alternative solution

We have derived the fields of a point charge in uniform motion either via special relativity or via Liénard-Wiechert potentials by setting $\vec{a}=0$.

In fact, they can be derived much more easily as the general Coulomb law from eqs. (1P) & (2P) as follows. Starting from (1P) & (2P),

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)U(x,y,z,t)=-\frac{\rho}{\epsilon_0} \quad \text{--(1P)}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}(x,y,z,t)=-\mu_0\vec{J} \quad \text{--(2P)}$$

For a moving point charge with $\vec{v}=v\hat{x}$, one has

$$\rho(\vec{r},t)=q\,\delta(x-vt)\,\delta(y)\,\delta(z) \quad \text{--(79)}$$

$$\vec{J}(\vec{r},t)=\rho\vec{v}=q\vec{v}\,\delta(x-vt)\,\delta(y)\,\delta(z) \quad \text{--(80)}.$$

Hence $U(x,y,z,t)$ & $\vec{A}$ must be in the form

$$U(x,y,z,t)=U(x-vt,\,y,\,z) \quad \text{--(81)}$$
$$\vec{A}(x,y,z,t)=\vec{A}(x-vt,\,y,\,z)$$

By changing variables, $\xi = x - vt$, one has

$$\frac{\partial}{\partial x} U(x-vt, y, z) = \frac{\partial}{\partial \xi} U(\xi, y, z)$$

$$\frac{\partial}{\partial t} U(x-vt, y, z) = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} U(\xi, y, z)$$

$$= -v\frac{\partial}{\partial \xi} U(\xi, y, z)$$

Hence eq. (1) becomes

$$\left[(1-\beta^2)\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] U(\xi, y, z) = -\frac{q}{\varepsilon_0}\delta(\xi)\delta(y)\delta(z)$$

$\llcorner$ -- (2)

where $\beta = v/c$

Now, if one sets $x' = \frac{\xi}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\beta^2}}(x - vt)$

which is essentially the Lorentz transformation, one gets

$$\because \delta(\xi) = \delta(x'\sqrt{1-\beta^2}) = \frac{1}{\sqrt{1-\beta^2}}\delta(x')$$

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

$$\therefore \left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) U(x', y, z) = -\frac{1}{\varepsilon_0}\frac{q}{\sqrt{1-\beta^2}}\delta(x')\delta(y)\delta(z)$$

$\llcorner$ -- (3)

Eq. (3) is the same as the equation of U for a point charge $q' = \frac{q}{\sqrt{1-\beta^2}}$ at $x' = 0$, $y = 0$, $z = 0$!

$= \gamma q$

The solution is the Coulomb law.

$$\therefore\ V(x', y, z) = \frac{1}{4\pi\epsilon_0}\frac{q'}{\sqrt{x'^2+y^2+z^2}}$$

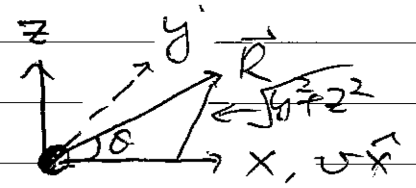
Back in original coordinates, one obtains

$$V(x,y,z,t) = \frac{1}{4\pi\epsilon_0}\frac{\gamma q}{\sqrt{\gamma^2(x-vt)^2+y^2+z^2}} \quad \cdots (84)$$

where $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$.

Similarly, $$\vec{A}(x,y,z,t) = \frac{1}{4\pi\epsilon_0}\frac{\gamma q v \hat{x}}{\sqrt{\gamma^2(x-vt)^2+y^2+z^2}} \quad \cdots (85)$$

Setting $\vec{R} = \vec{r} - \vec{v}t = (x-vt,\ y,\ z)$,



$$\sin^2\theta = \frac{y^2+z^2}{R^2} = \frac{y^2+z^2}{(x-vt)^2+y^2+z^2}$$

$$\frac{\gamma}{\sqrt{\gamma^2(x-vt)^2+y^2+z^2}} = \frac{1}{\sqrt{(x-vt)^2+\frac{1}{\gamma^2}(y^2+z^2)}}$$

$$= \frac{1}{\sqrt{(x-vt)^2+(1-\beta^2)(y^2+z^2)}} = \frac{1}{\sqrt{R^2-\beta^2(y^2+z^2)}} = \frac{1}{R\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}}$$

$$\therefore\ V(x,y,z,t) = \frac{1}{4\pi\epsilon_0}\frac{q}{R\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}}$$

$$\vec{A}(x,y,z,t) = \frac{\mu_0}{4\pi} \frac{q\vec{v}}{R\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}}$$

Which recover eqs (54) !

By taking $-\nabla V - \frac{\partial \vec{A}}{\partial t}$, one can check

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0} \frac{1-v^2/c^2}{(1-\frac{v^2}{c^2}\sin^2\theta)^{3/2}} \frac{\hat{R}}{R^2} \text{ is}$$

also recovered.

The Heaviside - Feynman formula

The Liénard - Wiechert potentials and fields are expressed in terms of retarded time t_r & v_r, a_r.

It's convenient to express fields in current time t. This leads to the Heaviside - Feynman formula which we shall derive in below.

We start from eq. (38)

$$V(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\int dt' \frac{\delta\left(t'-t+\frac{|\vec{r}-\vec{r}_0(t')|}{c}\right)}{|\vec{r}-\vec{r}_0(t')|}$$

and similarly, $\vec{A}(\vec{r},t) = \frac{\mu_0 q}{4\pi}\int dt' \frac{\vec{v}(t')\,\delta\left(t'-t+\frac{|\vec{r}-\vec{r}_0(t')|}{c}\right)}{|\vec{r}-\vec{r}_0(t')|}$

Set $\vec{R}(t') = \vec{r} - \vec{r}_0(t')$, one gets

$$\vec{E}(\vec{r},t) = -\frac{q}{4\pi\epsilon_0}\nabla\int dt' \frac{\delta\left(t'-t+\frac{R(t')}{c}\right)}{R(t')}$$

$$-\frac{\mu_0 q}{4\pi}\frac{\partial}{\partial t}\int dt' \frac{\vec{v}(t')\,\delta\left(t'-t+\frac{R(t')}{c}\right)}{R(t')} \quad \cdots (86)$$

$$R(t') = |\vec{r}-\vec{r}_0(t')| = \left[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right]^{1/2}$$

$$\therefore \nabla R(t') = \frac{(x-x_0,\, y-y_0,\, z-z_0)}{R} = \hat{R}(t')$$

$$\nabla\frac{1}{R(t')} = -\frac{1}{R^2}\nabla R = -\frac{\hat{R}(t')}{R^2(t')}$$

$$\nabla\frac{\delta\left(t'-t+\frac{R(t')}{c}\right)}{R(t')} = \left[\nabla\delta\left(t'-t+\frac{R(t')}{c}\right)\right]\frac{1}{R(t')} + \delta\left(t'-t+\frac{R(t')}{c}\right)$$

$$\times\nabla\frac{1}{R(t')}$$

$$= \frac{d\delta(u)}{du}\nabla u\cdot\frac{1}{R(t')} - \frac{\hat{R}(t')}{R^2(t')}\delta\left(t'-t+\frac{R(t')}{c}\right), \quad u = t'-t+\frac{R(t')}{c}$$

$$= \delta'\left(t'-t+\frac{R(t')}{c}\right)\cdot\frac{1}{R(t')}\frac{1}{c}\nabla R(t') - \frac{\hat{R}(t')}{R^2(t')}\delta\left(t'-t+\frac{R(t')}{c}\right)$$

$$= -\frac{\partial}{\partial t}\delta\left(t'-t+\frac{R(t')}{c}\right)\frac{\hat{R}(t')}{cR(t')} - \frac{\hat{R}(t')}{R^2(t')}\delta\left(t'-t+\frac{R(t')}{c}\right) \quad \cdots (87)$$

Combining $\frac{\partial}{\partial t}\delta\left(t'-t+\frac{R(t')}{c}\right)$ with the 2nd term in (86), (noting $\mu_0 = \frac{1}{\epsilon_0 c^2}$)

One obtains

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left[\int dt'\,\delta\left(t'-t+\frac{R(t')}{c}\right)\frac{\hat{R}(t')}{R^2(t')}\right. \quad \cdots (88)$$

$$\left.+\frac{\partial}{\partial t}\int dt'\,\delta\left(t'-t+\frac{R(t')}{c}\right)\frac{\hat{R}(t')-\vec{v}(t')/c}{cR(t')}\right]$$

Now, from the derivation of eq. (38) to eq. (41), one has

$$\int dt' \, \delta\left(t' - t + \frac{\mathscr{R}(t')}{c}\right) F(\mathscr{R}(t'), \vec{v}(t'))$$

$$= \frac{F(\mathscr{R}(t_r), \vec{v}(t_r))}{1 - \frac{\hat{\mathscr{R}}(t_r) \cdot \vec{v}(t_r)}{c}}$$

$$\equiv S F \Big|_{\text{retarded}, \; t' = t_r} \quad \text{--- (89)}$$

$S(t') \equiv 1 - \frac{\hat{\mathscr{R}}(t') \cdot \vec{v}(t')}{c}$ = scaling factor due to Galilean Doppler effect

for any F.

