

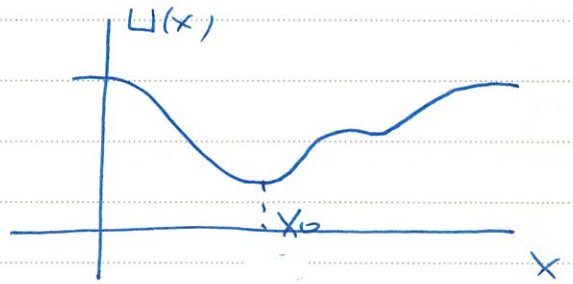
Oscillations and Chaotic motion

在 stable equilibrium 附近

$$\therefore U'(x_0) = 0$$

$$\therefore U(x) = U(x_0) + \frac{1}{2}(x-x_0)^2 U''(x_0) + \dots$$

$$U''(x_0) > 0$$



如果運動之範圍不大，我們可以忽略高次項

$$\text{平衡位置} \rightarrow x_0, \quad \therefore x - x_0 \rightarrow x$$

$$\text{set } U'' = k$$

$$\text{則 } U(x) = \frac{1}{2} k x^2$$

($U(x_0)$ 只是一常數，為位能原點，可被忽略)

$$F = -\frac{dU}{dx} = -kx$$

此即所謂的虎克定律

以能量形式

x 的運動已在上一單元解過，為周期、振盪

通稱為簡諧運動 (Simple harmonic motion)

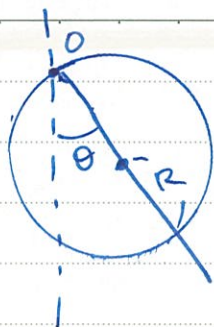
一般而言， x 不需為 x 座標，只要能量有

$$\text{以下的形式} \quad E = \frac{1}{2} \alpha \left(\frac{dq}{dt} \right)^2 + \frac{1}{2} \beta q^2$$

其中 q 為某一座標， $\alpha, \beta > 0$

$$\text{則為簡諧運動, } \omega = 2\pi\nu = \frac{2\pi}{T} = \sqrt{\frac{\beta}{\alpha}}$$

例



$$U = -mgR \cos \theta$$

$$T = \frac{1}{2} I \dot{\theta}^2 = \frac{1}{2} I \dot{\theta}^2$$

$$\sum \frac{1}{2} m_i v_i^2 = \frac{1}{2} \left(\sum m_i r_i^2 \right) \dot{\theta}^2$$

$$\therefore E = T + U = \frac{1}{2} I \dot{\theta}^2 - mgR \left(1 - \frac{\theta^2}{2} + \dots \right)$$

$$\approx \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} mgR \theta^2 - mgR$$

$$\therefore \omega = \sqrt{\frac{mgR}{I}}$$

* 二階微分方程式: 雖然能量守恒可以降階為一階, 簡諧運動是最簡單需要解二階微分方程式的問題。

一般而言 (見 Appendix C), 最廣的 線性 n 階 微分方程式是

$$a_n(t) \frac{d^n x}{dt^n} + a_{n-1}(t) \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_1(t) \frac{dx}{dt} + a_0(t) x = b(t)$$

線性: a_k 與 x 無關, $b(t) = 0$ 為 homogeneous

(i) 同時若 $X = X_1(t)$ 是解, 即 $A_n \frac{d^n X_1}{dt^n} + A_{n-1} \frac{d^{n-1} X_1}{dt^{n-1}} + \dots + A_0 X_1 = 0$

則 $\therefore \frac{d^k}{dt^k} C X_1(t) = C \frac{d^k}{dt^k} X_1(t)$, $C = \text{常數}$

$\tilde{X} = C X_1(t)$ 也是解

$$A_n \frac{d^n \tilde{X}}{dt^n} = C A_n \frac{d^n X_1}{dt^n}$$

$$A_n \frac{d^n \tilde{X}}{dt^n} + A_{n-1} \frac{d^{n-1} \tilde{X}}{dt^{n-1}} + \dots + A_0 \tilde{X}$$

$$= C \left[A_n \frac{d^n X_1}{dt^n} + A_{n-1} \frac{d^{n-1} X_1}{dt^{n-1}} + \dots + A_0 X_1 \right] = C \cdot 0 = 0 \dots \textcircled{1}$$

(ii) 若 X_1 及 X_2 皆為解, 則 $X = X_1 + X_2$ 也是解:

$$A_n \frac{d^n X_1}{dt^n} + A_{n-1} \frac{d^{n-1} X_1}{dt^{n-1}} + \dots + A_1 \frac{d X_1}{dt} + A_0 X_1 = 0$$

$$A_n \frac{d^n X_2}{dt^n} + A_{n-1} \frac{d^{n-1} X_2}{dt^{n-1}} + \dots + A_1 \frac{d X_2}{dt} + A_0 X_2 = 0$$

$$\frac{d^k X}{dt^k} = \frac{d^k}{dt^k} (X_1 + X_2) = \frac{d^k X_1}{dt^k} + \frac{d^k X_2}{dt^k}$$

$$A_n \frac{d^n X}{dt^n} + A_{n-1} \frac{d^{n-1} X}{dt^{n-1}} + \dots + A_0 X$$

$$= \left(A_n \frac{d^n X_1}{dt^n} + \dots + A_0 X_1 \right) + \left(A_n \frac{d^n X_2}{dt^n} + \dots + A_0 X_2 \right) = 0 \dots \textcircled{2}$$

(ii) 及 (iii) 的成立主要是 A_k 只與 t 有關之故!

如果 A_k 與 X 或 X 的微商有關, 就不再是線性。

例如 $A_0 = X$, 則 $A_0 X = X^2$ $(CX_1)^2 = C^2 X_1^2$

$$a_0 \hat{x}^2 \neq c \underbrace{a_0 x_1}_{x_1^2}$$

$$\underbrace{\hat{x}^2}_{c^2 x_1^2}$$

①式不成立。

同理, 若 $b(x) \neq 0$, (i) 及 (ii) 也不成立

此時稱為 inhomogeneous.

Inhomogeneous 方程式可由 homogeneous 解得出, 故

我們先開始
從 homogeneous

二階 線性 ODE

$$\frac{d^2 y}{dx^2} + a \frac{dy}{dx} + by = f(x)$$

homogeneous: $y''(x) + a(x)y' + b(x)y = 0 \quad \dots (3)$

除了 (i) 及 (ii) 外, 有以下 2 個重要性質。

(iii) 若 $y_1(x)$ 及 $y_2(x)$ 是 (3) 式的解,

則其 Wronskian $\equiv \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \equiv W(x)$

$= 0$ 或 $\neq 0$ (沒有根)

pf: $y_1'' + a(x)y_1'(x) + by_1 = 0 \quad - (4)$

$y_2'' + a(x)y_2'(x) + by_2 = 0 \quad - (5)$

$$\frac{dW}{dx} = \frac{d}{dx} (y_1 y_2' - y_1' y_2)$$

$$= y_1 y_2'' - y_1'' y_2$$

$$y_1 \times (5) - y_2 \times (4) \Rightarrow y_1 y_2'' - y_1'' y_2 \quad \rightarrow \frac{dW}{dx}$$

$$+ a(y_1 y_2' - y_1' y_2) = 0$$

W

$$\therefore \frac{dW(x)}{dx} + a(x)W = 0$$

$$\therefore W = W_0 e^{-\int a(x) dx}$$

\therefore 若 $W_0 = 0$, $W(x) \equiv 0$

$W_0 \neq 0$, $W(x)$ 恒 $\neq 0$

$$y_2 = c y_1, \quad W = y_1 y_2' - y_1' y_2 = c(y_1 y_1' - y_1' y_1) = 0$$

反之亦然, $\therefore W = 0$, $y_2 \propto y_1$, 此為线性相關
(証明略) 稱 y_1 及 y_2 為

$$(W=0 \Rightarrow \frac{y_1 y_2' - y_1' y_2}{y_1^2} = 0, \frac{d}{dx}(\frac{y_2}{y_1}) = 0) \quad (\text{linearly dependent})$$

否則稱為线性(無關) (linearly independent)
獨立

(V) 若 y_1 及 y_2 是 (3) 式 = 個線性独立的解,

則任意, (3) 式的解 $y(x)$ 中可表示為

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \quad \dots (6)$$

(6) 式反映了之前所述 = 微分 方程式有 =

個未定常數, 即為 c_1 及 c_2 .

$\therefore c_1$ 及 c_2 可由起始條件決定:

$$\left. \begin{aligned} y_0 &= c_1 y_1(0) + c_2 y_2(0) \\ y'_0 &= c_1 y'_1(0) + c_2 y'_2(0) \end{aligned} \right\} \begin{array}{l} \text{解聯立方程式} \\ \text{可求出 } c_1 \text{ 及 } c_2 \end{array} \quad \dots (7)$$

$\because y_1$ 及 y_2 線性獨立, $\therefore \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} \neq 0$ $\therefore c_1$ 及 c_2 中有解!

即任何 (3) 式的解 $y(x)$, 一旦 $y(0)$ 及 $y'(0)$ 系統
即被唯一決定
 $y(x)$, 而由 (7) 式所得之 c_1 及 c_2 所建構

出來的解 $c_1 y_1(x) + c_2 y_2(x)$ 在 $x=0$ 滿足 $y(0)$ 及 $y'(0)$

所以在 $x \neq 0$ 處, 由唯一性可知 $c_1 y_1(x) + c_2 y_2(x)$ 即為
解 $\therefore y(x) = c_1 y_1 + c_2 y_2$

\therefore 只要找出任意 = 個独立的解, (3) 式即被解決!

Inhomogeneous: 若 $f \neq 0$, 則我們中須解

$$y'' + ay' + b = f(x) \quad \dots (8)$$

如果我們猜出一個特解 $y_p(x)$ (particular solution)
 則 $y'' + ay' + by = f(x)$ ①

因為一般解滿足 ①, \therefore ① - ② 式可得

$$\frac{d^2}{dx^2}(y - y_p) + a \frac{d}{dx}(y - y_p) + b(y - y_p) = 0$$

即 $y - y_p$ 滿足 ③ 式 (homogeneous)

依據 (v), $y - y_p = c_1 y_1 + c_2 y_2$

$$\therefore y = c_1 y_1 + c_2 y_2 + y_p, \quad c_1 y_1 + c_2 y_2 = y_c$$

即 Inhomogeneous 微分方程式, 只需猜出 y_c = Complementary Solution
 特解 = 齊線性

三個解 y_1, y_2 及 y_p 即可!

* Simple harmonic oscillation

其運動方程式為

$$m \frac{d^2 x}{dt^2} + kx = 0$$

$$\therefore \ddot{x} + \omega_0^2 x = 0 \quad \omega_0^2 \equiv \frac{k}{m} \quad \dots (10)$$

猜解: $x = e^{\gamma t}$, $\therefore \ddot{x} = \gamma^2 x$ \therefore (10) $\Rightarrow 0$

$$\Rightarrow (\omega_0^2 + \gamma^2)x = 0, \quad \gamma^2 = -\omega_0^2, \quad \gamma = \pm \omega_0 i$$

$$\therefore x_1 = e^{i\omega t} \quad \text{或} \quad x_2 = e^{-i\omega t}$$

$$\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} e^{i\omega t} & e^{-i\omega t} \\ i\omega e^{i\omega t} & -i\omega e^{-i\omega t} \end{vmatrix} = -2i\omega \neq 0$$

$\therefore x_1$ 与 x_2 线性独立!

$$\begin{aligned} \text{故} \quad x(t) &= C_1 x_1 + C_2 x_2 = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \\ &= \alpha \cos \omega t + \beta \sin \omega t \end{aligned}$$

$$\alpha = C_1 + C_2, \quad \beta = iC_1 - iC_2$$

$$= \sqrt{\alpha^2 + \beta^2} \sin \omega t \cos \phi + \sin \phi \cos \omega t$$

$$= \sqrt{\alpha^2 + \beta^2} \sin(\omega t + \phi)$$

与用 能量守恒 得出解一致!

一般而言, 若 $a(x) = a$, $b(x) = b$ 为常数

$$\textcircled{3} \text{ 式} \quad y''(x) + a y' + b y = 0 \quad \text{--- (1)}$$

$$\text{可令 } y = e^{rx}$$

猜

$$\text{此时 } y'' = r^2 y$$

$$y' = r y$$

$$\therefore (r^2 + ar + b) y = 0$$

$\therefore r^2 + ar + b = 0$, 此为 auxiliary equation, 且通常有

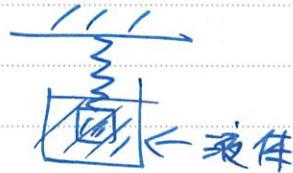
二個解 v_1 及 v_2 (除了 $a^2 - 4b = 0$ 外)

$$\therefore y = c_1 e^{v_1 x} + c_2 e^{v_2 x}$$

* Damped Oscillations

真實的振盪會逐漸的變小，主要是有阻力或摩擦之故。

如果阻力來自液體，如



(油壓振盪器)

(則阻力 = $-b\dot{x}$)

$$\therefore m\ddot{x} + b\dot{x} + kx = 0$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \quad \dots \textcircled{12}$$

此時之方程式即為 $\textcircled{11}$ 式之一特例。

(如果是摩擦力， $F = -\mu smg \text{sign}(\dot{x})$)

其運動情形，請參考習題.)

$$\therefore x = e^{r\tau}, \quad r \text{ 滿足}$$

$$r^2 + 2\beta r + \omega_0^2 = 0$$

$$r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}, \quad r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$$

$$X(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t}$$

$$= e^{-\beta t} (A_1 e^{\sqrt{\beta^2 - \omega^2} t} + A_2 e^{-\sqrt{\beta^2 - \omega^2} t})$$

很明顯的，可由 $\beta^2 - \omega^2 \geq 0$ 還原，將運動分為：

(i) underdamping $\omega^2 - \beta^2 \equiv \omega_1^2 > 0$

$$\text{此時 } X(t) = e^{-\beta t} (A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t})$$

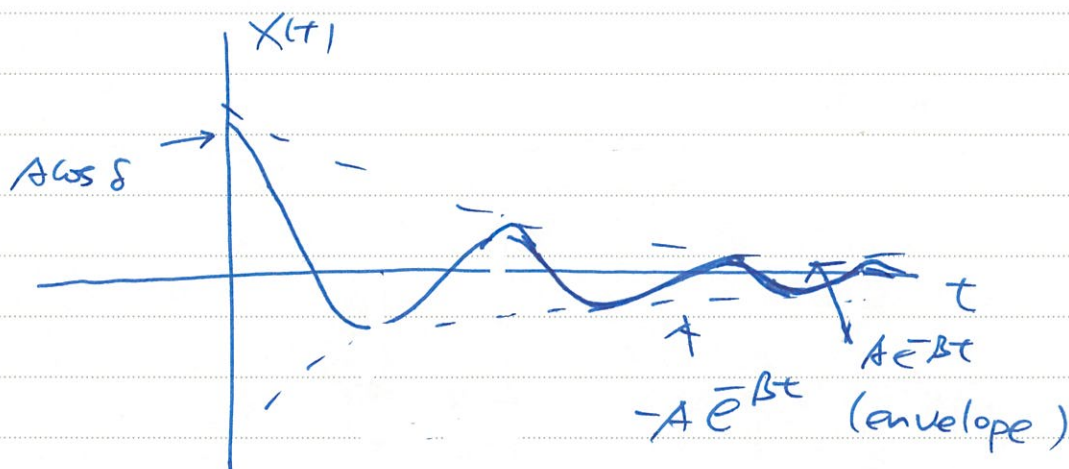
$$= e^{-\beta t} (\alpha \cos \omega_1 t + \beta \sin \omega_1 t)$$

$$= e^{-\beta t} \sqrt{\alpha^2 + \beta^2} (\cos \omega_1 t \cos \delta + \sin \omega_1 t \sin \delta)$$

$$\cos \delta = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}, \quad \sin \delta = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}$$

$$= A e^{-\beta t} \cos(\omega_1 t - \delta)$$

$\therefore X(t)$ 一般振盪一週，振幅衰減

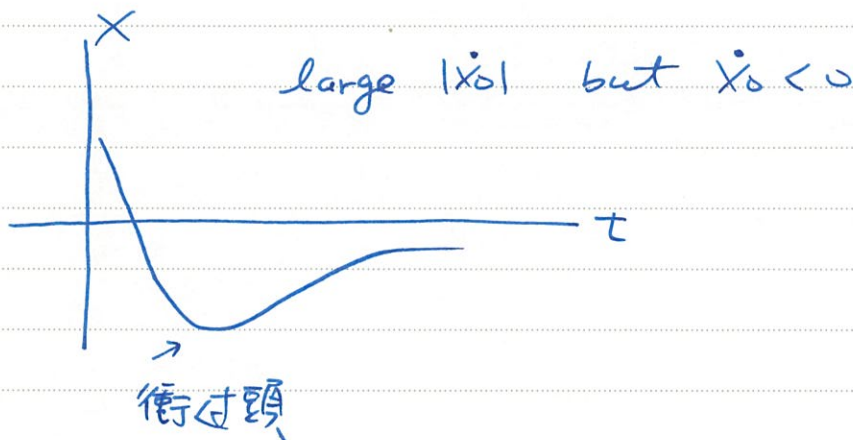
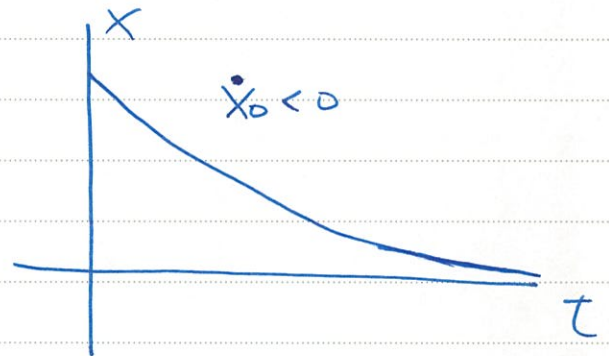
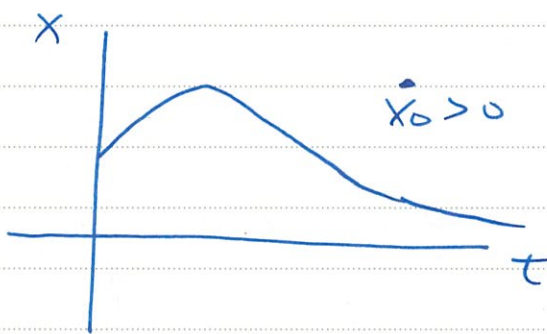


(ii) over damping $\omega_0^2 < \beta^2$, $\omega_2^2 \equiv \beta^2 - \omega_0^2$

此時完全沒有振盪, (γ_1 及 γ_2 皆為實數)

$$\begin{aligned}
 X(t) &= e^{-\beta t} (A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}) \\
 &= A_1 e^{-\underbrace{(\beta - \omega_2)}_{>0} t} + A_2 e^{-\underbrace{(\beta + \omega_2)}_{>0} t}
 \end{aligned}$$

依 $\left. \frac{d}{dt} X(t) \right|_{t=0} = \dot{X}_0 > 0$, < 0 有不同之行爲
($X_0 > 0$)



(iii) critical damping $\omega_0^2 = \beta^2$.