∴ Eq (88) becomes

$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\hat{\mathscr{R}}}{S\mathscr{R}^2}\Big|_{t'=t_r} + \frac{q}{4\pi\epsilon_0} \frac{d}{dt}\left[\frac{\hat{\mathscr{R}} - \vec{\beta}}{S c \mathscr{R}}\Big|_{t'=t_r}\right] \quad \text{--- (90)}$$

Similarly,

$$\vec{B}(\vec{r}, t) = \vec{\nabla} \times \vec{A}(\vec{r}, t) = \frac{\mu_0 q}{4\pi} \int dt' \left(\nabla \frac{\delta\left(t' - t + \frac{\mathscr{R}(t')}{c}\right)}{\mathscr{R}(t')}\right) \times \vec{v}(t')$$

Using eq. (87), one gets

$$\vec{B}(\vec{r}, t) = \frac{\mu_0 q}{4\pi} \int dt' \, \vec{v}(t') \times \frac{\hat{\mathscr{R}}(t')}{\mathscr{R}^2(t')} \delta\left(t' - t + \frac{\mathscr{R}(t')}{c}\right)$$

$$+ \frac{\mu_0 q}{4\pi} \int dt' \, \vec{v}(t') \times \frac{\hat{\mathscr{R}}(t')}{c\mathscr{R}(t')} \frac{\partial}{\partial t} \delta\left(t' - t + \frac{\mathscr{R}(t')}{c}\right)$$

$$= \frac{\mu_0 q}{4\pi} \left. \frac{\vec{v} \times \hat{\mathscr{r}}}{s \mathscr{r}^2} \right|_{t'=t_r} + \frac{\mu_0 q}{4\pi} \frac{d}{dt} \left[\left. \frac{\vec{v} \times \hat{\mathscr{r}}}{s c \mathscr{r}} \right|_{t'=t_r} \right]$$

$\llcorner$ -- (91)

Instead of being expressed in terms of t_r, one treats t_r as a function t.

$$\therefore \vec{\mathscr{r}}(t_r) \equiv \vec{R}_a(t) \quad \hat{\mathscr{r}}(t_r) \equiv \hat{R}_a(t) \quad (\vec{R}_a = \text{apparent position})$$

$$= \vec{r} - \vec{r}_0(t_r(t))$$

From eq. (43), $c(t - t_r) = |\vec{r} - \vec{r}_0(t_r)| = \mathscr{r}(t_r)$

$$\therefore \quad t_r = t - \frac{\mathscr{r}(t_r)}{c} \quad \text{-- (91)}$$

Combing eqs. (52) & (P1), one gets

$$\frac{dt_r}{dt} = \frac{1}{1 - \frac{1}{c} \vec{v}_r \cdot \hat{\mathscr{r}}(t_r)} = \frac{1}{s} = 1 - \frac{1}{c} \frac{d\mathscr{r}}{dt}$$

$$= 1 - \frac{1}{c} \frac{dR_a(t)}{dt} \quad \text{-- (P1)-1}$$

$$\therefore \frac{1}{s} = 1 - \frac{1}{c} \frac{dR_a}{dt} \quad \text{--- (P2)}$$

Similarly, $\vec{\beta} \big|_{t=t_r} = \frac{1}{c} \vec{v}_r = \frac{-1}{c} \frac{d}{dt_r} (\vec{r}_0 - \vec{r}_0(t_r))$

$$= -\frac{1}{c} \frac{d\vec{\mathscr{r}}(t_r)}{dt_r} = \frac{-1}{c} \frac{dt}{dt_r} \frac{d}{dt} \vec{\mathscr{r}}(t_r)$$

$$= -\frac{s}{c} \frac{d}{dt} \vec{R}_a(t) \quad \text{--- (P3)}$$

Substituting eqs (P2) & (P3) into eq. (P0), we obtain

$\vec{E}$ & $\vec{B}$ due to a moving charge

It's more transparent to derive eqs. (72) & (73) from eqs (P0) & (P1), not from the Liénard – Wiechert potentials directly.

For this purpose, we first expand eq. (P0) explicitly.

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{\mathscr{R}}}{s\mathscr{R}^2} + \frac{1}{cs\mathscr{R}}\left(\frac{d\hat{\mathscr{R}}}{dt} - \frac{d\vec{\beta}}{dt}\right) - \frac{(\hat{\mathscr{R}}-\vec{\beta})}{c\mathscr{R}^2 s^2}\left(\mathscr{R}\frac{ds}{dt} + s\frac{d\mathscr{R}}{dt}\right)\right]_{t'=t_r} \quad \text{(P2)}$$

$\because$ $s = 1 - \hat{\mathscr{R}}\cdot\vec{\beta}$, $\frac{ds}{dt} = -\frac{d\hat{\mathscr{R}}}{dt}\cdot\vec{\beta} - \hat{\mathscr{R}}\cdot\frac{d\vec{\beta}}{dt}$

$\therefore$ We need to compute $\frac{d\hat{\mathscr{R}}}{dt}$, $\frac{d\mathscr{R}}{dt}$ & $\frac{d\vec{\beta}}{dt}$.

We first note that $\frac{d}{dt}(\cdots) = \frac{d}{dt_r}\frac{dt_r}{dt}$

$\frac{dt_r}{dt} = \frac{1}{1-\vec{\beta}\cdot\hat{\mathscr{R}}} = \frac{1}{s}$ (eq. (S2))

$\therefore$ Computing $\frac{d\hat{\mathscr{R}}}{dt_r}$, $\frac{d\mathscr{R}}{dt_r}$ & $\frac{d\vec{\beta}}{dt_r} = \dot{\vec{\beta}}$ are sufficient.

(i) $\mathscr{R} = |\vec{r} - \vec{r}_0(t_r)|$, $\frac{d\mathscr{R}}{dt_r} = \frac{\partial\mathscr{R}}{\partial\mathscr{R}_x}\frac{d\mathscr{R}_x}{dt_r} + \frac{\partial\mathscr{R}}{\partial\mathscr{R}_y}\frac{d\mathscr{R}_y}{dt_r} + \frac{\partial\mathscr{R}}{\partial\mathscr{R}_z}\frac{d\mathscr{R}_z}{dt_r}$

$= \nabla\mathscr{R}\cdot\frac{d\vec{\mathscr{R}}}{dt_r} = -\hat{\mathscr{R}}\cdot\vec{v} = -c\hat{\mathscr{R}}\cdot\vec{\beta}$, $\therefore$ $\frac{d\mathscr{R}}{dt} = -\frac{c}{s}\hat{\mathscr{R}}\cdot\vec{\beta}$ (CP3)

(ii) $\frac{d\hat{\mathcal{R}}}{dt_r} = \frac{d}{dt}\left(\frac{\vec{\mathcal{R}}}{\mathcal{R}}\right) = \frac{1}{\mathcal{R}^2}\left[\frac{d\vec{\mathcal{R}}}{dt_r}\mathcal{R} - \vec{\mathcal{R}}\frac{d\mathcal{R}}{dt_r}\right]$

$\underset{\uparrow \\ (i)}{=} \frac{1}{\mathcal{R}^2}\left[-\vec{v}\mathcal{R} + \vec{\mathcal{R}}(\vec{\mathcal{R}}\cdot\vec{v})\right] = \frac{1}{\mathcal{R}}\left[\hat{\mathcal{R}}\times(\hat{\mathcal{R}}\times\vec{v})\right]$

$= \frac{c}{\mathcal{R}}\left[\hat{\mathcal{R}}\times(\hat{\mathcal{R}}\times\vec{\beta})\right]$

$\therefore \frac{d\hat{\mathcal{R}}}{dt} = \frac{c}{s\mathcal{R}}\left[\hat{\mathcal{R}}\times(\hat{\mathcal{R}}\times\vec{\beta})\right]$ --- (C94)

Using eqs. (C93) & (C94), we obtain

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left\{\frac{\hat{\mathcal{R}}}{s\mathcal{R}^2} + \frac{1}{cs\mathcal{R}}\left[\frac{c}{s\mathcal{R}}\hat{\mathcal{R}}\times(\hat{\mathcal{R}}\times\vec{\beta}) - \frac{\dot{\vec{\beta}}}{s}\right]\right.$$
$$+ \frac{(\hat{\mathcal{R}}-\vec{\beta})}{c\mathcal{R}^2 s^2}\mathcal{R}\left[\vec{\beta}\cdot\frac{c}{s\mathcal{R}}\hat{\mathcal{R}}\times(\hat{\mathcal{R}}\times\vec{\beta}) + \frac{\hat{\mathcal{R}}\cdot\dot{\vec{\beta}}}{s}\right]$$
$$\left. - \frac{(\hat{\mathcal{R}}-\vec{\beta})}{\cancel{c}\mathcal{R}^2 s^2}\cancel{s}\cdot\frac{-1}{\cancel{s}}c\,\hat{\mathcal{R}}\cdot\vec{\beta}\right\} \quad \text{--- (C95)}$$

terms without $\dot{\vec{\beta}}$ $\left(O\left(\frac{1}{\mathcal{R}^2}\right)\right)$ in (C95)

$$= \frac{q}{4\pi\epsilon_0}\left\{\frac{\hat{\mathcal{R}}}{s\mathcal{R}^2} + \frac{1}{s^2\mathcal{R}^2}\underbrace{\hat{\mathcal{R}}\times(\hat{\mathcal{R}}\times\vec{\beta})}_{\hat{\mathcal{R}}(\hat{\mathcal{R}}\cdot\vec{\beta})-\vec{\beta}} + \frac{(\hat{\mathcal{R}}-\vec{\beta})}{\mathcal{R}^2 s^3}\underbrace{\vec{\beta}\cdot\hat{\mathcal{R}}\times(\hat{\mathcal{R}}\times\vec{\beta})}_{(\hat{\mathcal{R}}\cdot\vec{\beta})^2-\beta^2}\right.$$
$$\left. + \frac{(\hat{\mathcal{R}}-\vec{\beta})}{\mathcal{R}^2 s^2}\hat{\mathcal{R}}\cdot\vec{\beta}\right\}$$