此時 $\gamma_1 = \gamma_2 = \omega_0$, 只得到一個解,

另一解?

另一解可以由 $e^{r_1 t}$ 及 $e^{r_2 t}$ 組合得到

$$X_1 = e^{r_1 t}$$

$\xrightarrow{r_1 \rightarrow r_2}$

$$X_1(t) = e^{r_1 t} = e^{-\beta t}$$

$$X_2 = \frac{e^{r_1 t} - e^{r_2 t}}{r_1 - r_2}$$

$\xrightarrow{r_1 \rightarrow r_2}$

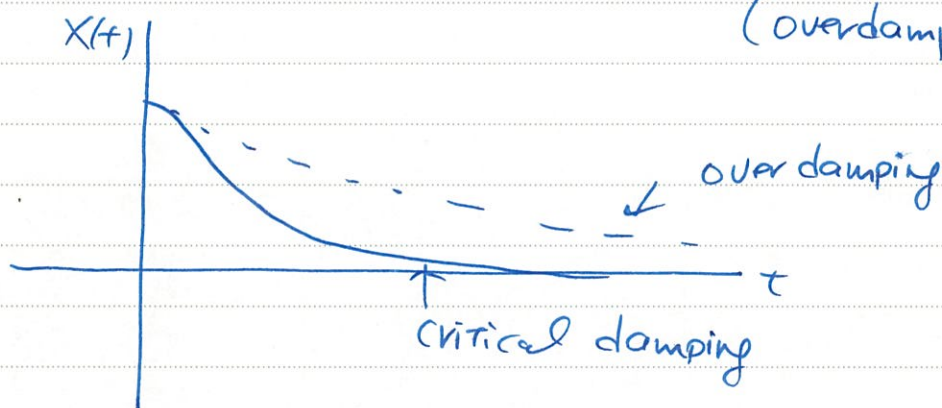
$$\lim_{r_1 \rightarrow r_2} \frac{\frac{d}{dr_1} e^{r_1 t}}{\frac{d}{dr_1} (r_1 - r_2)} = t e^{r_1 t} = t e^{-\beta t}$$

$$\therefore X(t) = A e^{-\beta t} + B t e^{-\beta t}$$

$$= (A + B t) e^{-\beta t}$$

decay 比較快

(overdamping $e^{-\frac{(\beta - \omega_d)t}{\tau}}$ to βt !)



* Phase diagrams

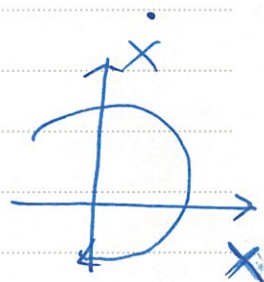
由於運動方程式是二階微分方程式，故

對在某一時刻 t ， $X(t)$ 及 $\dot{X}(t)$ 可決定之後的

所有運動，故可以在 $X - \dot{X}$ 平面上考慮

物體之運動，在 $X - \dot{X}$ 平面上所有可能

運動所形成之路徑稱為 phase diagrams



它是一個簡潔表示運動的方程式

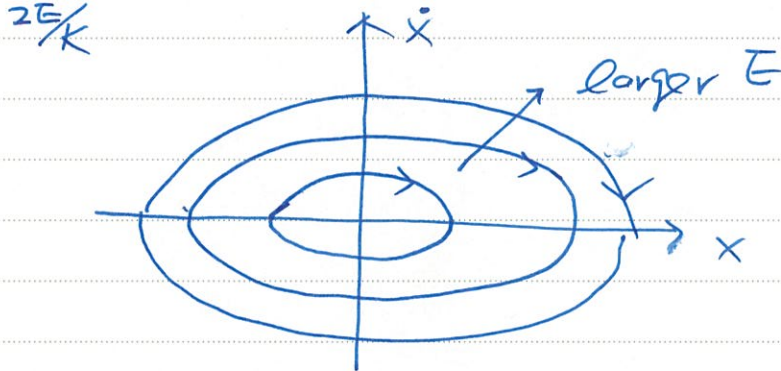
不同的路徑代表不同的起始條件，故不相交！

Free oscillations

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 = \frac{1}{2} k A^2 \quad \leftarrow \text{振幅}$$

$$\therefore \frac{x^2}{A^2} + \frac{\dot{x}^2}{A^2 k/m} = 1$$

$$\frac{x^2}{A^2} + \frac{\dot{x}^2}{\omega_0^2 A^2} = 1 \quad \Rightarrow \text{椭圆} \quad \left[\frac{2E}{k} \right]$$



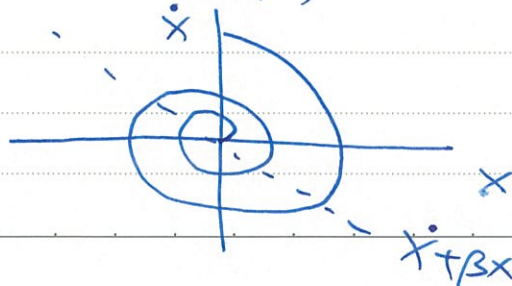
Counter clockwise: $\therefore \begin{cases} x > 0 \text{ 時} \\ \dot{x} > 0 \end{cases}$ \dot{x} decreases.

Underdamping: $\frac{dE}{dt} = m \ddot{x} \dot{x} + k x \dot{x} = \dot{x} (-\beta \dot{x}) = -\beta \dot{x}^2 < 0$

Details: $x(t) = A e^{-\beta t} \cos(\omega t - \delta)$ \therefore

$$\begin{aligned} \dot{x}(t) &= -A e^{-\beta t} (\beta \cos(\omega t - \delta) + \omega \sin(\omega t - \delta)) \\ &= -\beta x(t) - A \omega e^{-\beta t} \sin(\omega t - \delta) \end{aligned}$$

$$\therefore \left(\frac{x}{A}\right)^2 + \left(\frac{\dot{x} + \beta x}{A \omega}\right)^2 = e^{-2\beta t}$$



Overdamping

$$X(t) = A_1 e^{-(\beta - \omega_2)t} + A_2 e^{-(\beta + \omega_2)t}$$

$$\equiv A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}$$

$$\dot{X} = -A_1 \beta_1 e^{-\beta_1 t} - A_2 \beta_2 e^{-\beta_2 t}$$

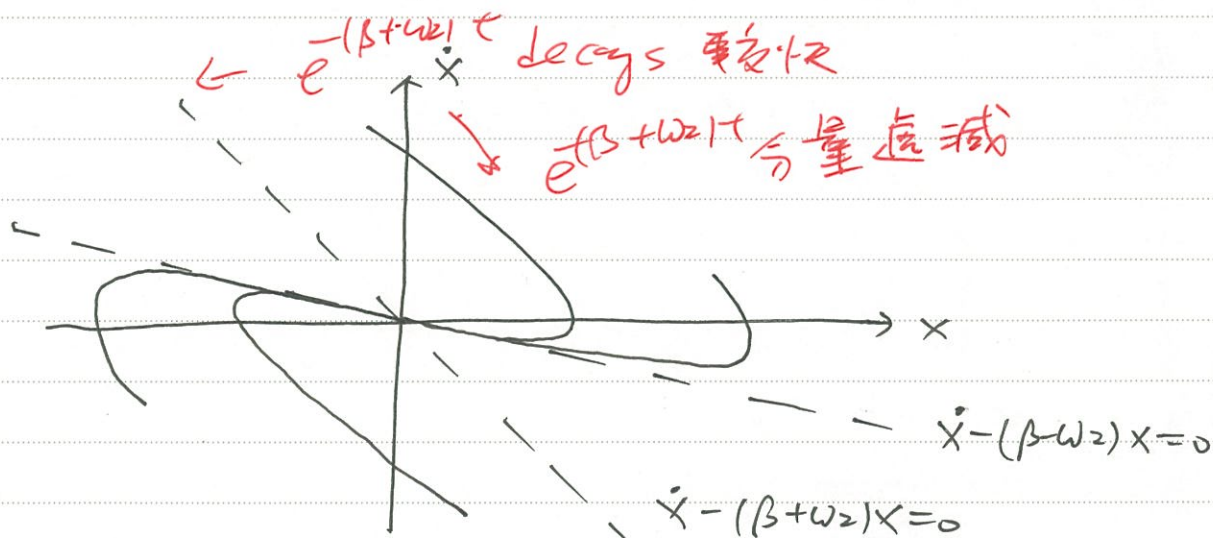
$$\therefore \frac{\dot{X}}{X} = - \frac{A_1 \beta_1 + A_2 \beta_2 e^{-(\beta_2 - \beta_1)t}}{A_1 + A_2 e^{-(\beta_2 - \beta_1)t}}$$

$$\beta_2 - \beta_1 = 2\omega_2 > 0 \quad \therefore e^{-(\beta_2 - \beta_1)t} \rightarrow 0 \text{ as } t \rightarrow \infty$$

$$\therefore \text{if } A_1 \neq 0, \quad \left. \frac{\dot{X}}{X} \right|_{t \rightarrow \infty} = -\beta_1$$

$$\dot{X} - (\beta - \omega_2)X = 0 \text{ as } t \rightarrow \infty$$

$$A_1 = 0, \quad \frac{\dot{X}}{X} = -\beta_2, \quad \dot{X} - (\beta + \omega_2)X = 0$$

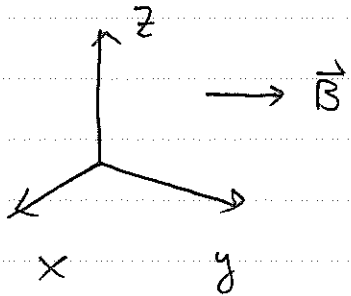


$$\text{Critical damping} \quad \left. \frac{\dot{X}}{X} \right|_{t \rightarrow \infty} \rightarrow -\beta \quad (\text{与上图相似})$$

高維振盪 (Oscillations in higher dimension)

許多振盪發生在高維，如在均勻磁場

中帶電質點之運動：



$$m \ddot{\vec{r}} = q \vec{v} \times \vec{B}$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ q\dot{x} & q\dot{y} & q\dot{z} \\ 0 & B & 0 \end{vmatrix}$$

$$= (-qB\dot{z}, 0, qB\dot{x})$$

$$\therefore m \ddot{x} = -qB\dot{z}$$

$$m \ddot{y} = 0 \quad \Rightarrow \quad y = y_0 + v_{y(0)} t$$

$$m \ddot{z} = qB\dot{x}$$

$$\ddot{x} = -\omega \dot{z}$$

$$\omega = \frac{qB}{m}$$

$$\ddot{z} = \omega \dot{x}$$

--- (13)

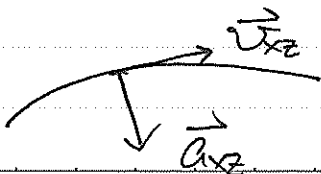
即

$$\dot{v}_x = -\omega v_z$$

換句話說

$$\dot{v}_z = \omega v_x$$

$$\vec{a}_{xz} = \begin{pmatrix} \dot{v}_x \\ \dot{v}_z \end{pmatrix} \perp \begin{pmatrix} v_x \\ v_z \end{pmatrix} \quad \dots (14)$$



只有垂直 \vec{v}_x 之加速，反映了 $\vec{v} \perp \vec{B}$. $\vec{v} \perp \vec{v}$ 之事實

磁場只改變 \vec{v} 之方向，並不改 \vec{v} 之大小

沒有輸入能量！ ($\frac{d\vec{v}}{dt} \cdot \vec{v} = 0, \therefore \frac{d|\vec{v}|^2}{dt} = 0$)

(4)式之解可由 $\dot{v}_x = -\omega v_z = -\omega^2 v_x$ 得到 (5)

$$\dot{v}_z = +\omega v_x = -\omega^2 v_z$$

在 v 的二次微分後， v_x 及 v_z decouple!

$$\therefore v_x = A_0 \cos \omega t + B_0 \sin \omega t$$

$$v_z = A_0' \cos \omega t + B_0' \sin \omega t$$

雖然 (5) 式好像說 v_x 及 v_z decouple，但

v_x 及 v_z 還是有關係 (14) 式)， A_0, B_0 與 A_0', B_0' 並不獨立。

由 $\dot{v}_x = -\omega v_z$ 可得

$$-A_0 \omega \sin \omega t + B_0 \omega \cos \omega t = -\omega A_0' \cos \omega t - \omega B_0' \sin \omega t$$

$$\therefore A_0 = B_0', \quad A_0' = -B_0$$

$$\therefore v_x = A_0 \cos \omega t + B_0 \sin \omega t = \frac{dx}{dt}$$

$$v_z = -B_0 \cos \omega t + A_0 \sin \omega t = \frac{dz}{dt}$$

$$\begin{aligned} x - x_0 &= \int_0^t A_0 \cos \omega t' dt' + \int_0^t B_0 \sin \omega t' dt' \\ &= \frac{A_0}{\omega} \sin \omega t + \frac{B_0}{\omega} (1 - \cos \omega t) \end{aligned}$$

$$z - z_0 = \frac{-B_0}{\omega} \sin \omega t + \frac{A_0}{\omega} (1 - \cos \omega t)$$

$$\therefore v_x^0 = A_0, \quad v_z^0 = -B_0$$

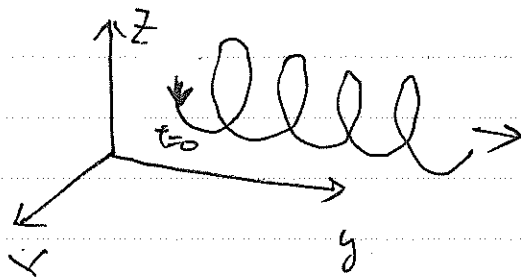
$$\therefore X - X_0 - \frac{v_z^0}{\omega} = \frac{v_x^0}{\omega} \sin \omega t + \frac{v_z^0}{\omega} \cos \omega t$$

$$z - z_0 - \frac{v_x^0}{\omega} = +\frac{v_z^0}{\omega} \sin \omega t - \frac{v_x^0}{\omega} \cos \omega t$$

$$\frac{1}{2} v_x^0 = 0$$

$$\left(X - X_0 - \frac{v_z^0}{\omega} \right)^2 + (z - z_0)^2 = \left(\frac{v_z^0}{\omega} \right)^2$$

$$= \left(\frac{m v_z^0}{8B} \right)^2$$



helix (cyclotron motion)

$$\text{radius} = \frac{m v_z^0}{8B}$$

In general,

$$\left(X - X_0 + \frac{v_z^0}{\omega} \right)^2 + \left(z - z_0 - \frac{v_x^0}{\omega} \right)^2 = \frac{v_x^0{}^2 + v_z^0{}^2}{\omega^2}$$

$$\Rightarrow \text{helix with radius} = \frac{m v_{\perp}^0}{8B}$$

$$\omega = \frac{8B}{m} = \text{cyclotron frequency}$$

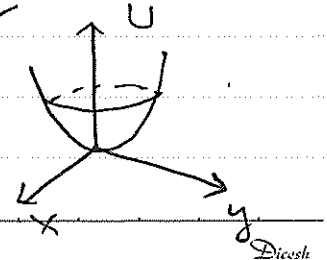
Harmonic oscillations in 2D - Lissajour curve

$$\vec{F} = -k\vec{r} \quad \text{in 2D} \quad \therefore m\ddot{\vec{r}} = -k\vec{r}$$

$$\therefore m\ddot{x} = -kx$$

$$m\ddot{y} = -ky$$

$$U = \frac{1}{2}kr^2$$



$$x(t) = A \cos(\omega t - \alpha)$$

$$y(t) = B \cos(\omega t - \beta)$$

----- (15)

α 与 β 由 $x(0)$ 及 $y(0)$ 决定

由 (15) 可以消去 t

$$y(t) = B \cos(\omega t - \alpha + \alpha - \beta)$$

$$= B \cos(\omega t - \alpha) \underbrace{\cos(\alpha - \beta)}_{\delta} - B \sin(\omega t - \alpha) \underbrace{\sin(\alpha - \beta)}_{\delta}$$

$$= \frac{B}{A} \cos \delta x - B \sqrt{1 - \cos^2(\omega t - \alpha)} \sin \delta$$

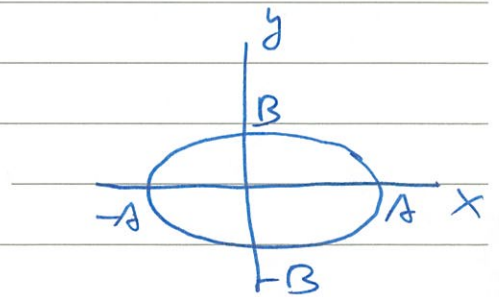
$$= \frac{B}{A} x \cos \delta - B \sqrt{1 - \left(\frac{x}{A}\right)^2} \sin \delta$$

$$\therefore \left(y - \frac{B}{A} x \cos \delta\right)^2 = B^2 \sin^2 \delta \left(1 - \left(\frac{x}{A}\right)^2\right)$$

$$A^2 y^2 + B^2 x^2 - 2ABxy \cos \delta = A^2 B^2 \sin^2 \delta \quad \text{在 } xy \text{ 平面上}$$

为一椭圆

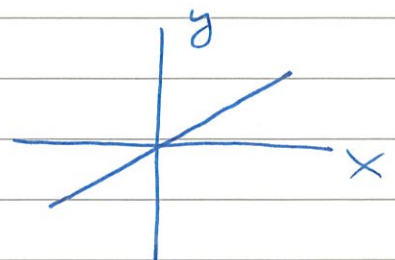
$$\delta = \pm \frac{\pi}{2} \text{ 时} \quad \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$



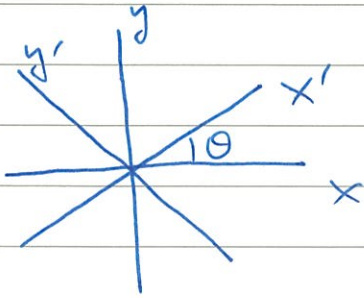
$$\delta = 0 \text{ 时} \quad A^2 y^2 + B^2 x^2 - 2ABxy = 0$$

$$(Bx - Ay)^2 = 0$$

$$y = \frac{B}{A} x$$



In general, δ 代表旋轉後的橢圓:



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\therefore \frac{x^2}{A^2} + \frac{y^2}{B^2} = \frac{1}{A^2} (\cos\theta x' - \sin\theta y')^2 + \frac{1}{B^2} (\sin\theta x' + \cos\theta y')^2$$

$$= \left(\frac{1}{A^2} \cos^2\theta + \frac{\sin^2\theta}{B^2} \right) x'^2 + \left(\frac{\sin^2\theta}{A^2} + \frac{\cos^2\theta}{B^2} \right) y'^2$$

$$- \underbrace{\left(\frac{2}{A^2} \sin\theta \cos\theta - \frac{2}{B^2} \sin\theta \cos\theta \right)}_{\propto \cos\delta!} x' y' = 1$$

see Fig 3.2 (書本) for rotated ellipse.

Lissajour curve

如果有 anisotropy, 即 $F_x = -k_x x$ $k_x \neq k_y$
 $F_y = -k_y y$

$$\text{則 } \ddot{x} + \frac{k_x}{m} x = 0$$

$$\ddot{y} + \frac{k_y}{m} y = 0$$

$$\therefore \ddot{x} + \omega_x^2 x = 0 \quad \omega_x \neq \omega_y$$

$$\ddot{y} + \omega_y^2 y = 0$$

$$\therefore x(t) = A \cos(\omega_x t - \alpha)$$

$$y(t) = B \cos(\omega_y t - \beta)$$

此時在 x, y 平面上之曲線比較複雜，

不再是橢圓，稱為 Lissajous curve.

當 $\frac{\omega_x}{\omega_y} = \text{有理數} \equiv \frac{n_x}{n_y}$ (稱為 commensurable) ($n_x, n_y = 1$)

則 $x(t) = x(t+T)$
可找到 T 使得 $y(t) = y(t+T)$.

取 $\left. \begin{array}{l} \omega_x T = 2\pi n_x \\ \omega_y T = 2\pi n_y \end{array} \right\} \frac{\omega_x}{\omega_y} = \frac{n_x}{n_y}$ 即可滿足

反之亦真。

$$\left(\frac{\omega_x}{2\pi n_x} = \frac{\omega_y}{2\pi n_y} \equiv T \right)$$

因此當 ω_x 與 ω_y commensurate 時，曲線是封閉的。
(closed)

反之 ω_x 與 ω_y incommensurate 時，即 $\frac{\omega_x}{\omega_y}$ 為無理數，

T 不存在， \therefore 曲線是開放的！

即曲線永不通過同一點（通過時，速度也與之前通過時一樣）！

事實上，可以證明曲線遲早會無限小靠近任一給定點。
(J. Haag, Oscillation Motions, p62)

封閉的 Lissajous curve 一般比橢圓複雜，

在給定 ω_x/ω_y 下，不同的 n_x, n_y 也有不同的圖形，

(見 Fig. 3-4)

* 強迫振盪 (forced oscillation)

正弦力與共振 (sinusoidal driving force & resonance)

如果施力為 $F_0 \cos \omega t$

則運動方程不再是 homogeneous, 為 inhomogeneous

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos \omega t$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A \cos \omega t \quad \dots (16)$$

$A = F_0/m$, 物理例子: 束縛在原子中之古典電子

由 = 階微分方程理論可知, 中須先找一

$X_p(t)$ (particular solution, 特別解), 則一般解

$$X(t) = \underbrace{X_c(t)}_m + X_p(t) \quad \dots (17)$$

Complementary solution

$$X_c(t) = e^{-\beta t} (A_1 e^{\sqrt{\beta^2 - \omega_0^2} t} + A_2 e^{-\sqrt{\beta^2 - \omega_0^2} t})$$

為已知, 但 $X_p(t) = ?$, 如何求出 $X_p(t)$?

首先, 在(17)式, 當 $t \rightarrow \infty$, $\therefore X_c \rightarrow 0 \quad \therefore X_c$

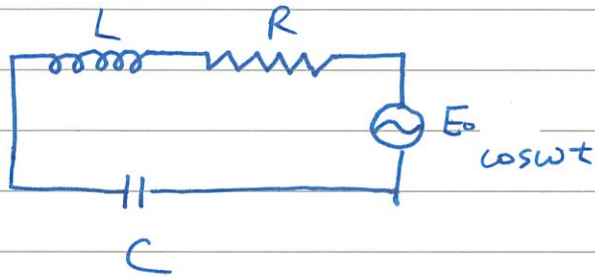
只描述暫態的行為 (transient effect)

因此, $X_p(t)$ 中須在 $t \rightarrow \infty$ 保持不為零

與 F_0 有關!

物理系統 - 電路的應用

(16) 式 不只是描述被施加外力之彈簧系統，
在電路上有完全之對應。



如左圖，RLC 電路

接上交流電 $E_0 \cos \omega t$ 。

$$\text{則 } V_L = L \frac{dI}{dt} = L \frac{d^2q}{dt^2}$$

$$V_R = RI = R \frac{dq}{dt}$$

$$V_C = q/C$$

故由 Kirchoff 定律可知 $V_L + V_R + V_C = E_0 \cos \omega t$

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + q/C = E_0 \cos \omega t \quad \dots (17)$$

與 (16) 式相比可知有以下之對應

$$x \leftrightarrow q$$

$$\dot{x} \leftrightarrow I = \dot{q}$$

$$m \leftrightarrow L$$

$$b \leftrightarrow R$$

$$k \leftrightarrow C$$

$$F \leftrightarrow E_0$$

疊加原理與富氏級數 (Fourier series)

(principle of superposition)

x 滿足之方程式雖然不是 inhomogeneous; 但

有一個有用的性質: 滿足 Superposition

$$\left(\frac{d^2}{dt^2} + a \frac{d}{dt} + b \right) x' = A \cos \omega t = F_1$$

Linear operator, \hat{L}

$$\frac{d}{dt}(x_1 + x_2) = \frac{dx_1}{dt} + \frac{dx_2}{dt}$$

$$b(x_1 + x_2) = bx_1 + bx_2$$

$$\frac{d^2}{dt^2}(x_1 + x_2) = \frac{d^2x_1}{dt^2} + \frac{d^2x_2}{dt^2}$$

$$\therefore \hat{L}(x_1 + x_2) = (\hat{L}x_1) + (\hat{L}x_2)$$

因此, 如果 $\hat{L}x_1 = F_1$

$$\hat{L}x_2 = F_2$$

$$\text{則 } \hat{L}(x_1 + x_2) = F_1 + F_2$$

更進一步的, 因為 $\frac{d}{dt} \alpha x_1 = \alpha \frac{dx_1}{dt}$, $\alpha = \text{常數}$

$$b(\alpha x_1) = \alpha (bx_1)$$

$$\frac{d^2}{dt^2}(\alpha x_1) = \alpha \frac{d^2x_1}{dt^2}$$

$$\therefore \hat{L}(\alpha x) = \alpha \hat{L}(x)$$

$$\begin{aligned}\mathcal{L}(\alpha_1 x_1 + \alpha_2 x_2) &= \alpha_1 \mathcal{L}x_1 + \alpha_2 \mathcal{L}x_2 \\ &= \alpha_1 F_1 + \alpha_2 F_2\end{aligned}$$

一般而言，若 $\mathcal{L}x_k = F_k(t)$

$$\text{則 } \mathcal{L}\left(\sum_{k=1}^N \alpha_k x_k\right) = \sum_{k=1}^N \alpha_k F_k(t) \quad \dots (19)$$

此為 principle of superposition. 只對 線性 算符成立!

(19) 式的用途 主要在於如果可以把任一 $F(t)$ 以 固定的 $\{F_n(t)\}$ 展開， $F(t) = \sum_{n=1}^{\infty} \alpha_n F_n(t)$

則 求 $\mathcal{L}x(t) = F(t)$ 之解可被簡化

為求 $\mathcal{L}x_n(t) = F_n(t)$

例如：如果 $F(t)$ 為週期性 $F(t) = F(t+T)$
施力

一定 $F(t)$ 一般不一定 是 $F_0 \cos \omega t$, $\omega = \frac{2\pi}{T}$

$F_1 \cos 2\omega t$, $F_2 \cos 3\omega t$, ... 也滿足 $F(t+T) = F(t)$

如何處理 $F(t)$ 呢?

此時我們可利用 Fourier series expansion 符

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + b_n \sin n\omega t \quad \dots (20)$$

然後再單獨求 $\hat{L}(x_n(t)) = a_n \cos \omega t$ 之特解 $x_n(t)$

即為 (20) 富氏展開，其係數 a_n 及 b_n 可以由

$$a_n = \frac{2}{T} \int_0^T F(t') \cos n\omega t' dt'$$

$$b_n = \frac{2}{T} \int_0^T F(t') \sin n\omega t' dt'$$

求出

(仔細的理論在應教課程中有交待)

正弦力之特解

同樣的辦法可以適用解 (16) 式。

首先 $A \cos \omega t = \frac{A}{2} (e^{i\omega t} + e^{-i\omega t})$

\therefore 我們先求 $\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = \frac{A}{2} e^{i\omega t} \dots (21)$

之特解

$\therefore \frac{d}{dt} e^{rt} = r e^{rt}$

\therefore 可以試 $y = y_0 e^{rt}$ 代入 (21) 式，可得

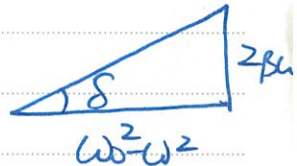
$$y_0 (r^2 + 2\beta r + \omega_0^2) e^{rt} = \frac{A}{2} e^{i\omega t} \dots (22)$$

$\therefore r = \omega i$

由 (22) 式可得 $(\omega_0^2 + 2i\beta\omega - \omega^2) y_0 = A/2$

$$\text{故 } y_0 = \frac{A}{2} \frac{1}{\omega_0^2 - \omega^2 + 2i\omega\beta}$$

$$\therefore \omega_0^2 - \omega^2 + 2i\omega\beta = \sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2} e^{i\delta}$$



$$\tan \delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}, \quad \delta = \tan^{-1} \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

$$\therefore y_0 = \frac{A}{2} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}} e^{-i\delta}$$

$$y(t) = \frac{A}{2} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}} e^{i(\omega t - \delta)}$$

$$\equiv \frac{D}{2} e^{i(\omega t - \delta)}$$

除了(21)式外, 求(16)式解, 還得求

$$\ddot{\tilde{y}} + 2\beta\dot{\tilde{y}} + \omega_0^2\tilde{y} = \frac{A}{2} e^{-i\omega t}$$

$$\text{与(21)相比, } \tilde{y} = y^* = \frac{D}{2} e^{-i(\omega t - \delta)}$$

$$\therefore x_p(t) = y + \tilde{y}$$

$$\therefore x_p(t) = y + y^* = D \cos(\omega t - \delta) \quad \dots (23)$$

$$D = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}} \quad \dots (24)$$

以上所建構之特別解 $X_p(t)$ 只在 $\beta \neq 0$ 且 $\omega \neq \omega_0$ 才成立!

$\omega = \omega_0$ 時, $\therefore D \rightarrow \infty$ 所以並不適用。

$\beta = 0$

然而 $\omega = \omega_0$, $\beta = 0$ 時之 $X_p(t)$ 為何呢?

要回答此問題, 必須了解, 若 $X_p(t)$

是特別解, 則 $X_p(t)$ 加上任何 $X_c(t)$ 仍是一個特別解!

因此, 當 $\beta = 0$ 時, 可以取 $X_p(t) = \frac{A \cos \omega t}{\omega_0^2 - \omega^2}$

(即將 (23) 式中 β 設為 0)

也可以加上 $X_c = -\frac{A}{\omega_0^2 - \omega^2} \cos \omega_0 t$

使得 $X_p(t) = A \frac{\cos \omega t - \cos \omega_0 t}{\omega_0^2 - \omega^2}$

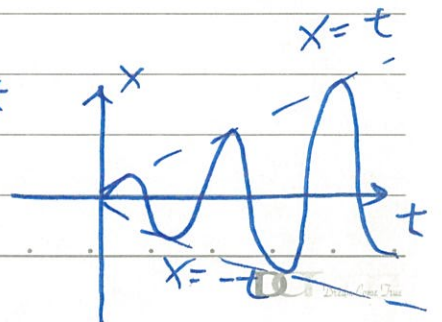
此特別解在 $\omega \rightarrow \omega_0$ 是存在的!

正如我們在等 critical damping 之解所用
的: 技巧, 取 $\omega \rightarrow \omega_0$

$X_p(t) \rightarrow A \frac{-t \sin \omega t}{-2\omega} \Big|_{\omega \rightarrow \omega_0}$ (分子及分母都对 ω 微分)

$\therefore \omega = \omega_0, \beta = 0$ 之特別解 $X_p = \frac{At}{2\omega} \sin \omega t$

振幅隨 t 大而增加!

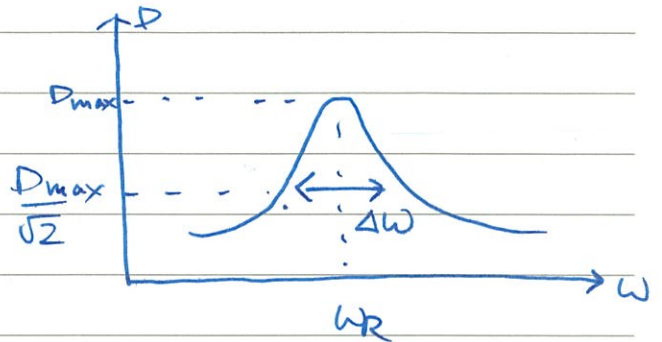


Resonance

D 与 ω 之關係如右圖

在 ω_R 達到最大，此時

$$\left. \frac{dD}{d\omega} \right|_{\omega=\omega_R} = 0$$



這種 D 在 ω 小範圍內有大數值之現象，稱為 resonance

而 ω_R 則稱為 resonant frequency

在實驗上，resonance 常被用以放大待測且

微弱的訊號，此時， $F(t)$ 為待測訊號，而 $x(t)$ 則為儀器的反應。為此，為量度共振的強弱，定

$$Q = \frac{\omega_R}{\Delta\omega} \approx \frac{\omega_0}{\Delta\omega} \text{ 為 quality factor!}$$

由 (24) 式可得 $\frac{dD}{d\omega} = \frac{dD}{d\omega^2} \cdot \frac{d\omega^2}{d\omega} = \dots$

$$\therefore \frac{dD}{d\omega} = 0 \Rightarrow \frac{dD}{d\omega^2} = 0 \quad \omega^2 = x$$

$$\frac{dD}{dx} = \frac{A}{[(\omega^2 - x)^2 + 4\beta^2 x]^{3/2}} \cdot \left(-\frac{1}{2}\right) \cdot (4\beta^2 - 2(\omega^2 - x)) = 0$$

$$\therefore x = \omega^2 - 2\beta^2 \quad \omega_R = \sqrt{\omega^2 - 2\beta^2} \quad \text{--- (25)}$$

\therefore 當 $\beta^2 > \frac{\omega^2}{2}$ ， ω_R 是虛數，不再有 resonance

對於 ζ damping 小的系統 ($\beta \ll \omega_0, \omega_R$)

$$\therefore D_R = \frac{A}{\sqrt{(2\beta)^2 + (2\beta)^2(\omega^2 - \beta^2)}} = \frac{A}{2\beta\sqrt{\omega_0^2 - \beta^2}}$$

$$\omega_R^2 = \omega_0^2 - \beta^2$$

$$\omega_0^2 - \omega_R^2 = 2\beta^2$$

$$\text{令 } \omega = \omega' \text{ 時 } D = \frac{D_R}{\sqrt{2}}$$

$$\therefore \frac{1}{\sqrt{2}} \frac{A}{2\beta\sqrt{\omega_0^2 - \beta^2}} = \frac{A}{\sqrt{(\omega_0^2 - \omega'^2)^2 + 4\omega'^2\beta^2}}$$

$$\therefore (\omega_0^2 - \omega'^2)^2 + 4\beta^2\omega'^2 = 8\beta^2(\omega_0^2 - \beta^2), \quad \omega' \equiv \omega'^2$$

$$\omega'^2 + (4\beta^2 - 2\omega_0^2)\omega' + \omega_0^4 - 8\beta^2(\omega_0^2 - \beta^2) = 0$$

$$\omega' = \omega'^2 = \omega_0^2 - 2\beta^2 \pm \sqrt{4\beta^2\omega_0^2 - 4\beta^2}$$

$$\therefore \omega'^2 = \omega_0^2 - 2\beta^2 \pm 2\beta\omega_0 \left(1 - \frac{\beta^2}{\omega_0^2}\right)^{\frac{1}{2}}$$

$$\approx \omega_R^2 \pm 2\beta\omega_0 \approx \omega_0^2 \pm 2\beta\omega_R$$

$$\therefore \omega' = \omega_R \left(1 \pm \frac{2\beta}{\omega_R}\right)^{\frac{1}{2}} = \omega_R \left(1 \pm \frac{\beta}{\omega_R}\right)$$

$$\therefore \Delta\omega = 2\beta, \quad Q = \frac{\omega_R}{2\beta} \approx \frac{\omega_0}{2\beta}$$

Real systems: ordinary mechanic system $Q = 0.11$ to 100

石英振盪 $Q \rightarrow 10^4$

Cavity $Q \rightarrow 10^4 - 10^5$

原子能腔之輻射盪 (radiation damping) $Q \sim 10^7$

gas laser $Q = 10^4$ (largest known)

power factor 与 δ 的 意义

由以上可知 當力 = $A/2 e^{i\omega t}$ 時

系統的反應可以慢一些, $X \propto e^{i(\omega t - \delta)}$

即力 達到最大時, X 並未達到最大

其間之差距由 δ 由描述

由於有相差存在, 力所作的功就不完全輸入系統 $X(t)$!