$\hat{\mathcal{R}}(1-2\hat{\mathcal{R}}\cdot\vec{\beta}+(\hat{\mathcal{R}}\cdot\vec{\beta})^2)$

$$= \frac{q}{4\pi\epsilon_0}\cdot\frac{1}{s^3\mathcal{R}^2}\left\{\hat{\mathcal{R}}(1-\hat{\mathcal{R}}\cdot\vec{\beta})^2 + \left[\hat{\mathcal{R}}(\hat{\mathcal{R}}\cdot\vec{\beta})-\vec{\beta}\right](1-\hat{\mathcal{R}}\cdot\vec{\beta})\right.$$
$$+ (\hat{\mathcal{R}}-\vec{\beta})\left[\cancel{(\hat{\mathcal{R}}\cdot\vec{\beta})^2}-\beta^2\right]$$
$$\left. + (\hat{\mathcal{R}}-\vec{\beta})\underbrace{(\hat{\mathcal{R}}\cdot\vec{\beta})(1-\hat{\mathcal{R}}\cdot\vec{\beta})}_{-\cancel{(\hat{\mathcal{R}}\cdot\vec{\beta})^2}+\hat{\mathcal{R}}\cdot\vec{\beta}}\right\}$$

$\uparrow$
$s = 1-\hat{\mathcal{R}}\cdot\vec{\beta}$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{s^3\mathcal{R}^2}\left\{\hat{\mathcal{R}}(1-\hat{\mathcal{R}}\cdot\vec{\beta})\left[1-\cancel{\hat{\mathcal{R}}\cdot\vec{\beta}}+\cancel{\hat{\mathcal{R}}\cdot\vec{\beta}}\right] - \vec{\beta}(1-\hat{\mathcal{R}}\cdot\vec{\beta})\right.$$
$$\left. + (\hat{\mathcal{R}}-\vec{\beta})(\hat{\mathcal{R}}\cdot\vec{\beta}-\beta^2)\right\}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{s^3 r^2} \left\{ (\hat{r}-\vec{\beta})(1-\hat{r}\cdot\vec{\beta}) + (\hat{r}-\vec{\beta})(\hat{r}\cdot\vec{\beta}-\beta^2) \right\}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{s^3 r^2} (\hat{r}-\vec{\beta})(1-\beta^2) \Big|_{ret}$$

$$\therefore O\left(\frac{1}{r^2}\right) = \frac{q}{4\pi\epsilon_0} \frac{(\hat{r}-\vec{\beta})(1-\beta^2)}{s^3 r^2} \Big|_{ret} \quad \cdots \text{CP5} - 1$$

$O\left(\frac{1}{r}\right)$ ($\dot{\vec{\beta}}$ term) in CP5

$$= \frac{q}{4\pi\epsilon_0} \left\{ -\frac{1}{c s^2 r} \dot{\vec{\beta}} + \frac{(\hat{r}-\vec{\beta})}{c s^3 r} (\hat{r}\cdot\dot{\vec{\beta}}) \right\}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{c s^3 r} \left\{ (\hat{r}-\vec{\beta})(\hat{r}\cdot\dot{\vec{\beta}}) - \dot{\vec{\beta}} \underbrace{(1-\hat{r}\cdot\vec{\beta})}_{s} \right\}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{c s^3 r} \hat{r} \times \left[(\hat{r}-\vec{\beta}) \times \dot{\vec{\beta}} \right] \Big|_{ret} \quad \cdots \text{CP5} - 2$$

$$\therefore \vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0} \left[\frac{(\hat{r}-\vec{\beta})(1-\beta^2)}{s^3 r^2} + \frac{\hat{r} \times [(\hat{r}-\vec{\beta}) \times \dot{\vec{\beta}}]}{c s^3 r} \right] \Big|_{ret}$$

Now, setting $s^3 r^3 = (1-\hat{r}\cdot\vec{\beta})^3 r^3 \quad \cdots \text{CP6}$

$$= \left[\vec{r}\cdot(c\hat{r}-\vec{v}) \right]^3 \cdot \frac{1}{c^3}$$

and define $\vec{u} = c\hat{r} - \vec{u}$

C96 becomes

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left\{(c^2-v^2)\vec{u}+\vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right\}\Big|_{ret}$$

which recovers eq. 72 — C97

Similarly, starting from eq. 91, by comparing to eq. 90, one replaces $\hat{\mathscr{r}} \to \vec{v}\times\hat{\mathscr{r}}$

$$\hat{\mathscr{r}}-\vec{\beta} \to \vec{v}\times(\hat{\mathscr{r}}-\vec{\beta}) = \vec{v}\times\hat{\mathscr{r}}$$

∴ C96 implies

$$\vec{B}(\vec{r},t) = \frac{\mu_0 q}{4\pi}\left[\frac{(\vec{v}\times\hat{\mathscr{r}})(1-\beta^2)}{s^3\mathscr{r}^2} + \frac{(\vec{v}\times\hat{\mathscr{r}})(\dot{\vec{\beta}}\cdot\hat{\mathscr{r}})+cs\dot{\vec{\beta}}\times\hat{\mathscr{r}}}{cs^3\mathscr{r}}\right]_{ret}$$

C95 - 2

C92

$\left(-\frac{1}{cs\mathscr{r}}\dot{\vec{\beta}}\right.$ in C95 - 1 is from $\frac{1}{cs\mathscr{r}}\,\frac{-d\vec{\beta}}{dt}$ in C92 and is replaced by $\frac{1}{cs\mathscr{r}}\frac{d}{dt}(\vec{v}\times\hat{\mathscr{r}})$ $\hat{\mathscr{r}}-\vec{\beta}$

$$= \frac{1}{cs^2\mathscr{r}}\left(\frac{d}{dt_r}\vec{v}\right)\times\hat{\mathscr{r}} = \frac{s}{s^3\mathscr{r}}\dot{\vec{\beta}}\times\hat{\mathscr{r}}\Big)$$

see over

$\left(\frac{d\hat{\mathscr{r}}}{dt_r}\right.$ is already taken care of)

$$= \frac{q}{4\pi\epsilon_0 c^2}\left[\frac{(\vec{v}\times\hat{\mathscr{r}})(1-\beta^2)}{s^3\mathscr{r}^2} + \frac{(\vec{\beta}\times\hat{\mathscr{r}})(\dot{\vec{\beta}}\cdot\hat{\mathscr{r}})+s\dot{\vec{\beta}}\times\hat{\mathscr{r}}}{s^3\mathscr{r}}\right]_{ret}$$

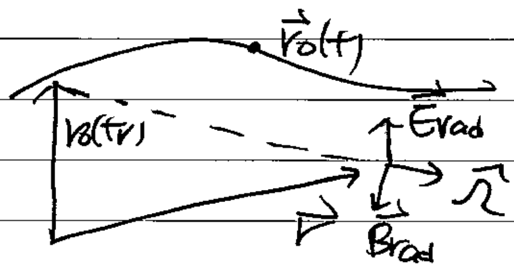
$\mu_0 = \frac{1}{\epsilon_0 c^2}$ — C98

Comparing (C96) & (C98) ; we obtain

$$\vec{B}(\vec{r},t) = \frac{1}{c}\vec{\Omega} \times \vec{E}(\vec{r},t)$$ which

reproduces eq. (74).

⇒ go to $\boxed{10-52}$



$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left\{ \frac{\hat{R}_a(t)}{R_a^2(t)}\left(1 - \frac{1}{c}\frac{dR_a(t)}{dt}\right)\right.$$

$$+ \frac{d}{dt}\left[\frac{1}{c}\frac{\hat{R}_a}{R_a}\left(1 - \frac{1}{c}\frac{dR_a}{dt}\right)\right.$$

$$\left.\left. + \frac{1}{c}\frac{1}{\cancel{q}R_a} \times \frac{\cancel{q}}{c}\frac{d}{dt}\vec{R}_a\right]\right\}$$

$$= \frac{q}{4\pi\epsilon_0}\left\{\frac{\hat{R}_a(t)}{R_a^2(t)} + \left[\frac{-1}{c}\frac{\hat{R}_a}{R_a^2}\frac{dR_a}{dt} + \frac{1}{c}\frac{d}{dt}\left(\frac{\hat{R}_a}{R_a}\right)\right]\right.$$

$$\left. + \frac{d}{dt}\left[-\frac{1}{c^2}\frac{\hat{R}_a}{R_a}\frac{dR_a}{dt} + \frac{1}{c^2}\frac{1}{R_a}\frac{d\vec{R}_a}{dt}\right]\right\}$$