我們可以計算功率來看 δ 之角色:

$P(t)$ (功率) = $\dot{X}(t) F(t)$ 之 平均值 ($t \rightarrow \infty$ 時)

$$= \langle D \frac{d}{dt} \cos(\omega t - \delta) \cdot A \cos \omega t \rangle$$

$$\langle \dots \rangle = \frac{1}{T} \int_0^T dt (\dots), \quad \omega = \frac{2\pi}{T}$$

$$\therefore \frac{d}{dt} \cos(\omega t - \delta) = -\omega \sin(\omega t - \delta) = -\omega \cos \delta \sin \omega t + \omega \sin \delta \cos \omega t$$

$$\int_0^T dt \cos \omega t \sin \omega t = \frac{1}{2} \int_0^T dt \sin 2\omega t = \frac{1}{2\omega} \int_0^{2\pi} d\theta \sin 2\theta = 0$$

$$\int_0^T dt \cos^2 \omega t = \frac{1}{2} \int_0^T dt (1 + \cos 2\omega t) = T/2$$

$$\therefore \frac{1}{T} \int_0^T dt \cos^2 \omega t = \frac{1}{2}$$

$$\therefore P_{av} = \frac{AD\omega}{2} \sin \delta = \frac{A^2\omega}{2} \frac{1}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2}} \sin \delta$$

$\delta = 0$ 時 $P_{av} = 0$ 沒有功之輸入。

註：有些書是 $\frac{\pi}{2} - \delta = \beta_0$ 為相位差 (phase lag)

$$\text{此時 } P_{av} = \frac{A^2 \omega}{2} \frac{1}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\beta^2 \omega^2}} \cos \beta_0$$

$$\therefore \tan \delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

$$\omega = 0, \delta = 0$$

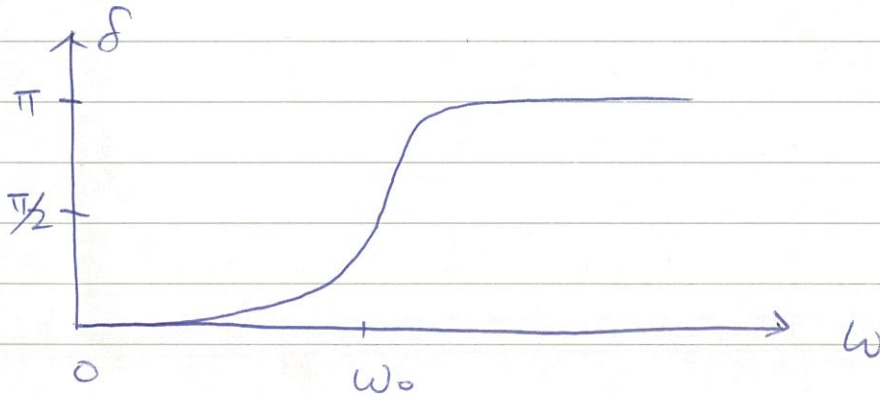
$$\omega \rightarrow \omega_0^-, \delta \rightarrow \frac{\pi}{2}^-, \tan \delta \rightarrow \infty$$

$$\omega \rightarrow \omega_0^+, \tan \delta \rightarrow -\infty, \delta \rightarrow \frac{\pi}{2}^+$$

$$\text{(如果取 } \delta = \tan^{-1} \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

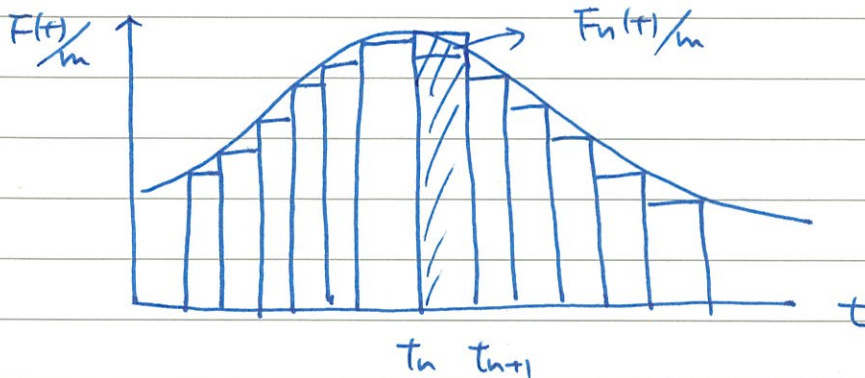
formally, $\delta \rightarrow -\frac{\pi}{2}$ 不連續)

$$\omega \rightarrow \infty, \tan \delta \rightarrow 0^-, \delta \rightarrow \pi$$



* 对 Impulse 力 之 反应 及 应用

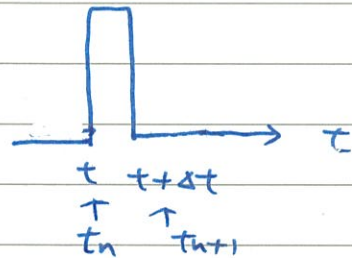
由於任何力 $F(t)$ 可視為許多 Impulse 之疊加



所以只要會解

$$\ddot{X}_n + 2\beta\dot{X}_n + \omega_0^2 X_n = I_n$$

$$I_n(t) = \frac{F_n(t)}{m}$$



則任意 $F(t)$ 对

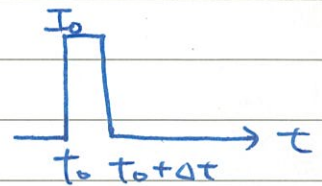
$$X = \sum_{n=-\infty}^{\infty} X_n(t)$$

Response to Impulse function (underdamping)

$$\Delta t \rightarrow 0$$

$$\ddot{X} + 2\beta\dot{X} + \omega_0^2 X = I(t)$$

$$I_0 \Delta t = \frac{F_0 \Delta t}{m} = b$$



因 $I(t)$ 而造成之 $X(t)$ 之变化 $\equiv \Delta X$

$$\Delta X(t) = A e^{-\beta(t-t_0)} \cos \omega_1(t-t_0)$$

$$+ B e^{-\beta(t-t_0)} \sin \omega_1(t-t_0)$$

for $t > t_0 + \Delta t$

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2}$$

L... (26)

由於 Δt 很小 $\rightarrow 0$ $\therefore \Delta X(t)$ 在 $(t_0, t_0 + \Delta t)$ 之

行為不重要 (damping 也沒有作用)
在 $t < t_0$, $\Delta X(t) = 0$, 在 $t > t_0 + \Delta t$,

$F(t)$ 之作用為改變動量, 即在 t_0 時, 粒子

獲得動量 $p_0 = F \Delta t = m \Delta v_0 = m b$

$$\therefore v_0 = b \quad \therefore \int_{t_0}^{t_0 + \Delta t} v \, dt = \Delta X(t_0 + \Delta t) = v_0 \Delta t$$

在 $\Delta t \rightarrow 0$ 需滿足

$$\therefore \textcircled{26} \text{ 之解 } \Delta X(t_0) = b \quad \text{且 } \Delta X(t_0) = 0 \quad \dots \textcircled{27}$$

代入 $\textcircled{26}$ 式

$$\therefore A = 0 \quad b = \omega_1 B \quad B = \frac{b}{\omega_1}$$

$$\therefore \Delta X(t) = \frac{b}{\omega_1} e^{-\beta(t-t_0)} \sin \omega_1(t-t_0) \quad \text{for } t > t_0$$

$$= 0 \quad t < t_0 \quad \dots \textcircled{28}$$

for an impulse delivered at $t = t_0$

而 $X(t) = X_c(t) + \Delta X(t)$, 其中 $X_c(t)$ 為沒有 $F \Delta t$ 之 impulse 的解。若指

現在 $b = \frac{F(t_0)}{m} \Delta t$

定 $X(t_0) = X_0$, $\dot{X}(t_0) = \dot{X}_0$, 則 $X_c(t) = e^{-\beta(t-t_0)} \left(X_0 \cos \omega_1(t-t_0) + \frac{\dot{X}_0 + \beta X_0}{\omega_1} \sin \omega_1(t-t_0) \right)$

$$\therefore \Delta X_n(t) = \frac{F(t_n)}{m \omega_1} \Delta t e^{-\beta(t-t_n)} \sin \omega_1(t-t_n)$$

$$X(t) = \sum_{n=-\infty}^{\infty} \frac{F(t_n)}{m \omega_1} \Delta t e^{-\beta(t-t_n)} \sin \omega_1(t-t_n)$$

$$t = t \rightarrow \int_{-\infty}^t dt' \frac{F(t')}{m \omega_1} e^{-\beta(t-t')} \sin \omega_1(t-t') \quad \dots \textcircled{29}$$

(設 $X(t_0) = 0$, $\therefore \sum_n \Delta X_n = X(t)$)

若定義 $G(t, t') = \frac{1}{m\omega_1} \begin{cases} e^{-\beta(t-t')} \sin \omega_1(t-t') & t > t' \\ 0 & t < t' \end{cases}$

則 (29) 式 可以被寫成

$$X(t) = \int_{-\infty}^{\infty} dt' G(t, t') F(t')$$

$$= \int_{-\infty}^t dt' G(t, t') F(t') \quad \dots (30)$$

描述了在 $t' = -\infty$ 時 $X(t') = 0$ 之 $X(t)$ 對 一般力 $F(t)$ 的反應

$G(t, t')$ 與 $F(t)$ 無關, 一般稱為 Green's function (格林函數)

而 $G = 0$ for $t < t'$ 代表了 causality (因果律), 即反應 $X(t)$ 發生在施力之後!

以上的作用對 一般 線性, inhomogeneous 方程式 $\hat{L} X(t) = F(t)$

皆適用, 基本上, 我們在找 \hat{L}^{-1} , 使得

$$X(t) = \hat{L}^{-1} F(t)$$

只是: \hat{L}^{-1} 的較複雜性, $\hat{L}^{-1} F(t) = \int_{-\infty}^{\infty} dt' G(t, t') F(t')$

此方法一般稱為 Green's method

例. $X(0)=0$, $\dot{X}(0)=0$ 求 $F(t) = F_0 e^{\gamma t}$, $t > 0$ 後
 $\begin{matrix} \text{初値} \\ \text{後} \end{matrix} = 0 \quad t \leq 0$

$\dot{X}(t)$

由格林函数法可知

$$X(t) = \frac{F_0}{m\omega_1} \int_0^t e^{-\gamma t'} e^{-\beta(t-t')} \sin\omega_1(t-t') dt'$$

set $z = \omega_1(t-t')$

$$= -\frac{F_0}{m\omega_1^2} \int_{\omega_1 t}^0 dz \sin z e^{\gamma(t-\frac{z}{\omega_1})} e^{-\beta\frac{z}{\omega_1}}$$

$$= \frac{F_0 e^{-\gamma t}}{m\omega_1^2} \int_0^{\omega_1 t} dz \sin z e^{\frac{\gamma-\beta}{\omega_1} z}$$

$$\frac{1}{2i} (e^{i z} - e^{-i z})$$

$$= \frac{F_0}{m\omega_1^2} e^{-\gamma t} \frac{1}{2i} \left[\frac{1}{\frac{\gamma-\beta}{\omega_1} + i} (e^{\frac{\gamma-\beta}{\omega_1} \omega_1 t} e^{i \omega_1 t} - 1) \right.$$

$$\left. - \frac{1}{\frac{\gamma-\beta}{\omega_1} - i} (e^{\frac{\gamma-\beta}{\omega_1} \omega_1 t} e^{-i \omega_1 t} - 1) \right]$$

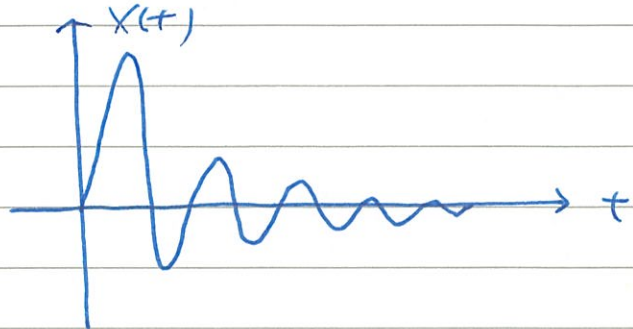
$$= \frac{F_0}{m\omega_1^2} e^{-\gamma t} \left[\frac{1}{1 + \left(\frac{\gamma-\beta}{\omega_1}\right)^2} + \frac{e^{(\gamma-\beta)t}}{2i} \frac{\left(\frac{\gamma-\beta}{\omega_1} - i\right) e^{i \omega_1 t} - \left(\frac{\gamma-\beta}{\omega_1} + i\right) e^{-i \omega_1 t}}{1 + \left(\frac{\gamma-\beta}{\omega_1}\right)^2} \right]$$

$$= \frac{F_0/m}{(\gamma-\beta)^2 + \omega_1^2} \left[e^{-\gamma t} - e^{-\beta t} (\cos\omega_1 t - \frac{\gamma-\beta}{\omega_1} \sin\omega_1 t) \right]$$

$X(t)$ 之行為大致可依 $\beta < \gamma$, $\beta = \gamma$ 及 $\beta > \gamma$ 區分

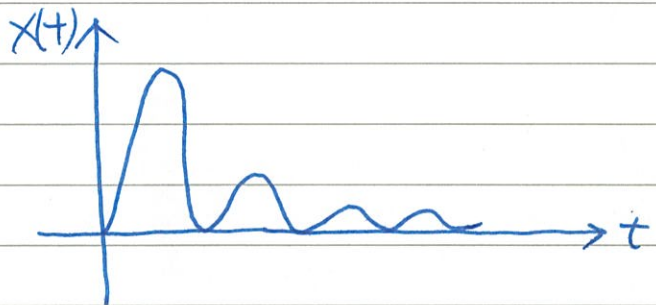
$\beta < \gamma$ 時, $F(t)$ 衰減很快, 系統可

振盪.

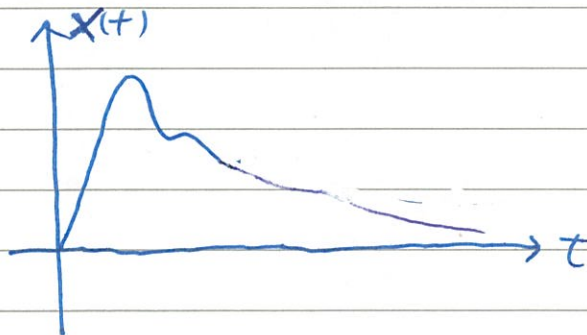


$$\beta = \gamma \quad X(t) = \frac{F_0}{m\omega^2} e^{-\beta t} (1 - \cos \omega t) > 0$$

$$T = \frac{2\pi}{\omega}$$



$\beta > \gamma$, 系統被 $F(t)$ 帶著衰減, 故無法振盪



注意 $\gamma \rightarrow 0$ 時 $F(t)$ 為 F_0 謂

的 step function. 此時, β 總是 $> \gamma$

$$X(t) \rightarrow \frac{F_0 m}{\omega^2 + \beta^2} \left[1 - e^{-\beta t} \left(\cos \omega t + \frac{\beta}{\omega} \sin \omega t \right) \right]$$

$$= \frac{F_0}{m\omega^2} \left[1 - e^{-\beta t} \left(\cos \omega t + \frac{\beta}{\omega} \sin \omega t \right) \right] \rightarrow \frac{F_0}{m\omega^2} \text{ as } t \rightarrow \infty$$

以上是 $x(0)$ 及 $\dot{x}(0) = 0$ 之解。

如果 $\dot{x}(0) = v_0$, $x(0) = x_0 \neq 0$, 則

解中須修正。

首先之前所得解應為 $\Delta x(t)$ (為 $F(t)$ 之

x 的改變), 即 $x_p(t)$ 但 $x_p(0)$ 及 $\dot{x}_p(0)$ 皆為 0!

因此, 一般起始條件之解應為

$$x(t) = x_p(t) + A e^{-\beta t} \cos \omega_1 t + B e^{-\beta t} \sin \omega_1 t$$

set $t=0$ 可得 ($x_p(0)=0$, $\dot{x}_p(0)=0$)

$$x_0 = A$$

$$v_0 = -\beta A + \omega_1 B \quad \therefore B = \frac{x_0 \beta + v_0}{\omega_1}$$

$$\therefore x(t) = e^{-\beta t} \left(x_0 \cos \omega_1 t + \frac{x_0 \beta + v_0}{\omega_1} \sin \omega_1 t \right)$$

$$+ \frac{F_0/m}{(\delta - \beta)^2 + \omega_1^2} \left[e^{-\delta t} - e^{-\beta t} \left(\cos \omega_1 t - \frac{\delta - \beta}{\omega_1} \sin \omega_1 t \right) \right]$$

例: $F(t) = F_0 e^{i\omega t}$

$$x(t) = \frac{F_0}{m\omega_1} \int_{-\infty}^t e^{i\omega t'} e^{-\beta(t-t')} \sin \omega_1(t-t') dt'$$

$$\int_{-\infty}^t x = t - t'$$

$$x(t) = \frac{F_0}{m\omega_1} \int_{-\infty}^0 -dx \sin \omega_1 x e^{-\beta x} e^{i\omega(t-x)}$$

$$= \frac{F_0}{m\omega_1} e^{i\omega t} \int_0^{\infty} dx e^{-(\beta+i\omega)x} \frac{1}{2i} (e^{i\omega_1 x} - e^{-i\omega_1 x})$$

$$\text{利用 [7]} \int_0^{\infty} dx e^{\alpha x} = \frac{1}{\alpha}$$

$$\text{可得 } X(t) = \frac{F_0}{2im\omega_1} e^{i\omega t} \left(\frac{1}{\beta+i\omega-i\omega_1} - \frac{1}{\beta+i\omega+i\omega_1} \right)$$

$$= \frac{F_0}{2im\omega_1} \times \frac{2i\omega_1}{(\beta+i\omega)^2 + \omega_1^2} e^{i\omega t}$$

$$\because \omega_1^2 = \omega_0^2 - \beta^2$$

$$\begin{aligned} \therefore (\beta+i\omega)^2 + \omega_1^2 &= \beta^2 - \omega^2 + \omega_1^2 + 2i\beta\omega \\ &= \omega_0^2 - \omega^2 + 2i\beta\omega \end{aligned}$$

$$\therefore X(t) = \frac{F_0}{\omega_0^2 - \omega^2 + 2i\beta\omega} e^{i\omega t} \quad \text{与之前的解一致}$$

補充: $G(t, t')$ 滿足

$$\left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega^2\right) G(t, t') = \frac{\delta(t-t')}{m} \quad (\text{即 } F(t) = \delta(t-t'))$$

其中 $\delta(t-t')$ 是所謂的 Dirac, delta function:

$$\begin{aligned} \delta(t-t') &= \infty & t=t' \\ &= 0 & \text{otherwise} \end{aligned}$$

$$\text{且 } \int dt \delta(t-t') = 1$$

首先, 我們可以用 $G(t, t')$ 寫為

$$G(t, t') = \theta(t-t') \frac{1}{m\omega} e^{-\beta(t-t')} \sin \omega(t-t') \equiv \theta(t-t') X_c(t-t')$$

$$\text{其 } \theta(t-t') \text{ 為 step function, } \theta(t-t') = \begin{cases} 1 & t > t' \\ 0 & t < t' \end{cases}$$

$$\therefore \frac{d}{dt} \theta(t-t') = \delta(t-t')$$

$$\therefore 2\beta \frac{d}{dt} G(t, t') = 2\beta \delta(t-t') X_c(t-t') + 2\beta \theta(t-t') \frac{d}{dt} X_c$$

$$\begin{aligned} \text{同理 } \frac{d^2}{dt^2} G(t, t') &= \delta'(t-t') X_c + 2\delta(t-t') \frac{dX_c}{dt} \\ &\quad + \theta(t-t') \frac{d^2 X_c}{dt^2} \end{aligned}$$

$$\begin{aligned} \therefore \left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega^2\right) G(t, t') &= \delta'(t-t') X_c + 2\delta(t-t') \frac{dX_c}{dt} \\ &\quad + \theta(t-t') \left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega^2\right) X_c \end{aligned}$$

$$\therefore \left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega^2\right) X_c = 0, \text{ 且 } X_c(t-t')|_{t=t'} = 0$$

$$\frac{d}{dt} X_c(t-t') \Big|_{t=t'} = \frac{1}{m}$$

$$\begin{aligned} \therefore \int \delta'(t-t') x_c(t-t') dt &= \delta(t-t') x_c(t-t') \Big|_{-\infty}^{\infty} \\ &\quad - \int \delta(t-t') \frac{dx_c}{dt} dt \end{aligned}$$

$$\therefore \delta'(t-t') x_c = - \delta(t-t') \frac{dx_c}{dt} \rightarrow -\frac{1}{m} \delta(t-t')$$

$$\therefore \left(\frac{d^2}{dt^2} + 2\beta \frac{d}{dt} + \omega_0^2 \right) G(t-t')$$

$$= -\frac{1}{m} \delta(t-t') + \frac{2}{m} \delta(t-t') = \frac{1}{m} \delta(t-t')$$

* 非線性振盪与 Chaotic motion

Deterministic view

微分方程式最重要的定理之一為 "unique theorem".
 對解牛頓運動方程式而言, 是指一旦 $x(0)$ 及 $\dot{x}(0)$
 給定 $x(t)$ 即為唯一之決定!