Now, the 2nd term

$$\frac{-1}{c}\frac{\hat{R}_a}{R_a^2}\frac{dR_a}{dt} + \frac{1}{c}\frac{d}{dt}\frac{\hat{R}_a}{R_a} = \frac{R_a}{c}\frac{d}{dt}\left(\frac{\hat{R}_a/R_a}{R_a}\right)$$

$$= \frac{R_a}{c}\frac{d}{dt}\left(\frac{\hat{R}_a}{R_a^2}\right)$$

The 3rd term, by setting $\vec{R}_a = R_a\hat{R}_a$, becomes

$$-\frac{1}{c^2}\frac{d}{dt}\left[-\frac{\hat{R}_a}{R_a}\frac{dR_a}{dt} + \frac{1}{R_a}\frac{dR_a\hat{R}_a}{dt}\right]$$

$$= +\frac{1}{c^2}\frac{d^2\hat{R}_a}{dt^2}$$

$$\therefore\ \vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{R}_a(t)}{R_a^2(t)} + \frac{R_a(t)}{c}\frac{d}{dt}\left(\frac{\hat{R}_a}{R_a^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}\hat{R}_a(t)\right]$$

L - 94

Similarly, $\because$ eq. (93) implies

$$\vec{v_r} = -s\frac{d}{dt}\vec{R_a}(t)$$

$$\therefore \left.\frac{\vec{v}\times\hat{\mathcal{R}}}{s\mathcal{R}^2}\right|_{t'=t_r} = -\frac{d}{dt}\vec{R_a}(t)\times\frac{\hat{R_a}(t)}{R_a^2(t)}$$

$$\vec{v_a} = \frac{d\,\vec{r_0}(t_r(t))}{dt} = -\frac{d}{dt}\vec{R_a}(t)$$

$$\therefore \left.\frac{\vec{v}\times\hat{\mathcal{R}}}{s\mathcal{R}^2}\right|_{t'=t_r} = \vec{v_a}(t)\times\frac{\hat{R_a}(t)}{R_a^2(t)}$$

$$\left.\frac{\vec{v}\times\hat{\mathcal{R}}}{sc\mathcal{R}}\right|_{t'=t_r} = \frac{\vec{v_a}(t)}{c}\times\frac{\hat{R_a}(t)}{R_a(t)}$$

$$\therefore \vec{B}(\vec{r},t) \underset{\text{eq.(91)}}{\uparrow}{=} \frac{\mu_0 q}{4\pi}\vec{v_a}\times\frac{\hat{R_a}}{R_a^2} + \frac{\mu_0 q}{4\pi}\frac{d}{dt}\left(\frac{\vec{v_a}(t)}{c}\times\frac{\hat{R_a}(t)}{R_a(t)}\right)$$

$$\underset{\vec{R_a}=R_a\hat{R_a}}{\uparrow}{=} \frac{\mu_0 q}{4\pi}\left\{\underbrace{\frac{\hat{R_a}}{R_a^2}\times\frac{d}{dt}(R_a\hat{R_a})}_{=\frac{\hat{R_a}}{R_a}\times\frac{d\hat{R_a}}{dt}} + \frac{1}{c}\frac{d}{dt}\left[\underbrace{\frac{\hat{R_a}}{R_a}\times\frac{d}{dt}(R_a\hat{R_a})}_{=\hat{R_a}\times\frac{d}{dt}\hat{R_a}}\right]\right\}$$

$$= \frac{\mu_0 q}{4\pi}\left[\frac{\hat{R_a}}{R_a}\times\frac{d\hat{R_a}}{dt} + \frac{\hat{R_a}}{c}\times\frac{d^2\hat{R_a}}{dt^2}\right] \quad \text{--- (95)}$$

$$= \frac{d}{dt}\left(\frac{\hat{R_a}}{R_a^2}\times \cancel{R_a\hat{R_a}}^{\,0}\right) - \left(\frac{d}{dt}\frac{\hat{R_a}}{R_a^2}\right)\times\vec{R_a}$$

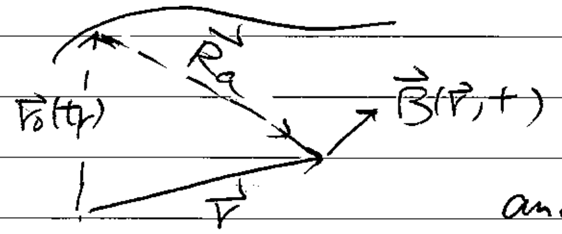
Comparing eqs (95) & (94), one gets

$$\vec{B} = \epsilon\mu_0\cdot c\cdot\hat{R_a}\times\vec{E}$$

$$\therefore \vec{B}(r,t) = \frac{\hat{R_a}(t)}{c}\times\vec{E}(r,t) \quad \text{--- (96)}$$

Eqs. (94) & (95) are known as the Heaviside-Feynman formula.

Eq. (96) shows that $\vec{B}(\vec{r},t)$ is always perpendicular to $\vec{E}(\vec{r},t)$ & the apparent position $\vec{R}_a$



The Heaviside-Feynman formula yields interesting and useful interpretations of fields generated for a moving charge: ...

(i) The first term $\frac{q}{4\pi\epsilon_0}\frac{\hat{R}_a}{R_a^2}$ in eq. (94) gives the Coulomb field produced by the charge at the apparent position $\vec{R}_a(t)$ or retarde position $\vec{\mathcal{R}}(t_r)$.

(ii) The second term $\frac{R_a}{c}\frac{d}{dt}\left(\frac{q}{4\pi\epsilon_0}\frac{\hat{R}_a}{R_a^2}\right)$ takes the time derivative of Coulomb field and multiplies with the delay $\frac{R_a}{c} = t - t_r = \Delta t$.

$\because\ f(x+\Delta x) - f(x) = \Delta f = f'(x)\Delta x$

$$\therefore\ \frac{R_a}{c}\frac{d}{dt}\left(\frac{q}{4\pi\epsilon_0}\frac{\hat{R}_a}{R_a^2}\right) = \Delta t\,\frac{d}{dt}\left(\frac{q}{4\pi\epsilon_0}\frac{\hat{R}_a}{R_a^2}\right) = \Delta\left(\frac{q}{4\pi\epsilon_0}\frac{\hat{R}_a}{R_a^2}\right)$$

is the correction of the Coulomb's law due to the finite propagation speed (c) of EM fields.

$\therefore$ first + 2nd terms in eq. (P4) gives

the corrected Coulomb law

and hence first + 2nd terms $\sim \frac{1}{R_a^2}$

as $R_a \to \infty$.

(iii) the most interesting term is the 3rd

term which contains radiation field.

and decays as $\frac{1}{R_a}$ as $R_a \to \infty$

First, $\frac{d^2}{dt^2}\hat{R_a}$ vanishes if there is no

acceleration.

Furthermore, if the acceleration is along $\hat{R_a}$

direction, $\frac{d^2\hat{R_a}}{dt^2} = 0$ as well.

$\therefore$ The largest contribution of the acceleration

is from acceleration $\perp \hat{R_a}$!

Now, $\because \hat{R_a}^2 = 1 \quad \therefore \frac{d\hat{R_a}}{dt} \cdot \hat{R_a} = 0$

$\frac{d\hat{R_a}}{dt} \perp \hat{R_a}$.

Consider the case when the region that the charge q moves $<< R_a$. $\because \hat{R_a} = \frac{\vec{R_a}^{\perp} + \vec{R_a}^{\parallel}}{R_a} \approx \frac{\vec{R_a}^{\perp}}{R_a}$

$\therefore |\vec{R_a}^{\perp}| << R_a$ (only $\vec{R_a}^{\perp}$ contributes)

$$\frac{R_a^{\parallel}}{R_a} \sim 1 \qquad R_a = \sqrt{R_a^{\perp 2} + (R_a^{\parallel})^2}$$

$$\frac{dR_a}{dt} \cong \frac{\vec{R_a}^{\perp} \cdot \frac{d\vec{R_a}^{\perp}}{dt}}{R_a} \sim O\left(\frac{1}{R_a}\right)$$

$$\therefore \frac{d\hat{R_a}}{dt} = \frac{d}{dt}\frac{\vec{R_a}}{R_a} = \frac{\frac{d\vec{R_a}^{\perp}}{dt}}{R_a} + O\left(\frac{1}{R_a^2}\right)$$

$$\frac{d^2\hat{R_a}}{dt^2} = \frac{1}{R_a}\ddot{\vec{R_a}}^{\perp} + O\left(\frac{1}{R_a^2}\right)$$

$$\cong -\frac{1}{R_a}\frac{d^2}{dt^2}\vec{r_{0\perp}}(t_r) \qquad \sim O\left(\frac{1}{R_a}\right)$$

$\therefore R_a \to \infty$

$$\vec{E} = -\frac{q}{4\pi\epsilon_0 c^2 R_a}\ddot{\vec{r}}_{0\perp}\left(t - \frac{R_a}{c}\right) \quad \text{-- (9f)}$$

One can further approximate $R_a = \Omega(t_r(t))$

by $\Omega(t)$ (i.e. neglect $\frac{R_a}{c}$ in $t_r = t_0 - R_a/c$)

then

$$\vec{E} = -\frac{q}{4\pi\epsilon_0 c^2 \Omega(t)}\underbrace{\ddot{\vec{r}}_{0\perp}(t)}_{a_\perp(t)} \quad \text{--- (9f)}$$

where $\Omega(t) = |\vec{r} - \vec{r_0}(t)|$

Hence $\vec{E}$ at large distance from charges moving in a local region is determined by <u>acceleration</u> <u>in perpendicular to the direction to observer.</u>

Non-relativistic & quasi-static limit

The Heaviside-Feynman formula enables one to discuss the quasi-static / non-relativistic limits more easily.