因此, 一定在 $t=0$ 時, 我們知道宇宙之^{質點} $x(0)$ 及
 $\dot{x}(0)$, 則宇宙之命運即已決定了, 這就是

所謂的 deterministic view of nature, 最
 先由著名的法國數學家^{Laplace} 提出的觀念。

Brief history of chaos (混沌)

以上的觀念, 之後受到嚴厲的考驗,
 特別是 Chaos 現象之發現。

Chaos^{現象} 是由 Jacques Hadamard 在 1898 年
 首先發現, 他發現^{雖然} 粒子之起始條件唯

一決定之後的 $x(t)$, 但^{雖然} 小小的不確定可以

導致^{之後} $\Delta x(t)$ 很大的差別: $|\Delta x(t)| \sim e^{\lambda t} |\Delta x(0)|$
 $\lambda = \text{Lyapunov exponent} \geq 0$

Chaos 理論後來經 Henri Poincaré (研究 3 體
 運動) 及 20 世紀中快速的發展 (Edward Lorenz,

1961), 到現在已經是確認的觀念: sensitivity to
 initial conditions may lead to unpredictability of $x(t)$!

其名稱為數學家 James A. Yorke 取的。

主要造成 chaos 現象是 "non-linearity"

∴ 如果是 linear 的系統，則依據微分_分程式

之理論

$$\Delta X(t) \sim \Delta C_1 X_1(t) + \Delta C_2 X_2(t)$$

$$\Delta X(t) = e^{\lambda t} \Delta X(0)$$

$$(\text{如 } X(t) = C_1 \cos \omega t + C_2 \sin \omega t, \Delta X(t) \text{ 是有限時!})$$

∴ 非線性才是主因。

亂流現象 (turbulence)

Lorentz 的工作 主要在研究大氣^(流体)的現象。

其後在 1970 後期，許多的研究人員開始將

流体中 random 的現象與 chaos 聯繫，

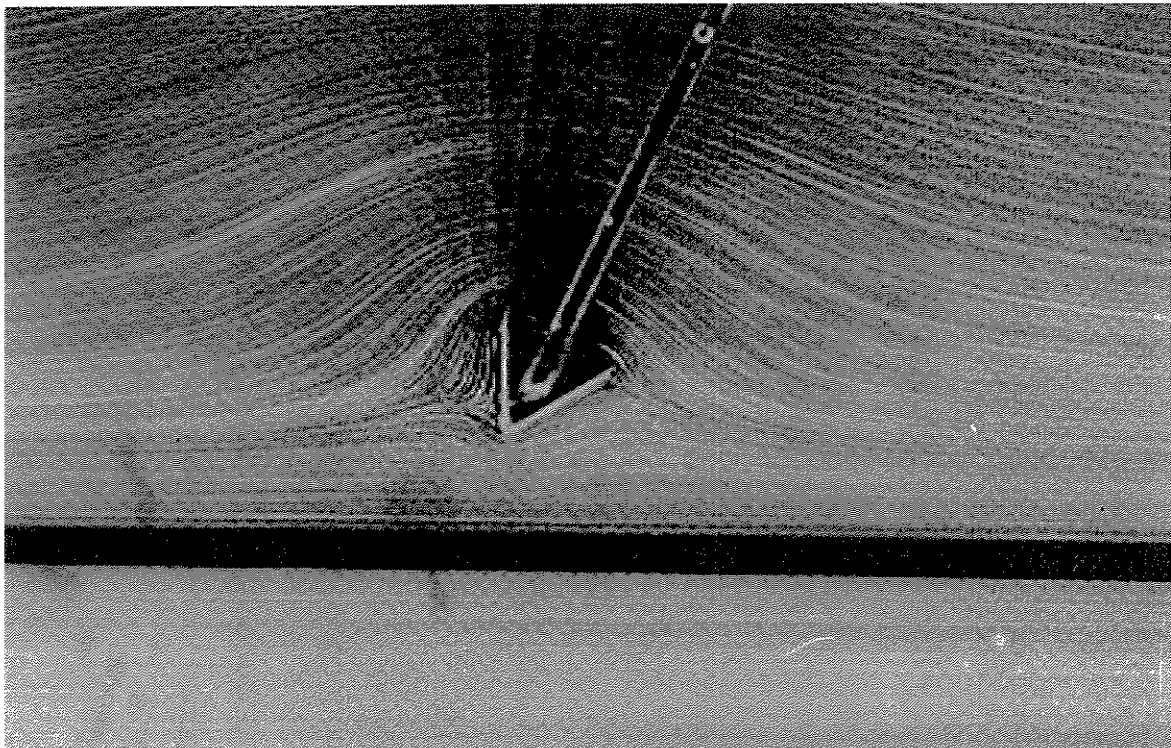
雖然這中間的聯繫未完整的建立，

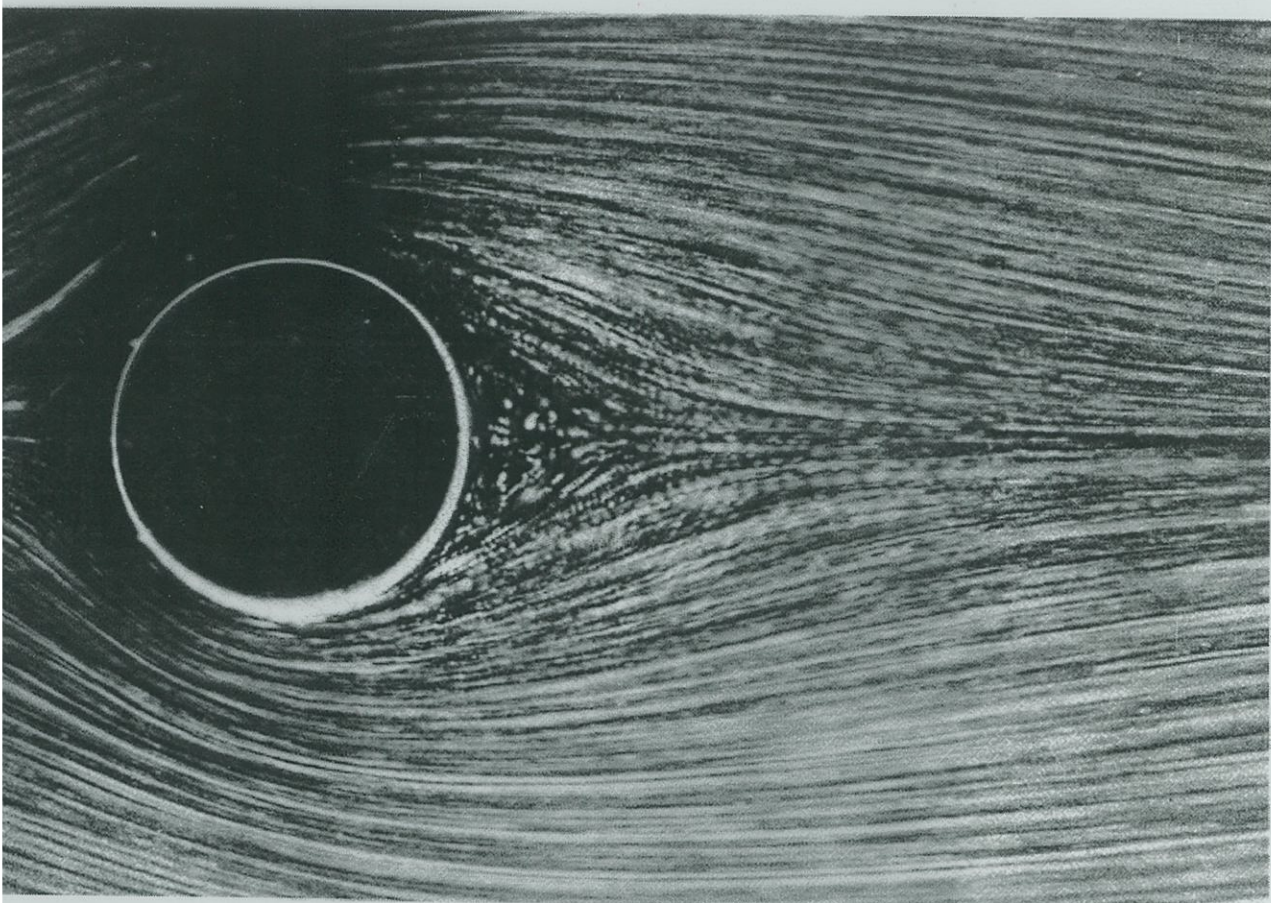
一般相信，turbulence ^{也是} 非線性

及對 initial conditions sensitive 的結果，

只是這時有空間的 pattern 牽涉其中。

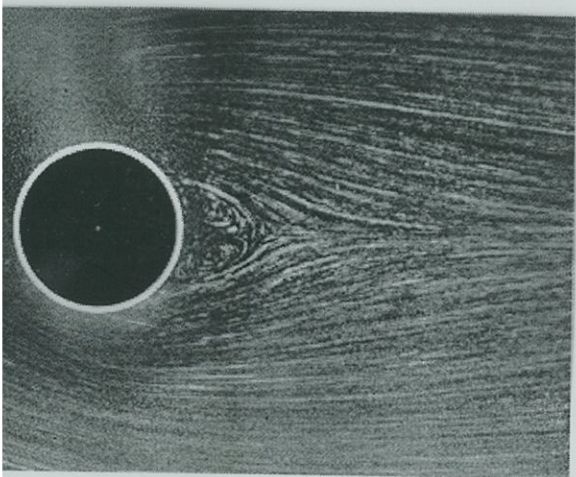
Fig. 3-1



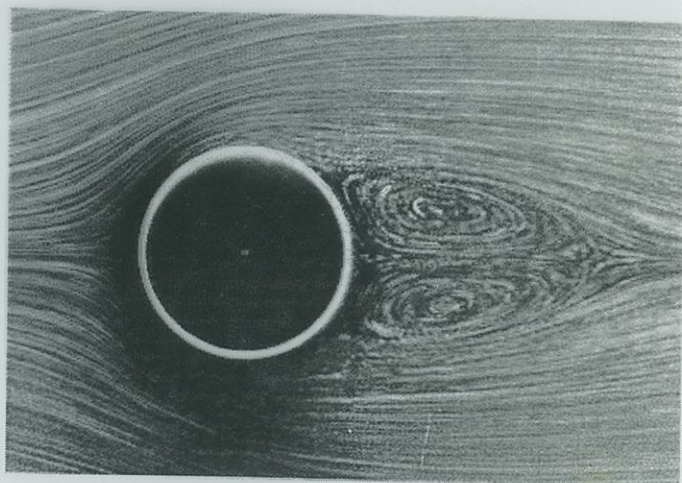


Circular cylinder at $R=9.6$. Here, in contrast to $R=24$, the flow has clearly separated to form a pair of rotating eddies. The cylinder is moving through a tank of water containing aluminum powder, and is illuminated

by a sheet of light below the free surface. Extrapolation of such experiments to unbounded flow suggests separation at $R=4$ or 5, whereas most numerical computations give $R=5$ to 7. Photograph by Sadatoshi Taneda

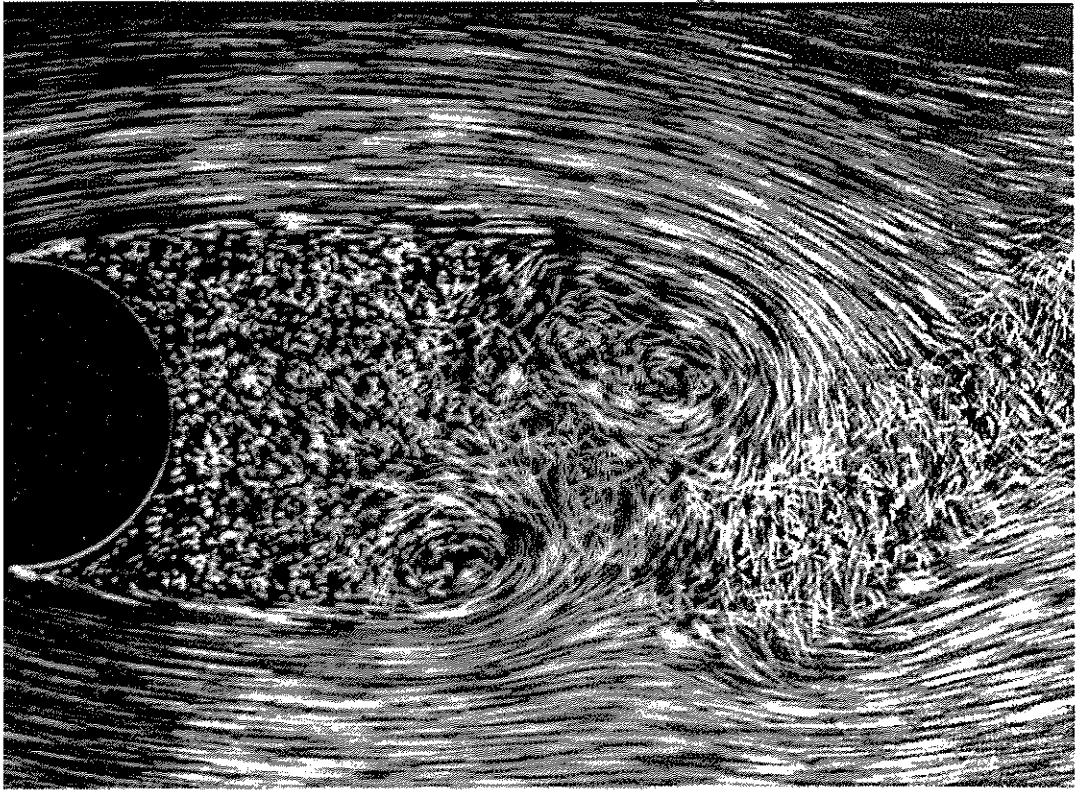


Circular cylinder at $R=13.1$. The standing eddies are elongated in the flow direction as the speed increases. Their length is found to increase linearly with Reynolds number until the flow becomes unstable above $R=13.1$. Photograph by Sadatoshi Taneda, 1956a



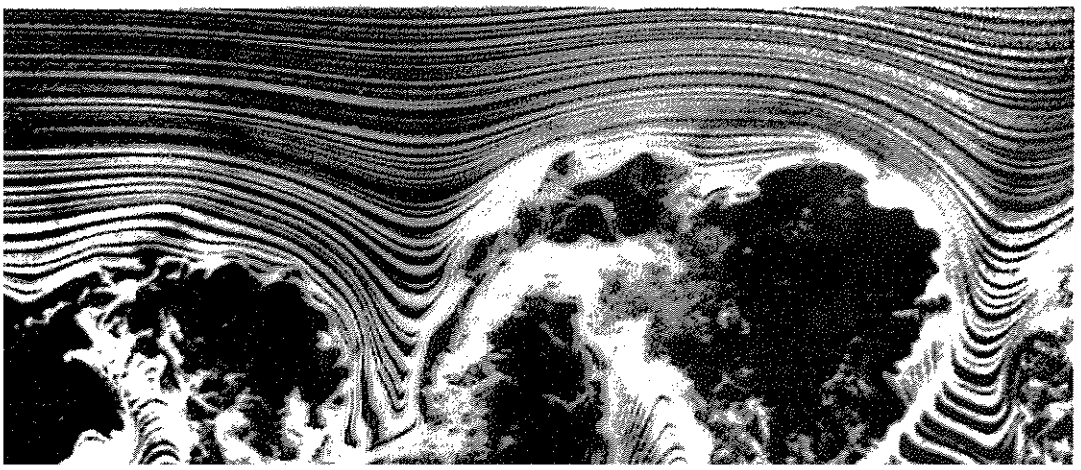
42. Circular cylinder at $R=26$. The downstream distance to the cores of the eddies also increases linearly with Reynolds number. However, the lateral distance between the cores appears to grow more nearly as the square root. Photograph by Sadatoshi Taneda, 1956a

Fig 33



$Re = 2000$. At this Reynolds number, the flow is at the limit of a boundary layer. It is laminar, and breaks up into a series of vortices, moving forward as

the Reynolds number is increased, have now attained their upstream limit, ahead of maximum thickness. Visualization is by air bubbles in water. ONERA photograph, Werlé & Gallon 1977



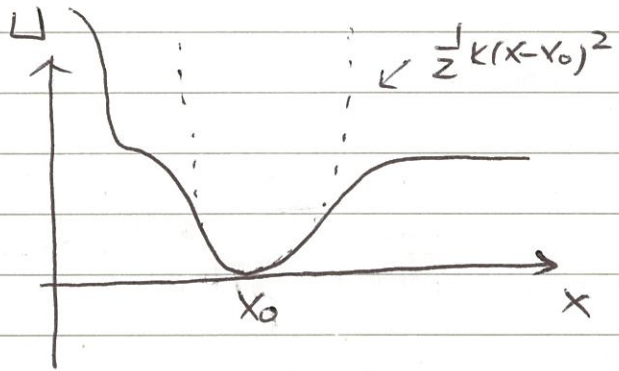
non-linear oscillations

典型造成 non-linear 之原因為為 anharmonics

如圖所示, 當 $x - x_0$

不小時, $U(x)$

$$= U(x_0) + \frac{1}{2} U''(x_0) (x - x_0)^2 + \frac{1}{3!} U'''(x_0) (x - x_0)^3 + \dots$$



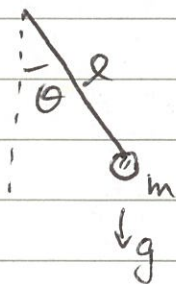
之高次項不可再被忽略, 故 $m\ddot{x} = -kx - \beta x^2 + \dots$

i.e. $m\ddot{x} = F(x) - k(x)x$

$k(x)$ 與 x 有關。

例一 $k(x) = k(1 + \beta x^2)$, 此為 Duffing equation.

例二. Plane pendulum



$$ml^2 \ddot{\theta} = -mg \sin \theta l$$

$$\ddot{\theta} + \omega_0^2 \sin \theta = 0 \quad \omega_0^2 = g/l$$

只有 $\theta \rightarrow 0$, $\ddot{\theta} + \omega_0^2 \theta = 0$ 才約成立。

nature of multi-harmonics

非線性微分方程式一般沒有固定的解法,

在一維時, 可以由能量守恆簡化而得解 (integrable).
 能量守恆

但有 inhomogeneous 項，則有完全不同之行為。

$$\text{以 } \ddot{x} = -\omega_0^2 x + \epsilon x^3 + f \cos \omega t \text{ 為例} \dots (31)$$

$\epsilon=0$ 時，我們期待 x_0 也以 ω 振盪。

但 $\epsilon \neq 0$ 時，則有所謂的 harmonics 產生。

如我可以試 $x_1 = A \cos \omega t$ 代入 (31) 式之右手邊，

$$\begin{aligned} \text{則 } \ddot{x}_2 &= -\omega_0^2 A \cos \omega t + \epsilon A^3 \cos^3 \omega t + f \cos \omega t \\ &= -(A\omega_0^2 - \frac{3}{4}\epsilon A^3 - f) \cos \omega t + \frac{1}{4}\epsilon A^3 \cos 3\omega t \end{aligned}$$

$$\cos^3 \omega t = \frac{3}{4} \cos \omega t + \frac{1}{4} \cos 3\omega t$$

可見 $\cos 3\omega t$ (higher harmonics) 被產生了！
→ 指 \cos, \sin 之高次 ω 項

因此， x 不會是單純的 $\cos \omega t$ 形態！

非線性 ^{可能} 會產生 3ω (或 2ω) ...

↑
倍頻

註：倍頻或多頻之產生並非一定，如算三次作業 (3-f) cycloid

Plane Pendulum: 即為一例：經過常轉換可成為線性！

↓ 不再與振幅無關 一般非線性振盪仍有固定的 T，只是不是 $\cos \omega t / \sin \omega t$

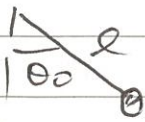
如果沒有推動外力，plane pendulum 雖是 non-linear

仍可以解。設 $t=0, \theta = \theta_0, \dot{\theta} = 0$ (最大值)

見下例

3-39-1

$$\text{則 } E = U = mgl(1 - \cos \theta_0)$$



Perturbation theory:

比較有系統的解 $\ddot{x} + \omega^2 x = \varepsilon x^3 + f \cos \omega t$

是以 ε 為參數，展開：

$$\frac{1}{\varepsilon} \ddot{x} \quad x(t) = x_1(t) + x_2(t)\varepsilon + x_3(t)\varepsilon^2 + \dots$$

代入上式，可得

$$(\ddot{x}_1 + \omega^2 x_1 - f \cos \omega t) + \varepsilon (\ddot{x}_2 + \omega^2 x_2 - x_1^3) + \varepsilon^2 (\dots) + \dots = 0$$

$\because \varepsilon$ 為參數， \therefore 係數 = 0

$$\text{故 } \ddot{x}_1 + \omega^2 x_1 = f \cos \omega t$$

$$x_1 = A \cos \omega t$$

$$A = \frac{f}{\omega_0^2 - \omega^2}$$

$$\ddot{x}_2 + \omega^2 x_2 = x_1^3 = A^3 \cos^3 \omega t$$

$$= \frac{3A^3}{4} \cos \omega t + \frac{A^3}{4} \cos 3\omega t$$

$$\cos^3 \omega t = \frac{3}{4} \cos \omega t + \frac{1}{4} \cos 3\omega t$$

$$\text{故 } x_2 = \alpha \cos \omega t + \beta \cos 3\omega t$$

$$\alpha = \frac{\frac{3}{4} A^3}{\omega^2 - \omega^2}, \quad \beta = \frac{\frac{A^3}{4}}{\omega^2 - (3\omega)^2}$$

如下去...，其中 3ω 之解稱為 3rd harmonic

of ω ... (2ω 則為 second harmonic of ω)

$$\therefore \underbrace{\frac{1}{2} m \dot{\theta}^2}_{\text{動能}} + mgL(1 - \cos\theta) = mg(1 - \cos\theta_0)$$

動能

$$\therefore 1 - \cos\theta = 2 \sin^2 \frac{\theta}{2}$$

$$\therefore \frac{1}{2} m \dot{\theta}^2 = 2mgL \left(\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2} \right)$$

$$\dot{\theta} = 2 \sqrt{\frac{g}{L}} \sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}$$

$$dt = \frac{1}{2} \sqrt{\frac{L}{g}} \frac{d\theta}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}} \quad \dots (32)$$

由 $\theta=0$ 到 $\theta=\theta_0$, 需時 $\frac{T}{4}$

$$\therefore T = 2 \sqrt{\frac{L}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}} \quad \dots (33)$$

一般解只有會積分 (32) 式即可得到。而 (33) 式則給出 T 與 θ_0 之關係。

$$\frac{1}{2} z = \frac{\sin \frac{\theta}{2}}{\sin \frac{\theta_0}{2}} \quad k = \sin \frac{\theta_0}{2}$$

$$dz = \frac{1}{2} \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta_0}{2}} d\theta = \frac{\sqrt{1-k^2 z^2}}{zk} d\theta$$

$$\text{則 } T = 4 \sqrt{\frac{L}{g}} \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2 z^2)}} \quad k^2 < 1 \quad \dots (34)$$

$F(k, x)$

Elliptic Integrals
of the 1st kind

註：許多不能釋出之積分，皆可化成橢圓積分
 以基本函數

之型式 (見 Appendix B)

此時只能以數值分析求解。

若 $|\theta_0| < \pi$, $(\sin \theta_0/2) = |k| < 1 \Rightarrow$ oscillatory motions

$$\therefore \frac{1}{\sqrt{1-k^2 z^2}} = 1 + \frac{k^2 z^2}{2} + \frac{3k^4 z^4}{8} + \dots$$

$$\begin{array}{c} \uparrow \qquad \qquad \uparrow \\ \left(\frac{1}{2}(-k^2 z^2)\right) \quad \left(\frac{3}{2} = \frac{(\frac{1}{2})(\frac{1}{2})}{2!} = \frac{3}{8}\right) \end{array}$$

$$\therefore T = 4\sqrt{\frac{g}{l}} \int_0^1 \frac{dz}{\sqrt{1-z^2}} \left[1 + \frac{k^2 z^2}{2} + \frac{3k^4 z^4}{8} + \dots \right]$$

$$= 4\sqrt{\frac{g}{l}} \left[\frac{\pi}{2} + \frac{k^2}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} + \frac{3k^4}{8} \cdot \frac{3}{8} \cdot \frac{\pi}{2} + \dots \right]$$

↑

--- (35)

$$\int_0^1 \frac{dz}{\sqrt{1-z^2}} = \theta \Big|_0^{\pi/2} = \frac{\pi}{2}$$

$z = \sin \theta$

$$\int_0^1 \frac{z^2 dz}{\sqrt{1-z^2}} = \int_0^1 -z d\sqrt{1-z^2} = -z\sqrt{1-z^2} \Big|_0^1 + \int_0^1 \sqrt{1-z^2} dz = \frac{\pi}{4}$$

$$\left(\text{in general, } \int_0^1 \frac{z^{2n} dz}{\sqrt{1-z^2}} = \int_0^{\pi/2} \sin^{2n} \theta d\theta \right)$$

$z = \sin \theta$

③式只有在 $k = \sin \theta_0/2$ 小. 1. 9. 7. 收斂才好。

$$\text{如 } \theta_0 \text{ 時 } k = \theta_0/2 - \theta_0^3/48$$

$$T = 2\pi \sqrt{\frac{l}{g}} \left(1 + \frac{1}{16} \theta_0^2 + \frac{11}{3072} \theta_0^4 + \dots \right)$$

約與 θ_0 無關。

Phase diagrams

我們可以從 phase diagram 定性了解非線性振盪。

以 plane pendulum 而言,

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + 2mg l \sin^2 \theta/2 = 2mg l \sin^2 \theta/2$$

$$\frac{\dot{\theta}^2}{2(g/l)} + 2 \sin^2 \theta/2 = 2 \sin^2 \theta_0/2$$

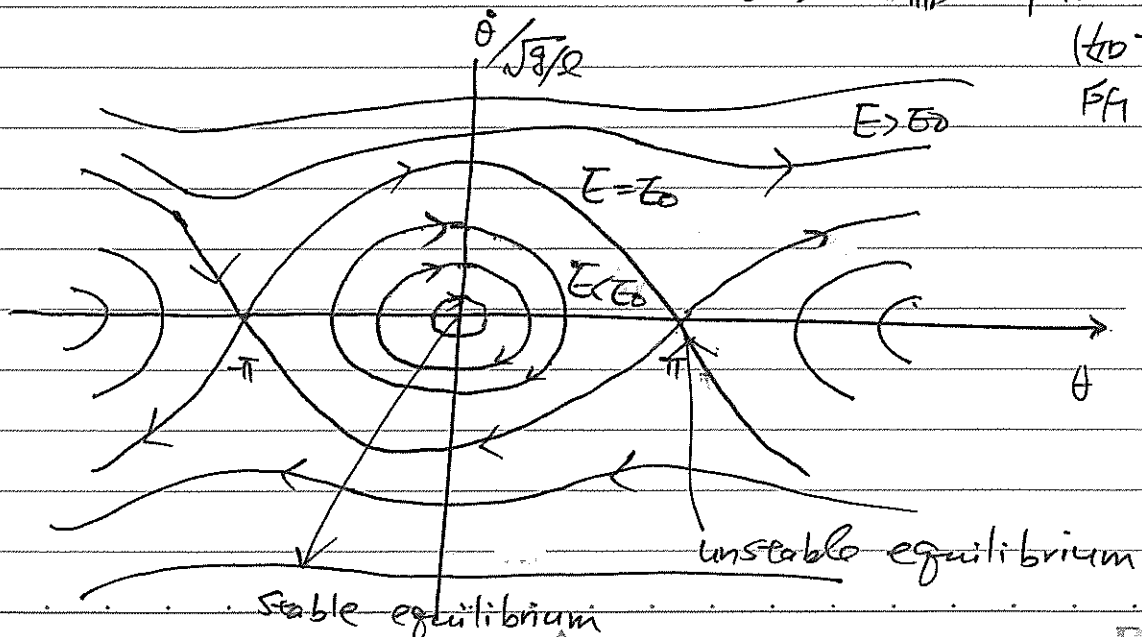
\therefore 對 $E < E_0 = 2mg l$, \Rightarrow bounded motions

$$E > E_0$$

un bounded motions

$E = E_0$ 分開 bounded 的 un bounded 運動, 稱為 separatrix

(如下圖所示)



注意 $E > 2mgL$ 時，在 $\theta = \pi$ 時 $\dot{\theta}^2$ 仍 ^{不為} 0，

代表了像 "rotator" 一樣的運動。

在 plane pendulum 的例子，若有 damping，

則系統最終會趨近 $\dot{\theta} = 0$ ， $\theta = 0$ ($2\pi, \dots$)

即 $\dot{\theta} = 0$ ， $\theta = 0$ 為 stable equilibrium 點 ($\frac{dU}{d\theta} = 0$)

另外， $\theta = \pi$ 也滿足 $\frac{dU}{d\theta} = \sin\theta = 0$ ，但

$\frac{d^2U}{d\theta^2} < 0$ \therefore 為 unstable equilibrium.

除非一開始 $\dot{\theta} = 0$ ， $\theta = \pi$ ，可以維持在 $\dot{\theta} = 0$ ， $\theta = \pi$

上，一般任何小的微擾皆將單擺推向 $\theta = 0$ ， $\dot{\theta} = 0$ 處

limit cycle

除了 stable, unstable 平衡點外，非線性系統常見
的穩定行為 ^為 limit cycle ^{另一}。

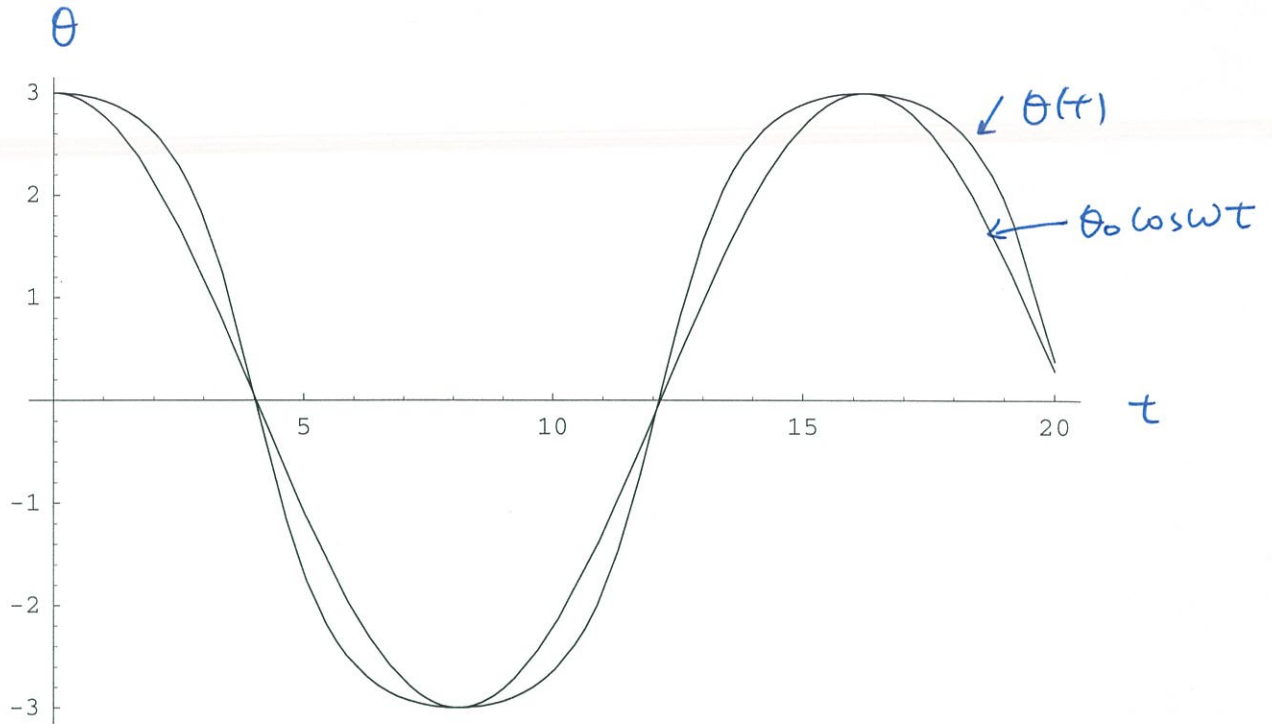
例: Van der Pol equation

$$\ddot{x} + \mu(x^2 - a^2)\dot{x} + \omega_0^2 x = 0 \quad \dots (36)$$

damping 係 x 有關係
^係 係數

即 $|x| > |a|$ 時，damping 係數 > 0

$\therefore x$ 與 \dot{x} 皆傾向變小。



Plane pendulum: $\theta_0 = 3$. 由上圖可知

$$\omega_0 = 1$$

$$\omega = \frac{2\pi}{T}$$

$\theta(t)$ 並不是單純的 $\cos \omega t$!

以 Fourier series 角度來說

$$\theta(t) = \sum_n a_n \cos n \omega t, \text{ 有高次 } 2\omega, 3\omega, 4\omega \text{ 之貢獻。}$$

反之, $|x| < |a|$ 時, damping coefficient < 0 .

為負阻現, x 與 \dot{x} 反而增加!

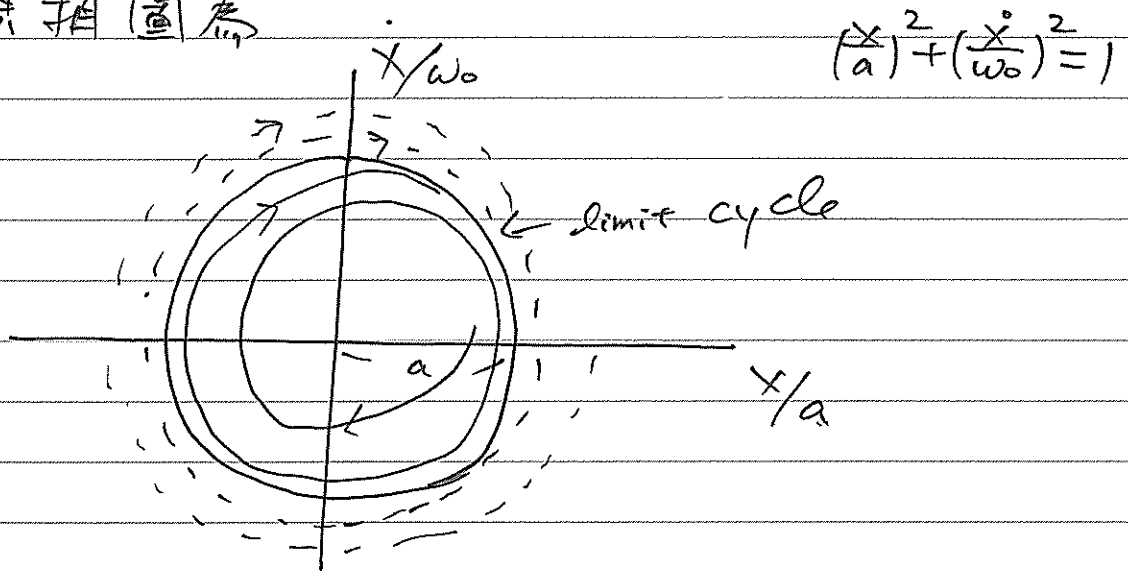
只有 $|x| = |a|$ 時, $\ddot{x} + \omega_0^2 x = 0$, $\frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 = E$

此時系統能量固定, 不隨時間而變

$$= \frac{1}{2} k a^2$$



因此, 其相圖為



所以, 除了 stable, unstable 穩定與外, } 一般而言,
 limit-cycle 也是一個 穩定的軌道 } 系統收斂
 的集合
 稱為 attractor

註: 要教值解 (36) 或 (31) 式, 現在已有
 許的套裝軟體可以使用。

例如 mathematica.