In the non-relativistic limit, $\frac{v_r}{c} \ll 1$, $s \simeq 1$.

$\because t_r = t - \frac{\mathscr{R}(t_r)}{c} \quad \therefore t = t_r + \frac{\mathscr{R}(t_r)}{c}$, $\vec{R}_a(t) = \vec{\mathscr{R}}[t_r(t)]$

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left\{\underbrace{\frac{\hat{R}_a}{R_a^2}}_{\frac{\hat{\mathscr{R}}(t_r)}{\mathscr{R}^2(t_r)}} + \underbrace{\frac{\mathscr{R}(t_r)}{c}\frac{d}{dt}\left(\frac{\hat{R}_a}{R_a^2}\right)}_{(t-t_r)\times\frac{d}{dt_r}\left(\frac{\hat{\mathscr{R}}}{\mathscr{R}^2}\right)\times\frac{1}{s}} + \frac{1}{c^2}\frac{d^2\hat{R}_a}{dt^2}\right\}$$

$\because s \simeq 1 \quad \therefore$ 2nd term $\approx (t-t_r)\frac{d}{dt_r}\left(\frac{\hat{\mathscr{R}}}{\mathscr{R}^2}\right)$

$\therefore$ 1st + 2nd term

$$\simeq \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{\mathscr{R}}(t_r)}{\mathscr{R}^2(t_r)} + (t-t_r)\frac{d}{dt_r}\left(\frac{\hat{\mathscr{R}}}{\mathscr{R}^2}\right)\right]$$

$$\simeq \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathscr{R}}(t)}{\mathscr{R}^2(t)}$$, where $\vec{\mathscr{R}}(t) = \vec{r} - \vec{r}_0(t)$

= instantaneous distance between observer & the charge

$\therefore$ In the non-relativistic limit,

$$\vec{E}(\vec{r},t) = \underbrace{\frac{q}{4\pi\epsilon_0}\frac{\hat{\mathscr{r}}(t)}{\mathscr{r}^2(t)}}_{} + \underbrace{\frac{q}{4\pi\epsilon_0}\frac{1}{c^2}\frac{d^2}{dt^2}}_{}\hat{R}_a(t) \quad \text{--- (99)}$$

instantaneous Coulomb interaction

due to acceleration of charge (radiation field)

Quasi-static limit

To consider the quasi-static limit, we have to consider fields due to a static/steady charge/current distribution.

We shall see that even though individual charges may be in accelerated motion, as long as the entire distribution is steady, there is no radiation. As a result,

$$\vec{E} = \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d\tau'\, \rho(\vec{r}')\frac{\hat{\mathscr{r}}}{|\vec{r}-\vec{r}'|^2}$$

$$\vec{B} = \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d\tau'\, \vec{J}(\vec{r}')\times\frac{\hat{\mathscr{r}}}{|\vec{r}-\vec{r}'|^2}$$

where, $\hat{\mathscr{r}} = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}$

Electric field of a charge distribution

We first rewrite the delay term (2nd term) in eq. (94) by first noting that

$$\frac{d}{dt}\left[R_a \frac{\hat{R}_a}{R_a^2}\right] = R_a \frac{d}{dt}\left(\frac{\hat{R}_a}{R_a^2}\right) + \frac{\hat{R}_a}{R_a^2}\frac{dR_a}{dt}$$

$$\therefore \; R_a \frac{d}{dt}\left(\frac{\hat{R}_a}{R_a^2}\right) = \frac{d}{dt}\left[\frac{\hat{R}_a}{R_a}\right] - \frac{\hat{R}_a}{R_a^2}\frac{dR_a}{dt}$$

$\therefore$ Eq. (P4) becomes

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\hat{R}_a}{R_a^2}\left[1 - \frac{1}{c}\frac{dR_a}{dt}\right] + \frac{q}{4\pi\epsilon_0}\frac{1}{c}\frac{d}{dt}\left[\frac{\hat{R}_a}{R_a} + \frac{1}{c}\frac{d\hat{R}_a}{dt}\right] \quad \text{--- (100)}$$

$\therefore$ For a distribution of charges q_i at $\vec{R}_{ai}$, we get

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N} q_i \frac{\hat{R}_{ai}}{R_{ai}^2}\left[1 - \frac{1}{c}\frac{dR_{ai}}{dt}\right]$$

$$+ \frac{1}{4\pi\epsilon_0}\frac{1}{c}\frac{d}{dt}\left\{\sum_{i=1}^{N} q_i \left[\frac{\hat{R}_{ai}}{R_{ai}} + \frac{1}{c}\frac{d\hat{R}_{ai}}{dt}\right]\right\} \quad \text{--- (101)}$$

where $\vec{R}_{ai} = \vec{r} - \vec{r}_i[t_r(t)]$

$d\,r_i(t_r)$: a cross section

$\therefore$ Now q_i can be written as $\rho(\vec{r}_i)\,d\tau_i$

charge at retarded position $\vec{r}_i$

$$\therefore \sum_{i=1}^{N} q_i \left[\frac{\hat{R}_{ai}}{R_{ai}} + \frac{1}{c}\frac{d\hat{R}_{ai}}{dt}\right]$$

$$= \int d\tau' \rho(\vec{r}')\left[\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^2} + \frac{1}{c}\vec{f}(\vec{r},\vec{r}',\vec{v}(\vec{r}'))\right] \quad \text{--- (102)}$$

where $\frac{d\hat{R}_{ai}}{dt} = \frac{d\hat{R}_{ai}}{dt_{ri}}\frac{dt_{ri}}{dt} \doteq \frac{1}{1 - \frac{1}{c}\vec{v}_{ri}\cdot\hat{R}_{ai}}\frac{d\hat{R}_{ai}}{dt_{ri}}$

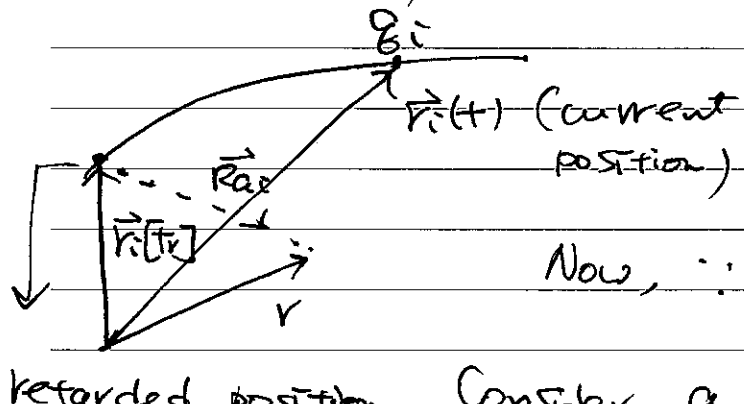
is a function of $\vec{v}_i(\vec{r}_i(t_r))$, $\vec{r}_i(t_r)$ & $\vec{r}$

$f(\vec{r}, \vec{r}_i(t_r), \vec{v}_i(\vec{r}_i(t_r)))$.

∴ As long as $\rho(\vec{r}')$ & $\vec{v}(\vec{r}')$ do not depend on time, even though individual charge's contribution may depend on time, the overall sum in eq. (102) does not depend on time.

$$\therefore \vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N} q_i \frac{\hat{R}_{ai}}{R_{ai}^2}\left[1 - \frac{1}{c}\frac{dR_{ai}}{dt}\right] \quad \text{— (103)}$$

for steady distribution of charges.



Now, $\because \dfrac{R_{ai}}{c} = \displaystyle\int_{r_i[t_r]}^{r_i[t]} \frac{dr}{v}$

Consider a change due to $t \to t+dt$

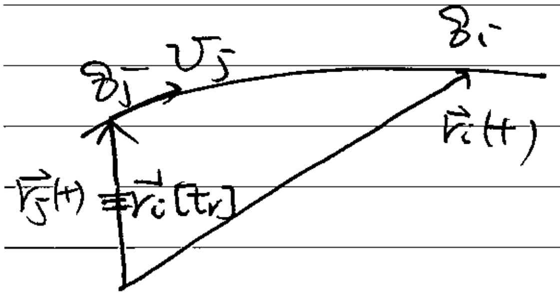
$$r_i[t] \to r_i[t] + dr_i(t)$$
$$r_i[t_r] \to r_i[t_r] + dr_i[t_r]$$

$$\therefore \frac{1}{c}\, dR_{ai} = \underbrace{\frac{dr_i(t)}{v_i}}_{= dt} - \frac{dr_i[t_r]}{v_{ri}}$$

$$\therefore\ 1-\frac{1}{c}\frac{dR_{ai}}{dt} = \frac{1}{v_i(t_r)}\frac{d}{dt}r_i(t_r) \quad \text{--- (104)}$$

At the present time t, q_i is at $\vec{r}_i(t)$.

At the retarded position $\underline{\vec{r}_i(t_r)}$, the charge becomes $\underline{q_j}$ as shown in below.