以下是以 mathematica 解 (36) 式的
 使用參考。

Define the Differential Equation

Name the equation "diffeq". Note the "==" in defining the equation.

```
diffeq := x''[t] +  $\alpha$  (x[t] * x[t] - 1) x'[t] +  $\omega_0^2$  * x[t] == 0
```

Set the parameters:

```
tmin = 0; tmax = 10000;
```

```
 $\alpha$  = 0.05;  $\omega_0$  = 1;
```

Choose among the initial conditions: a=theta(0)

```
xdot0 = 0.0;
```

```
x0 = 3.0
```

```
3.
```

Numerically solve the differential equation using NDSolve, specifying the initial conditions. Setting MaxSteps to a large number is needed if tmax is large.

```
solution = NDSolve[{diffeq, x[0] == x0, x'[0] == xdot0},  
  x, {t, tmin, tmax}, MaxSteps -> 200000];
```

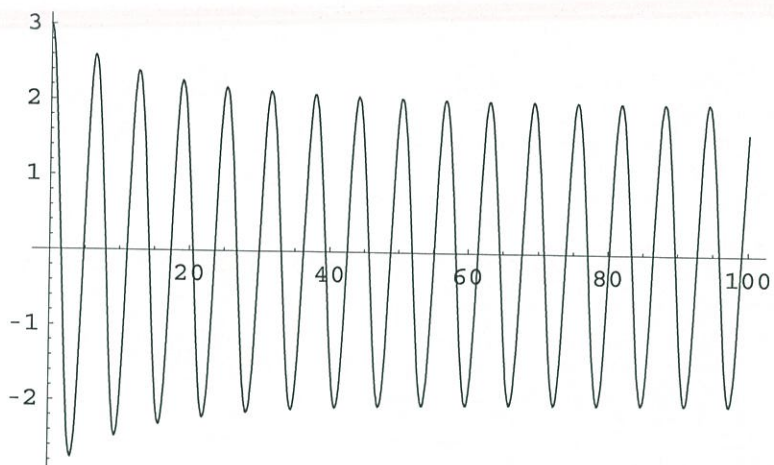
NDSolve returns an "interpolating function", which can be evaluated later at any time t in $t_{min} < t < t_{max}$.

Plot the Time Dependence and Phase Space

We can just use "Plot" with Evaluate and the "interpolating function" defined by "solution".

3-46

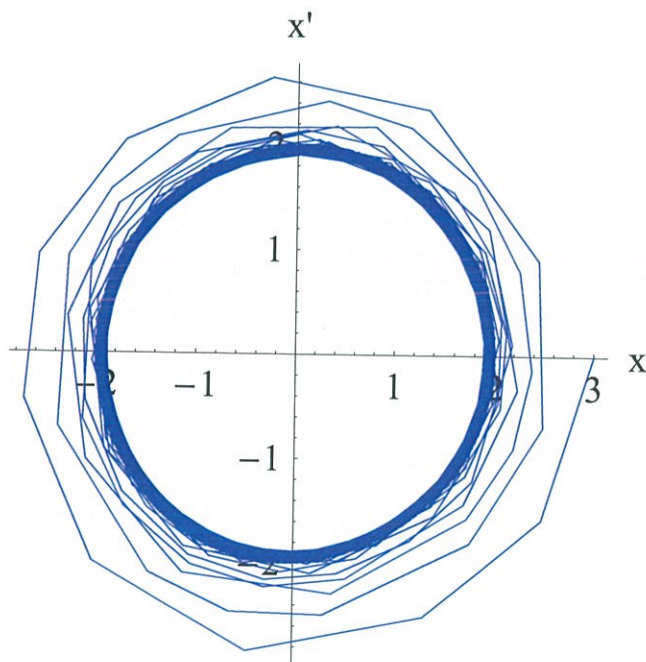
```
Plot[Evaluate[x[t] /. solution], {t, 0, 100}]
```



- Graphics -

Now do the phase space plot. You may need to change AxesOrigin to get the axes in a reasonable place.

```
ParametricPlot[Evaluate[{x[t], x'[t]} /. solution],
  {t, 0, 500}, PlotStyle -> Hue[.6], AspectRatio -> 1,
  TextStyle -> {FontFamily -> "Times", FontSize -> 14},
  ImageSize -> {250, 250}, AxesOrigin -> {0, 0}, AxesLabel -> {"x", "x'"}]
```



- Graphics -

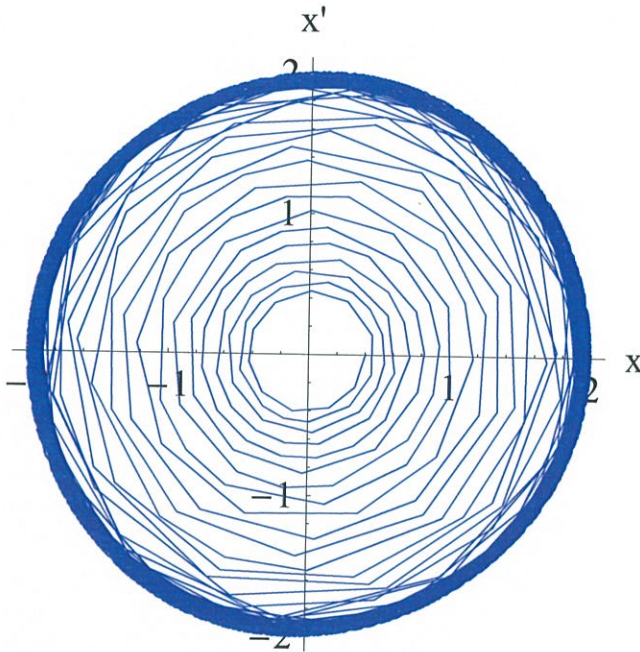
$x_0 = 0.4$

0.4

3-47

```
solution = NDSolve[{diffeq, x[0] == x0, x'[0] == xdot0},  
  x, {t, tmin, tmax}, MaxSteps -> 200000];
```

```
ParametricPlot[Evaluate[{x[t], x'[t]} /. solution],  
  {t, 0, 500}, PlotStyle -> Hue[.6], AspectRatio -> 1,  
  TextStyle -> {FontFamily -> "Times", FontSize -> 14},  
  ImageSize -> {250, 250}, AxesOrigin -> {0, 0}, AxesLabel -> {"x", "x'"}]
```



- Graphics -

* Chaos in pendulum

有許多 SS pendulum 的運動是 chaotic.

(forced, double, coupled, magnetic

見課本之 Fig. 4-17)

在此, 我們以 forced plane pendulum 為例.

$$I \ddot{\theta} = -b \dot{\theta} - mgL \sin \theta + Fa \cos \omega t, \quad I = ml^2$$

$$\therefore \ddot{\theta} = -\frac{b}{ml^2} \dot{\theta} - \frac{g}{l} \sin \theta + \frac{Fa}{ml^2} \cos \omega t \quad \text{--- (37)}$$

為了能以數值處理, 先將其變成沒有

單位: $x = \theta$

$$t' = \frac{t}{t_0} = \sqrt{\frac{g}{l}} t, \quad t_0 = \sqrt{l/g}$$

$$\therefore \ddot{\theta} = \frac{d^2 x}{dt'^2} \cdot \left(\frac{t'}{t}\right)^2 = \frac{g}{l} \ddot{x}$$

$$\ddot{\theta} = \sqrt{\frac{g}{l}} \ddot{x}$$

$$\text{(37) 式} \Rightarrow \ddot{x} = -c \dot{x} - \sin x + F \cos \omega t' \quad \text{--- (3P)}$$

$$\text{其中 } c = \frac{b}{ml^2} \sqrt{\frac{l}{g}}, \quad F = \frac{Fa}{ml^2} \frac{l}{g} = \frac{Fa}{mgL}$$

$$\omega = \omega a, \quad \sqrt{\frac{g}{l}}$$

(3P) 式可以以 mathematica 解, 當 F 為某些值

或範圍時, 相圖中每一處都被訪問過 (見 Fig. 4-19).

Define the Differential Equation

Name the equation "diffeq". Note the "==" in defining the equation.

```
diffeq :=  $\theta''[t] + \alpha \theta'[t] + \omega_0^2 \sin[\theta[t]] == fe * \cos[\omega_1 * t + \phi_e]$ 
```

Set the parameters:

```
tmin = 0; tmax = 1000;
```

```
 $\alpha = 0.05; \omega_0 = 1; \omega_1 = 0.7; \phi_e = 0.0;$ 
```

```
fe = 0.2;
```

Choose among the initial conditions: a=theta(0)

```
 $\theta_{dot}0 = 0.0;$ 
```

```
a = -0.3
```

```
-0.3
```

Numerically solve the differential equation using NDSolve, specifying the initial conditions. Setting MaxSteps to a large number is needed if tmax is large.

```
solution = NDSolve[{diffeq,  $\theta[0] == a, \theta'[0] == \theta_{dot}0$ },  
 $\theta, \{t, tmin, tmax\}, MaxSteps \rightarrow 20000];$ 
```

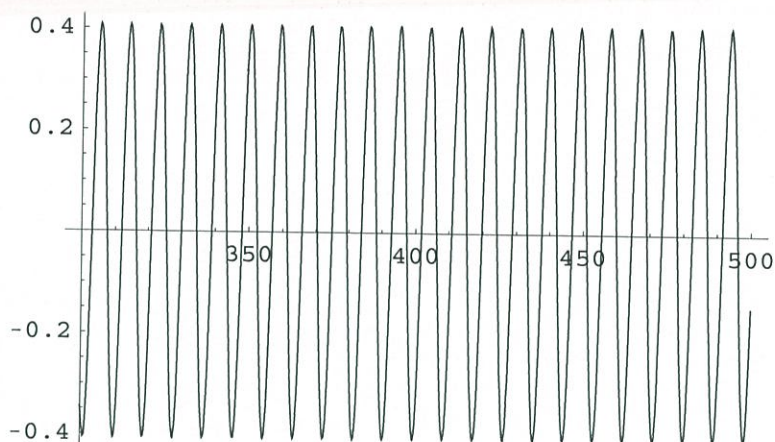
NDSolve returns an "interpolating function", which can be evaluated later at any time t in $tmin < t < tmax$.

Plot the Time Dependence and Phase Space

We can just use "Plot" with Evaluate and the "interpolating function" defined by "solution".

3-50

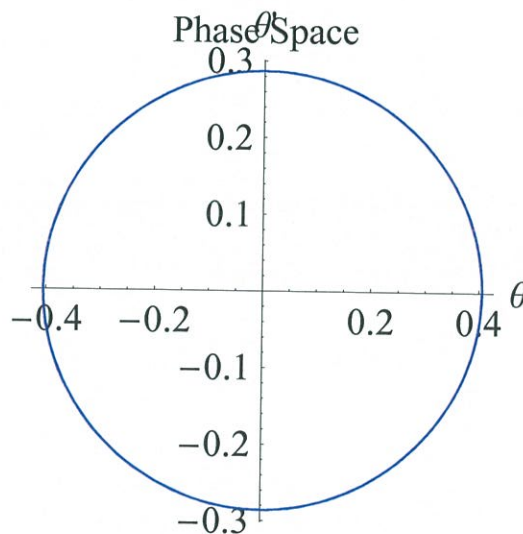
```
Plot[Evaluate[ $\theta[t]$  /. solution], {t, 300, 500}]
```



- Graphics -

Now do the phase space plot. You may need to change AxesOrigin to get the axes in a reasonable place.

```
ParametricPlot[Evaluate[ $\{\theta[t], \theta'[t]\}$  /. solution],
  {t, 300, 500}, PlotStyle -> Hue[.6], AspectRatio -> 1,
  TextStyle -> {FontFamily -> "Times", FontSize -> 14},
  ImageSize -> {250, 250}, PlotLabel -> StyleForm["Phase Space"],
  AxesOrigin -> {0, 0}, AxesLabel -> {" $\theta$ ", " $\theta'$ "}]
```



- Graphics -

3-51

Poincare Sections

The idea of a Poincare section is to plot a point in phase space once every period of the external force, $2\pi/(\text{external frequency})$. The resulting pattern gives information about the periodicity of the signal (or indicates chaos). Start the plot at a large enough time t ("tstart") so that the transients have died out.

Set the external period and how many periods we'll consider. Define a function `timeperiod[i]` giving the corresponding time as a function of the period number.

```
Textexternal := 2 Pi /  $\omega$ 1;

tstart = 10 * Textexternal;

numperiods = 50;

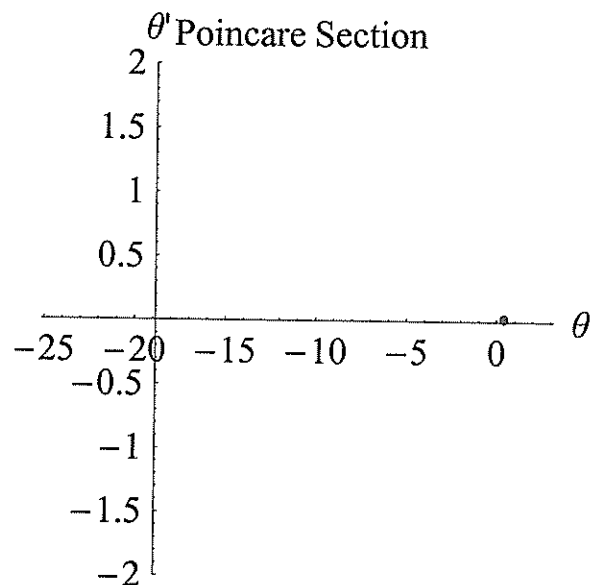
timeperiod[i_] := tstart + i * Textexternal
```

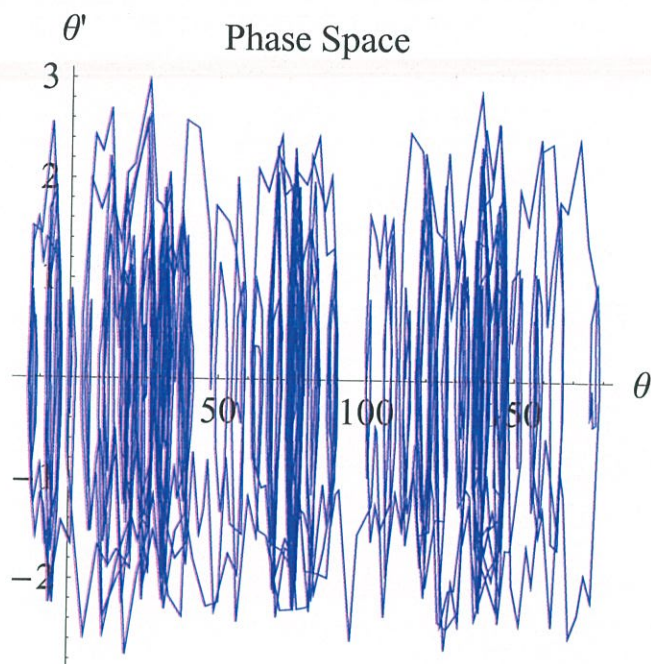
Just evaluate at the relevant points. `Flatten[expr,1]` strips off a layer of `{}`'s.

```
PoincarePts = Flatten[Table[
  Evaluate[{ $\theta$ [timeperiod[i]],  $\theta'$ [timeperiod[i]]} /. solution],
  {i, 10, numperiods}], 1];
```

`ListPlot` plots pairs of numbers in the `PlotRange`.

```
ListPlot[PoincarePts, AspectRatio  $\rightarrow$  1,
  TextStyle  $\rightarrow$  {FontFamily  $\rightarrow$  "Times", FontSize  $\rightarrow$  14},
  ImageSize  $\rightarrow$  {300, 300},
  PlotRange  $\rightarrow$  {{-8 * Pi, Pi}, {-2, 2}},
  PlotLabel  $\rightarrow$  StyleForm["Poincare Section"],
  AxesOrigin  $\rightarrow$  {-6 * Pi, 0}, AxesLabel  $\rightarrow$  {" $\theta$ ", " $\theta'$ "},
  PlotStyle  $\rightarrow$  {PointSize[0.015], RGBColor[1, 0, 0]}]
```





$$f_e = 0.6$$

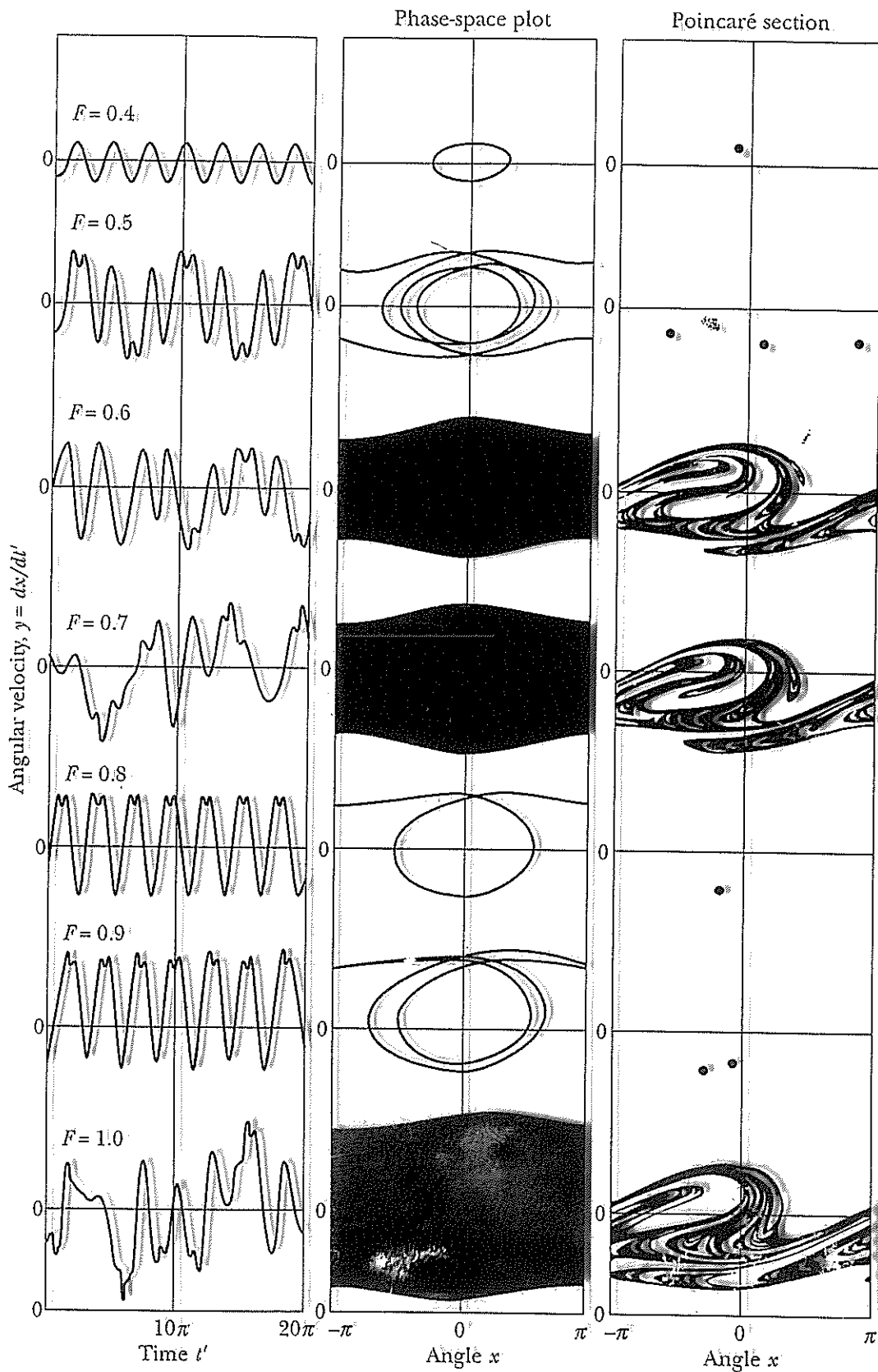


FIGURE 4.19 The damped and driven pendulum for various values of the driving force strength. The angular velocity versus time is shown on the left, and phase diagrams are in the center. Poincaré sections are shown on the right. Note that motion is chaotic for the driving force F values of 0.6, 0.7, and 1.0.