To maintain time-independent charge distribution, obviously,

$$\underline{v_j(t) \simeq v_i(t_r)}$$

& line charge density $q_j \to \underline{\frac{dq}{dr_j}}$ at t

= line charge density at t_r, $\underline{\frac{dq}{dr_i[t_r]}}$

$$\therefore\ 1-\frac{1}{c}\frac{dR_{ai}}{dt} = \frac{1}{v_j(t)}\frac{d}{dt}r_i(t_r)$$

$$q_i\left(1-\frac{1}{c}\frac{dR_{ai}}{dt}\right) = \frac{dq}{v_j}\frac{dr_i(t_r)}{dt} = \underbrace{\frac{dq}{dr_j}}_{\text{line charge density at } r_i[t_r],\ \text{time } t} d\,r_i(t_r)$$

$$= \underbrace{\frac{dq}{dr_i[t_r]}}_{\text{line charge at } r_i[t_r],\ \text{time } t_r} d\,r_i[t_r]$$

Hence eq.(103) becomes (steady distribution).

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N} \frac{dq}{dr_j(t)} \frac{\vec{r}-\vec{r}_i(t_r)}{|\vec{r}-\vec{r}_i(t_r)|^3} dr_i(t_r)$$

$$\because \vec{r}_i(t_r) = \vec{r}_j(t) \Rightarrow = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N} \frac{dq}{dr_j(t)} \frac{\vec{r}-\vec{r}_j(t)}{|\vec{r}-\vec{r}_j(t)|^3} dr_j(t)$$

$$\frac{dq}{dr_j(t)} dr_j(t) = \rho(r_j,t)\, d\tau_j \underset{\uparrow}{=} \rho(r_j)\, d\tau_j$$

time independent

$$\therefore \vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d\tau' \underbrace{\rho(\vec{r}',t)}_{=\rho(\vec{r}')} \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \vec{E}(\vec{r})$$

(for fixed t) $\rightarrow$ $\vec{r}(t_r(t))$ may depend on t $\downarrow$ but $\vec{r}_j(t)$ does not)

is correct for steady distribution of $\vec{J}$ & ρ

<u>Magnetic field of a current distribution</u>

Similarly, to find the magnetic field for a current distribution, we start from eq. (95) by rewriting

(i) $$\frac{\hat{R}_a}{R_a} \times \frac{d\hat{R}_a}{dt} = \vec{R}_a \times \frac{d}{dt}\left(\frac{\hat{R}_a}{R_a^2}\right)$$

$$= \frac{d}{dt}\left(\cancel{\vec{R}_a \times \frac{\hat{R}_a}{R_a^2}}\right)^{\to 0} - \frac{d\vec{R}_a}{dt} \times \frac{\hat{R}_a}{R_a^2}$$

$$= \frac{d\vec{r}_0(t_r)}{dt} \times \frac{\hat{R}_a}{R_a^2}$$

(ii) $$\hat{R}_a \times \frac{d^2\hat{R}_a}{dt^2} = \frac{d}{dt}\left(\hat{R}_a \times \frac{d\hat{R}_a}{dt}\right) - \cancel{\frac{d\hat{R}_a}{dt} \times \frac{d\hat{R}_a}{dt}}^{\to 0}$$

$\therefore\ \vec{B}(\vec{r},t) = \frac{\mu_0 q}{4\pi} \frac{d\vec{r}_0(t_r)}{dt} \times \frac{\hat{R}_a}{R_a^2} + \frac{\mu_0 q}{4\pi c}\frac{d}{dt}\left[\hat{R}_a \times \frac{d\hat{R}_a}{dt}\right]$

$\therefore$ For a distribution of charges q_i at $\vec{r}_i(t_{ri})$,

we obtain

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\sum_{i=1}^{N} q_i \frac{d\vec{r}_i[t_{ri}]}{dt} \times \frac{\hat{R}_{ai}}{R_{ai}^2}$$

$$+ \frac{\mu_0}{4\pi}\frac{d}{dt}\left(\sum_{i=1}^{N} \frac{q_i}{c}\hat{R}_{ai} \times \frac{d\hat{R}_{ai}}{dt}\right)$$

For a steady distribution,

$$\sum_{i=1}^{N} \frac{q_i}{c}\hat{R}_{ai} \times \frac{d\hat{R}_{ai}}{dt} = \frac{1}{c}\int d\tau' \rho(\vec{r}')$$

$\times$ function of $(\vec{r}, \vec{r}' \,\&\, \vec{v}(\vec{r}'))$

is time-independent.

$$\therefore \quad \frac{d}{dt}\sum_{i=1}^{N}\frac{q_i}{c}\hat{R}_{ai} \times \frac{d\hat{R}_{ai}}{dt} = 0$$

(for fixed t)

For the first term, imagine that during dt,

an amount dq_i passing $r_i(t_r)$ — (105)

That is, one replaces $\frac{q_i}{dt} \to \frac{dq_i}{dt} = \frac{dq_i}{dr_i[t_r]}\frac{dr_i[t_r]}{dt}$

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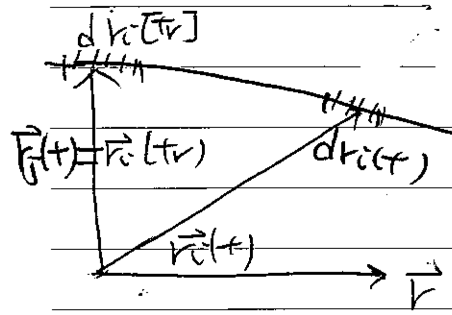
For a current path that passing $\vec{r}_i(t)$,



During dt, dq_i moves $d r_i(t)$. Correspondingly, there is the same amount dq_i passing the retarded position $\vec{r}_i(t_r)$, with segment $d r_i[t_r]$ $\left(t_r = t - \frac{R_a}{c} \right)$

$$\therefore \quad \frac{dq_i}{dr_i(t)} dr_i(t) = \frac{dq_i}{dr_i(t_r)} dr_i(t_r)$$

For the present time t, during dt, there is dq_j passing $\vec{r}_j(t) = \vec{r}_i(t_r)$. Since there is no charge accumulation between r_j & r_i,

$$\therefore \quad \frac{dq_j}{dr_j(t)} dr_j(t) = \frac{dq_i}{dr_i(t)} dr_i(t)$$

Combing the above two, one has $\frac{dq_j}{dr_j(t)} dr_j(t) = \frac{dq_i}{dr_i[t_r]} dr_i[t_r]$

Now,

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi} \sum_{i=1}^{N} \frac{q_i}{dt} \, d\vec{r}_i(t_r) \times \frac{\hat{R}_{ai}}{R_{ai}^2}$$

$$= \frac{\mu_0}{4\pi} \sum_{i=1}^{N} \underbrace{\frac{dq_i}{dr_i[t_r]} \frac{dr_i[t_r]}{dt}}_{\parallel} \, d\vec{r}_i(t_r) \times \frac{\vec{r} - \vec{r}_i(t_r)}{|\vec{r} - \vec{r}_i(t_r)|^3}$$

$$\frac{dq_j}{dr_j(t)} \frac{dr_j(t)}{dt} = I(r_j)$$

$\because \vec{r}_i(t_r) = \vec{r}_j(t)$

$$\therefore \vec{B} = \frac{\mu_0}{4\pi} \sum_{j=1}^{N} \underbrace{I(r_j)\ d\vec{r}_j(t)}_{\vec{J}(\vec{r}',t)\,d\tau'} \times \frac{\vec{r}-\vec{r}_j(t)}{|\vec{r}-\vec{r}_j(t)|^3}$$

(everything is in the form of function t)

$$= \frac{\mu_0}{4\pi} \int d\tau'\ \vec{J}(\vec{r}',t) \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

$\because \vec{J}$ is independent of t

$$\therefore \vec{B}(\vec{r},t) = \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d\tau'\ \vec{J}(\vec{r}') \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

recovering the Biot-Savart expression.

The Field tensor

The transformation rules of $\vec{E}$ & $\vec{B}$ in eq. (40) do not exhibit $\vec{E}$ or $\vec{B}$ as a 4-vector. Instead, they are mixed together.

How do we understand their transformation rules?

It turns out that they form an antisymmetric 2nd-rank tensor. (initiated by Oersted)

To see it, we first note that there are 6 components in $\vec{E}$ & $\vec{B}$, while for a given 2nd-rank tensor, $t^{\mu\nu}$, there are $4 \times 4 = 16$ components. Clearly, number of components are not matching:

$$t^{\mu\nu} = \begin{pmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{pmatrix}.$$

To match number of components, one requires that not all $t^{\mu\nu}$ are independent!

For instance, for a symmetric tensor,

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$$t^{\mu\nu} = t^{\nu\mu} \quad \cdots (117)$$

One only needs to specify 10 components

$$t^{\mu\nu} = \begin{pmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ & t^{11} & t^{12} & t^{13} \\ & \times & t^{22} & t^{23} \\ & & & t^{33} \end{pmatrix}.$$

For an antisymmetric tensor,

$$t^{\mu\nu} = -t^{\nu\mu} \quad \cdots (118)$$

$\therefore$ $t^{\mu\mu} = -t^{\mu\mu} = 0$. There are only 6 components which match $\vec{E}$ & $\vec{B}$ exactly

$$t^{\mu\nu} = \begin{pmatrix} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{pmatrix}.$$

Recall that for a 4-vector. (A^0, A^1, A^2, A^3), it transforms as

$$\bar{A}^{\mu} = \Lambda^{\mu}{}_{\nu} A^{\nu}$$

with $\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\beta = v/c$.

(eq. 21) $\cdots$ (118)-1

A 2nd-rank tensor behaves as "two vectors" being put together: $A^{\mu}B^{\nu}$.

Hence it transforms as

$$\bar{t}^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}\, t^{\lambda\sigma} \quad \text{---} (119)$$

$\therefore \quad \bar{t}^{01} = \Lambda^{0}_{\lambda}\Lambda^{1}_{\sigma}\, t^{\lambda\sigma}$

Using eq. (118)-1, $\because \Lambda^{0}_{\lambda} \neq$ only $\lambda = 0, 1$

$\Lambda^{1}_{\sigma} \neq 0$ only $\sigma = 0, 1$

$$\therefore \quad \bar{t}^{01} = \Lambda^{0}_{0}\Lambda^{1}_{0}\, t^{00} + \Lambda^{0}_{0}\Lambda^{1}_{1}\, t^{01} + \Lambda^{0}_{1}\Lambda^{1}_{0}\, t^{10} + \Lambda^{0}_{1}\Lambda^{1}_{1}\, t^{11} \quad \text{---} (120)$$

For antisymmetric tensors, $t^{00} = 0$, $t^{11} = 0$.

$t^{01} = -t^{10}$

$$\therefore \quad \bar{t}^{01} = (\Lambda^{0}_{0}\Lambda^{1}_{1} - \Lambda^{0}_{1}\Lambda^{1}_{0})\, t^{01}$$

$$= (\gamma^2 - \gamma^2\beta^2)\, t^{01} = t^{01}$$

Similarly, $\bar{t}^{02} = \Lambda^{0}_{\lambda}\Lambda^{2}_{\sigma}\, t^{\lambda\sigma}$, $\lambda = 0, 1$, $\sigma = 2$

$$= \Lambda^{0}_{0}\Lambda^{2}_{2}\, t^{02} + \Lambda^{0}_{1}\Lambda^{2}_{2}\, t^{12}$$

$$= \gamma t^{02} - \gamma\beta\, t^{12} = \gamma(t^{02} - \beta t^{12})$$

$$\bar{t}^{03} = \Lambda^0_{\ \lambda}\Lambda^3_{\ \delta}\, t^{\lambda\delta}, \quad \lambda = 0,1,\ \delta = 3$$

$$= \Lambda^0_{\ 0}\Lambda^3_{\ 3}\, t^{03} + \Lambda^0_{\ 1}\Lambda^3_{\ 3}\, t^{13}$$

$$= \gamma t^{03} - \gamma\beta t^{13}$$

$$= \gamma(t^{03} + \beta\, t^{31})$$

$$\bar{t}^{12} = \Lambda^1_{\ \lambda}\Lambda^2_{\ \delta}\, t^{\lambda\delta} \quad \lambda = 1,0,\ \delta = 2$$

$$= \Lambda^1_{\ 0}\Lambda^2_{\ 2}\, t^{02} + \Lambda^1_{\ 1}\Lambda^2_{\ 2}\, t^{12}$$

$$= -\gamma\beta t^{02} + \gamma t^{12}$$

$$= \gamma(t^{12} - \beta t^{02})$$

$$\bar{t}^{13} = \Lambda^1_{\ \lambda}\Lambda^3_{\ \delta}\, t^{\lambda\delta} \quad \lambda = 1,0,\ \delta = 3$$

$$= \Lambda^1_{\ 0}\Lambda^3_{\ 3}\, t^{03} + \Lambda^1_{\ 1}\Lambda^3_{\ 3}\, t^{13}$$

$$= -\gamma\beta t^{03} + \gamma t^{13}$$

$$\Rightarrow -\bar{t}^{31} = -\gamma\beta t^{03} - \gamma t^{31} \quad \therefore \quad \bar{t}^{31} = \gamma(t^{31} + \beta t^{03})$$

$$\bar{t}^{23} = \Lambda^2_{\ \lambda}\Lambda^3_{\ \delta}\, t^{\lambda\delta} \quad \lambda = 2,\ \delta = 3$$

$$= t^{23}$$

$\therefore$ The complete set of transformation rules is

$$\bar{t}^{01} = t^{01}, \quad \bar{t}^{02} = \gamma(t^{02} - \beta t^{12}), \quad \bar{t}^{03} = \gamma(t^{03} + \beta t^{31})$$

$$\bar{t}^{23} = t^{23}, \quad \bar{t}^{31} = \gamma(t^{31} + \beta t^{03}), \quad \bar{t}^{12} = \gamma(t^{12} - \beta t^{02}).$$

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Comparing eqs. (10) & (121), we can identify

01 component as $\frac{E_x}{c} \equiv F^{01}$

02 " as $\frac{E_y}{c} \equiv F^{02}$

03 " " $\frac{E_z}{c} \equiv F^{03}$

12 " " $B_z = F^{12}$

31 " " $B_y = F^{31}$

23 " " $B_x = F^{23}$.

Therefore, we can construct the field tensor $F^{\mu\nu}$ as

$$F^{\mu\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix} \quad \text{--- (122)}$$

Hence $\vec{E}/c$ & $\vec{B}$ together transform as a 2nd-tensor.

Note that only t^{01} & t^{23} are invariant under transformation, while B_x & E_x are invariant. Therefore, one can also identify 01 component as $B^x \equiv G^{01}$

02 " " $B^y \equiv G^{02}$

03 " " $B^z \equiv G^{03}$

then $\frac{E_z}{c} = -G^{12}$, $\frac{E_y}{c} = -G^{31}$

$\frac{E_x}{c} = -G^{23}$ ("−" sign see below)

Such contructed tensor is called the dual tensor $G^{\mu\nu}$:

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & \frac{E_z}{c} & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix} \quad \cdots (123)$$

It's relation to $F^{\mu\nu}$ is to replace $\vec{E}/c \to \vec{B}$, $\vec{B} \to -\vec{E}/c$ in $F^{\mu\nu}$.

This acturally reflects an invariant property of transformation rules of $\vec{E}$ & $\vec{B}$ in eq. (110):

Eq. (110) is invariant under

$$\vec{E}/c \to \vec{B}$$

$$\vec{B} \to -\vec{E}/c.$$

Which is a property of the Maxwell's equation with magnetic monopoles (see Ex 7.64. eq. 7.68 $\sin\alpha = 1$, $\cos\alpha = 0$)

Electrodynamics in tensor form

(i) As we have shown that $(c\rho, \vec{J})$ forms a 4-vector J^μ (contravariant) and $\left(\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ forms a covariant 4-vector ∂_μ

$$\therefore \partial_\mu J^\mu = \frac{\partial J^\mu}{\partial x^\mu} = 0 \quad \cdots (124)$$

represents the charge conservation; the continuity equation $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$

(ii) Using $F^{\mu\nu}$ & $G^{\mu\nu}$, one can rewrite the Maxwell's equations in a more compact form as

$$\partial_\nu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu \quad \cdots (125) \quad \text{(Source)}$$

$$\& \quad \partial_\nu G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \quad \cdots (126) \quad \text{(sourceless)}$$

Check: $\mu = 0$ in eq. (125) yields

$$\frac{\partial F^{0\nu}}{\partial x^\nu} = \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3}$$

$$= \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \frac{1}{c}\vec{\nabla}\cdot\vec{E}$$

$$\mu_0 J^0 = \mu_0 c\rho$$

$\therefore \mu = 0$ of eq. (125) represents the Gauss's law

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$$

$\mu = 1-3$ of Eq. (125) then represents the Maxwell-Ampere equation with source-current.

$\mu = 1$ $\quad \frac{\partial F^{1\nu}}{\partial x^\nu} = \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3}$

$= -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}$

$= \left(-\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B}\right)_x = \mu_0 J^1 = \mu_0 J_x$

$\mu = 2, 3$ gives y & z components.

Eq. (126) summarizes $\vec{\nabla}\cdot\vec{B} = 0$ & $\vec{\nabla}\times\vec{B} = -\frac{\partial \vec{E}}{\partial t}$ without source terms:

$\mu = 0$ $\quad \frac{\partial G^{0\nu}}{\partial x^\nu} = \frac{\partial G^{00}}{\partial x^0} + \frac{\partial G^{01}}{\partial x^1} + \frac{\partial G^{02}}{\partial x^2} + \frac{\partial G^{03}}{\partial x^3}$

$= \frac{\partial B^x}{\partial x} + \frac{\partial B^y}{\partial y} + \frac{\partial B^z}{\partial z}$

$= \vec{\nabla}\cdot\vec{B} = 0$

$\mu = 1$, $\quad \frac{\partial G^{1\nu}}{\partial x^\nu} = \frac{\partial G^{10}}{\partial x^0} + \frac{\partial G^{11}}{\partial x^1} + \frac{\partial G^{12}}{\partial x^2} + \frac{\partial G^{13}}{\partial x^3}$

$= -\frac{\partial B_x}{c\partial t} - \frac{1}{c}\frac{\partial E_z}{\partial y} + \frac{1}{c}\frac{\partial E_y}{\partial z}$

$= -\frac{1}{c}\left(\frac{\partial \vec{B}}{\partial t} + \vec{\nabla}\times\vec{E}\right)_x = 0$

Similarly, $\mu = 2, 3$ yield y & z components.

(iii) Lorentz force

The Lorentz force can be rewritten by using the proper force $K = \frac{dP}{d\tau}$ and $F^{\mu\nu}$.

$$\overset{\text{vector}}{\downarrow} K^{\mu} = q\,\eta_\nu \overset{\text{2nd-rank tensor}}{\downarrow} F^{\mu\nu} \quad \text{---} (127)$$

~~Where~~ $\vec{\eta}$ is the 4-vector.

$\mu = 1$, $K^1 = q\,\eta_\nu F^{1\nu}$

$$= q(\eta_0 F^{10} + \eta_1 F^{11} + \eta_2 F^{12} + \eta_3 F^{13})$$

$\because \eta_0 = -\eta^0$, $\eta_k = \eta^k$

$$\therefore K^1 = q[-\eta^0 F^{10} + \eta^1 F^{11} + \eta^2 F^{12} + \eta^3 F^{13}]$$

$$= q\left[\frac{-c}{\sqrt{1-u^2/c^2}}\left(-\frac{E_x}{c}\right) + \frac{u_y}{\sqrt{1-u^2/c^2}} B_z + \frac{u_z}{\sqrt{1-u^2/c^2}}(-B_y)\right]$$

$$= \frac{q}{\sqrt{1-u^2/c^2}}[\vec{E} + \vec{u} \times \vec{B}]_x$$

$\because K^1 = \frac{1}{\sqrt{1-u^2/c^2}} F_x \quad \therefore \overbrace{F_x = q[\vec{E} + \vec{u} \times \vec{B}]_x}^{\text{It yields}}$

Similarly, $\mu = 2$ & 3 yield y & z component.

$$\therefore \vec{K} = \frac{q}{\sqrt{1-u^2/c^2}}(\vec{E} + \vec{u} \times \vec{B})$$

$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$ is included in (127)

Eq. (127) makes it clear that why $\vec{E}$ & $\vec{B}$ together form a 2nd-rank tensor because it connects two vectors : $\vec{K}$ & $\vec{\eta}(\vec{u})$!

Note that $\mu = 0$ represents the energy conservation in power form. :

$$K^0 = \frac{1}{c}\frac{\vec{u}\cdot\vec{F}}{\sqrt{1-\frac{u^2}{c^2}}} = \gamma(-\eta^0 \underbrace{F^{00}}_{0} + \eta^1 F^{01} + \eta^2 F^{02} + \eta^3 F^{03})$$

$$= \gamma \frac{1}{\sqrt{1-u^2/c^2}}\, \vec{u}\cdot\frac{\vec{E}}{c}$$

i.e. $\vec{u}\cdot\vec{F} = \gamma\vec{E}\cdot\vec{u}$, which is ~~agree with the Lorentz force.~~

Relativistic potentials

As we have learned, $\vec{E}$ & $\vec{B}$ can be expressed in terms of a scalar potential V & a vector potential $\vec{A}$:

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla}\times\vec{A} \quad \cdots (128)$$

with the Lorentz condition

$$\vec{\nabla}\cdot\vec{A} = \frac{-1}{c^2}\frac{\partial V}{\partial t}$$

V & $\vec{A}$ satisfies a wave equation

$$\underbrace{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)}_{\equiv\ \Box^2\ \equiv\ \text{d'Alembertian}} \vec{A} = -\mu_0\vec{J} \quad \cdots (128)\text{-}1$$

$\Box^2 \equiv$ d'Alembertian.

& $\Box^2 V = -\mu_0 c^2 \rho = -\rho/\varepsilon_0 \quad \cdots (128)\text{-}2$

It's possible to include $\vec{A}$ & V in the tensor form. As we show, $\vec{E}$ & $\vec{B}$ transform ~~as a 2nd-rank tensor $F^{\mu\nu}$~~ ~~or~~ ~~product~~ of two vectors $\sim A^{\mu} B^{\nu}$

$\therefore$ It's clear that U & $\vec{A}$ also form a 4-vector.

Indeed, $A^{\mu} = (\frac{V}{c}, A_x, A_y, A_z)$

form a 4-vector ... (129)

potential

Using A^{μ}, one finds

$$F^{\mu\nu} = \frac{\partial A^{\nu}}{\partial X_{\mu}} - \frac{\partial A^{\mu}}{\partial X_{\nu}} \quad \cdots \quad (130)$$

with $X_{1,2,3} = X^{1,2,3}$ but $X_0 = -X^0 = -ct$.

Check: $\mu = 0$, $\nu = 1$

$$F^{01} = \frac{E_x}{c} = \frac{\partial A^1}{\partial X_0} - \frac{\partial A^0}{\partial X^1}$$

$$= -\frac{\partial A_x}{c\partial t} - \frac{\partial}{\partial x}\left(\frac{V}{c}\right) = \frac{-1}{c}\left(\frac{\partial \vec{A}}{\partial t} + \nabla V\right)_x$$

Similarly, $\mu = 0$, $\nu = 2$ & 3 yield y & z components

Similarly, $\mu = 1$, $\nu = 2$

$$F^{12} = B_z = \frac{\partial A^2}{\partial X_1} - \frac{\partial A^1}{\partial X_2}$$

$$= \frac{\partial A^2}{\partial X} - \frac{\partial A^1}{\partial y} = \frac{\partial}{\partial x}(A_y) - \frac{\partial}{\partial y}(A_x)$$

$$= (\vec{\nabla} \times \vec{B})_z$$

$\nu = 2$ & 3 yields y & z component of $\vec{B} = (\vec{\nabla} \times \vec{A})$

Hence eq. (130) represents eq. (128).

Note that under the gauge transformation

$$A^{\mu} \to A^{\mu} + \frac{\partial \lambda}{\partial x_{\mu}} \quad \cdots (131)$$

$\mu=0$ $\quad \frac{U'}{c} = \frac{U}{c} - \frac{\partial \lambda}{c \partial t}$

$\mu=1,2,3$ $\quad \vec{A}' = \vec{A}' + \nabla \lambda$

$$F^{\mu\nu} \to F^{\mu\nu} + \frac{\partial^2 \lambda}{\partial x_{\mu} \partial x_{\nu}} - \frac{\partial^2 \lambda}{\partial x_{\nu} \partial x_{\mu}}$$

$= F^{\mu\nu}$ is invariant!

The Lorentz condition can be expressed

as $\partial_{\mu} A^{\mu} = \frac{\partial A^{\mu}}{\partial x^{\mu}} = 0 \quad \cdots (132)$

$$\left(\mu=0 \quad \frac{\partial A^0}{\partial x^0} = -\frac{1}{c}\frac{\partial A^0}{\partial t} = -\frac{1}{c^2}\frac{\partial U}{\partial t} \right)$$

The Maxwell's equations

(125) become $\partial_{\nu} F^{\mu\nu} = \frac{\partial}{\partial x_{\mu}}\left(\frac{\partial A^{\nu}}{\partial x^{\nu}}\right) - \frac{\partial}{\partial x_{\nu}}\left(\frac{\partial A^{\mu}}{\partial x^{\nu}}\right) = \mu_0 J^{\mu}$

$\cdots (133)$

(126) is automatically satisfied.

In the Lorentz gauge condition, the first term in eq. (133) vanishes. Hence we get

$$-\frac{\partial^2}{\partial x_{\nu} \partial x^{\nu}} A^{\mu} = \mu_0 J^{\mu} \quad \cdots (134)$$

which reduces to eqs (128)-1 & (128)-2:

$$\frac{\partial^2}{\partial x_\nu x^\nu} = \frac{\partial^2}{\partial x_0 \partial x^0} + \frac{\partial^2}{\partial x_1 \partial x^1} + \frac{\partial^2}{\partial x_2 \partial x^2} + \frac{\partial^2}{\partial x_3 \partial x^3}$$

$$x_0 = -x^0 = -ct, \quad \because x_k = x^k = \quad k = 1, 2, 3$$

$$\therefore \quad = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \underbrace{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}_{\nabla^2}$$

Eq (134) becomes $\square^2 A^\mu = -\mu_0 J^\mu$ - - - (135)

which is the most elegant formulation of the Maxwell's equations.

Example: Relativistic potentials of a moving charge at constant velocity

In 10-58, eqs (84) & (85), we obtain that for a charge q moving with velocity $v\hat{x}$

$$\vec{A}(x,y,z,t) = \frac{\mu_0}{4\pi} \frac{\gamma q v \hat{x}}{\sqrt{\gamma^2(x-vt)^2 + y^2 + z^2}}$$

$$V(x,y,z,t) = \frac{1}{4\pi\epsilon_0} \frac{\gamma q}{\sqrt{\gamma^2(x-vt)^2 + y^2 + z^2}}$$

Clearly, $\because \mu_0 = \frac{1}{\epsilon_0 c^2}$, $\vec{A} = \frac{\vec{v}}{c^2} V$

$$\therefore (\vec{A})^2 - V^2/c^2 = \frac{q^2}{4\pi\epsilon_0} \frac{\gamma^2}{\gamma^2(x-vt)^2 + y^2 + z^2}\left[\frac{v^2}{c^4} - \frac{1}{c^2}\right]$$

$$\because \frac{\gamma^2}{c^2}(\beta^2 - 1) = -1/c^2 \quad \therefore (\vec{A})^2 - \left(\frac{V}{c}\right)^2 = -\frac{\mu_0}{4\pi} \frac{q^2}{\gamma^2(x-vt)^2 + y^2 + z^2}$$

$\gamma^2(x-vt)^2 + y^2 + z^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2$ = distance in the rest frame of q $\therefore$ is invariant, $(\vec{A})^2 - \left(\frac{V}{c}\right)^2$ is invariant

$\therefore \left(\frac{V}{c}, \vec{A}\right)$ is a 4-vector!