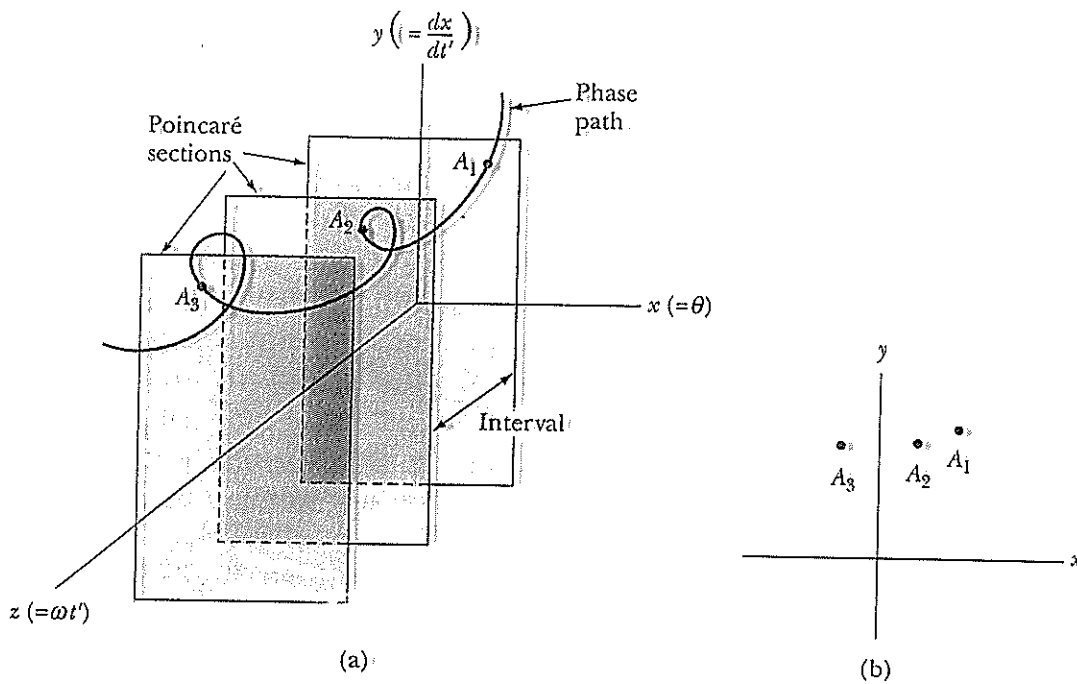
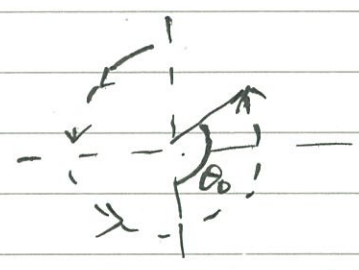
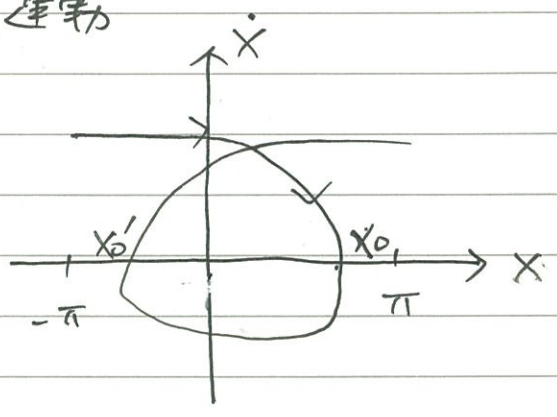


FIGURE 4.20 (a) Poincaré plot, a three-dimensional phase diagram, showing three Poincaré sections and the phase path. The sections are projections along the $y-x$ plane. (b) The points A_i are the phase path intersections with the section plots. They are plotted here on the $y-x$ plane to help visualize the motion in phase space.

Poincaré section

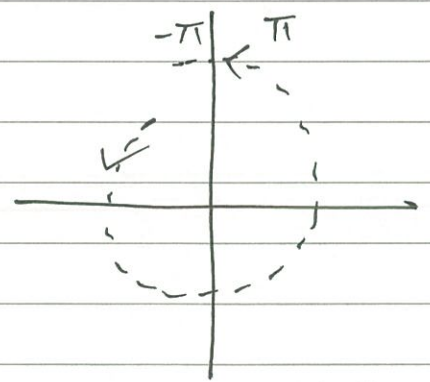
由 Fig 4-19 可以看到，系統在 $F=0.8$ 時，正如
plane pendulum 一樣作繞行一整圈 $(-\pi \rightarrow \pi)$ 之

運動



$x_0 \leftrightarrow x_0$ is max

返回 ($\dot{x} < 0$)



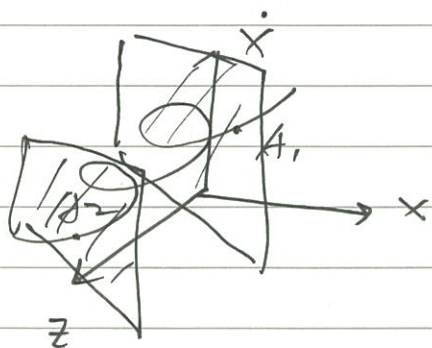
因此，Fig. 9 代表轉 = 圈
一個 cycle

~~所花時間~~
~~變 2 倍~~

~~為 periodic doubling~~
~~稱~~

如上的例子說明了相圖是靜態的，若要
動態包括進來，則除 x 及 x' 外，要再加入 $z = \omega t'$
形成 3D 的曲線。

由於 3D 曲線不易表現，^{提出} Poincaré 以 平行面



面曲線之相交 (如左圖)

A_1, A_2, \dots

再將 A_1, A_2, \dots 投影到

x, \dot{x} 平面上來展示動力
(dynamics)

這平重疊法稱為 Poincaré section plot.

以 Plane pendulum 為例，可以取

$t' = \frac{2\pi}{\omega} \times n$ 之平面 與 (x, \dot{x}, z) 之交點 作圖

(或 $\frac{2\pi}{\omega} \times \frac{n}{m}$)

$F = 0.5, T = \frac{2\pi}{\omega} \cdot 2$

$F = 0.5, T = \frac{2\pi}{\omega} \cdot \frac{3}{2}$

見 Fig 4-19. 可以展現 dynamics.

簡化

如 周期運動 \Rightarrow 只剩一點

periodic doubling \Rightarrow 二點

在 Chaotic region \Rightarrow 許多點，表示有許多周期牽涉
其中！

更重要的是有 strange (chaotic) attractor 的存在。
我們發現

即 Chaotic motions 中，雖然二相近之曲線會發散，
在 相空間中

但仍會回來 (才能將 phase space 填滿)，因此

Lorentz attractor

```
NDSolve[{x'[t] == 10*(y[t] - x[t]), y'[t] == -x[t]*z[t] + 28*x[t] - y[t],
z'[t] == x[t]*y[t] - 8*z[t]/3, x[0] == 1, y[0] == 15, z[0] == 10}, {x, y, z}, {t, 0, 50}]
```

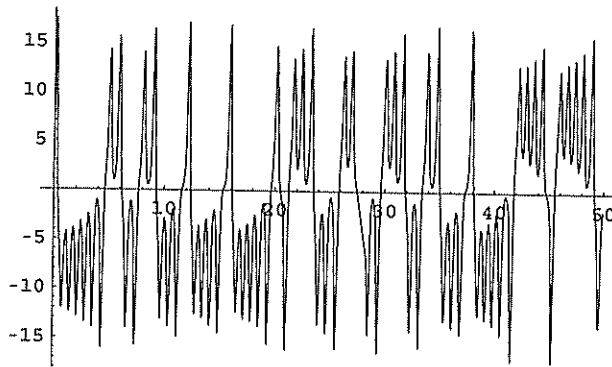
```
{{x -> InterpolatingFunction[{{0., 50.}}, <>],
y -> InterpolatingFunction[{{0., 50.}}, <>],
z -> InterpolatingFunction[{{0., 50.}}, <>]}}
```

$$\frac{dx}{dt} = 10(y-x)$$

$$\frac{dy}{dt} = -xz + 28x - y$$

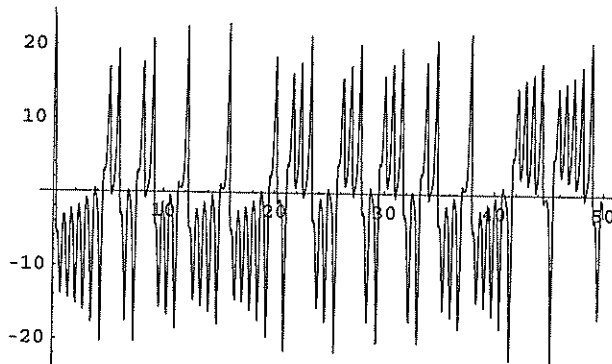
$$\frac{dz}{dt} = xy - \frac{8}{3}z$$

```
Plot[x[t] /. %1, {t, 0, 50}]
```



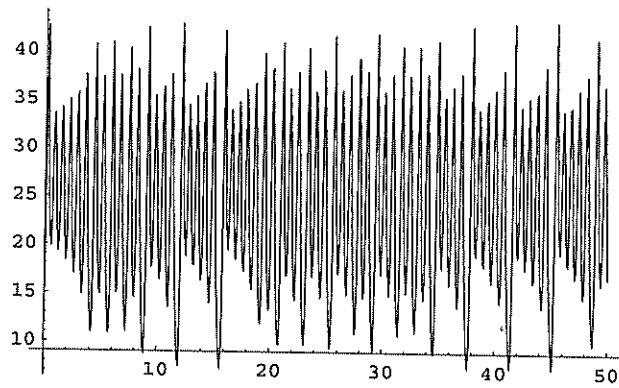
- Graphics -

```
Plot[y[t] /. %1, {t, 0, 50}]
```



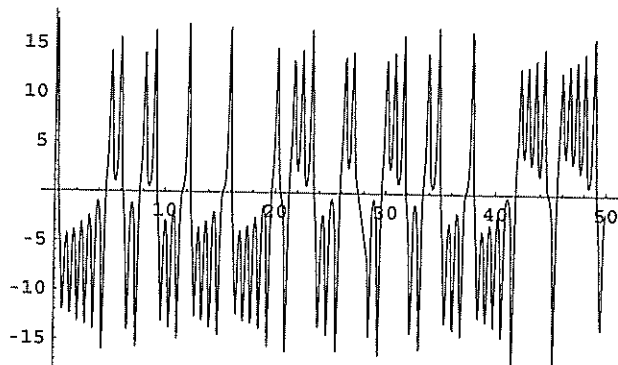
- Graphics -

```
Plot[z[t] /. %1, {t, 0, 50}]
```



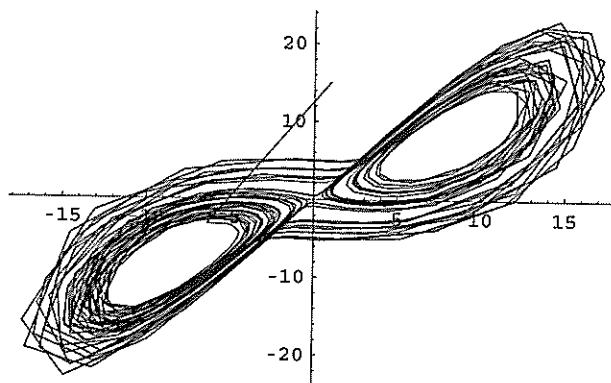
- Graphics -

```
Plot[Evaluate[x[t] /. %1], {t, 0, 50}]
```



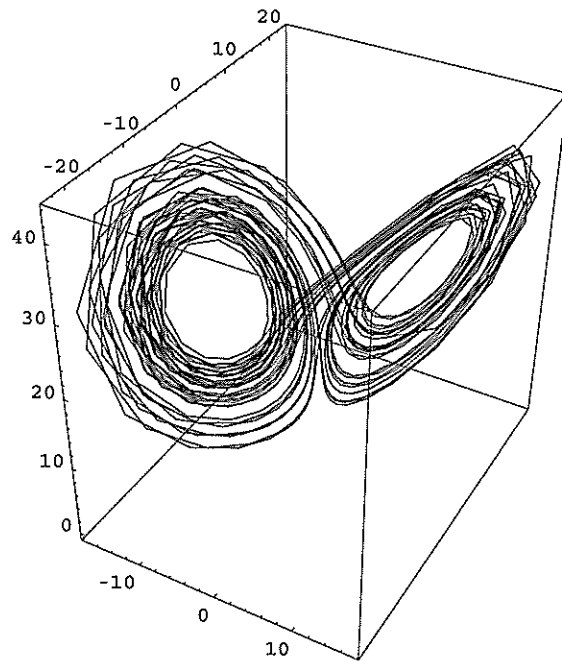
- Graphics -

```
ParametricPlot[{Evaluate[x[t] /. %1][[1]], Evaluate[y[t] /. %1][[1]]}, {t, 0, 50}]
```



- Graphics -

```
ParametricPlot3D[Evaluate[{x[t], y[t], z[t]} /. %1], {t, 0, 50}, PlotPoints -> 1000]
```



- Graphics3D -

```
Evaluate[x[0.12] /. %1]
```

```
{14.2638}
```

Poincaré map 与 map 的起源

有 chaos 的系统, 其主宰方程式皆可以

$$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}, \lambda), \quad \vec{x} = (x_1, x_2, \dots, x_d) \text{ 表示}$$

表示, 其中 λ 为 control parameter, 藉以调整以达 chaos.

例如: forced plane pendulum

$$\theta = x_1, \quad \dot{\theta} = x_2, \quad \ddot{\theta} = -\omega^2 \theta + F \cos z$$

$$\frac{dx_1}{dt} = x_2 \quad \lambda \quad \frac{dx_3}{dt} = \omega$$

$$\frac{dx_2}{dt} = -c x_2 - \sin x_1 + F \cos z$$

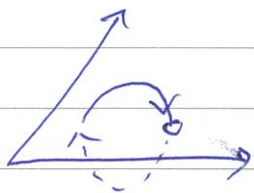
而视 \vec{x} 为 d 维空间中流动的质点, Poincaré map 基本上

Poincaré section, 为质点之轨迹与 $d-1$ hypersurface (如下图)

之交点所形成之 map:

$$\vec{x}(n+1) = \vec{G}(\vec{x}(n), \lambda), \quad \vec{x}(n) = (x_1(n), \dots, x_{d-1}(n))$$

(如左图所示)

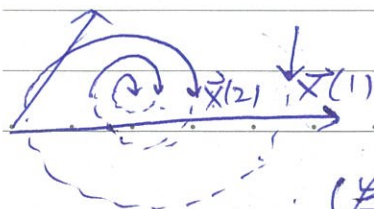


藉著研究 $\vec{x}(n+1) = \vec{G}(\vec{x}(n), \lambda)$

可得到真实运动之特徵

特别是 map 之群的特性与 同胚 即无

(为所谓的普适性, universality), 所以可以藉了



相空間之軌跡實際上拉開 (stretched) 及

折疊 (folding) 反覆進行, 但曲線不相交

以維持 (deterministic), 但可被預測的

程度大為降低! (例: Lorentz attractor)

見 3-55-1

* Logistic map

除了微分方程式有 chaotic behavior 外,

差分方程式也有對應的行為。

如果以 n 代表時間, X_n 代表對應之座標,

則 $X_{n+1} = f(X_n)$ 稱為一個 mapping (iterative map)

前圖的 Poincaré section plot 可視為 2D mapping.

$$(X_{n+1}, \dot{X}_{n+1}) = \vec{f}(X_n, \dot{X}_n)$$

最簡單又有 chaotic 行為的是所謂的 logistic equation:

(originally introduced by Pierre Francois Verhulst)

$$X_{n+1} = \alpha X_n (1 - X_n) \quad \dots \quad (39)$$

(如果沒有 $1 - X_n$, $X_{n+1} = \alpha X_n$, $\therefore \alpha = \text{growth rate}$, $1 - X_n$ 則是 restriction of growth)

$$\text{即 } f(x) = \alpha x (1 - x) \quad 0 \leq x \leq 1$$

許多的有關 chaos 重要的概念都可以 (39) 式說明。

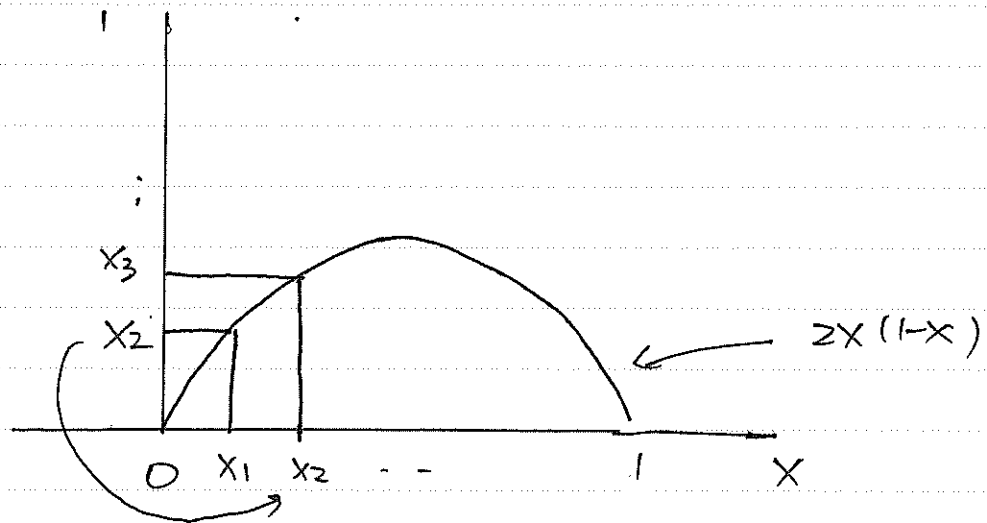
要了解經過多次 map 後 x 的行為, 一般使用

所謂的 logistic map 之圖示法。

logistic map: 例: $X_{n+1} = 2X_n(1-X_n)$

由 X_1 出發 得到 $X_2 = 2X_1(1-X_1)$
 再由 $X_2 \dots$ $X_3 = 2X_2(1-X_2)$
 \vdots \vdots

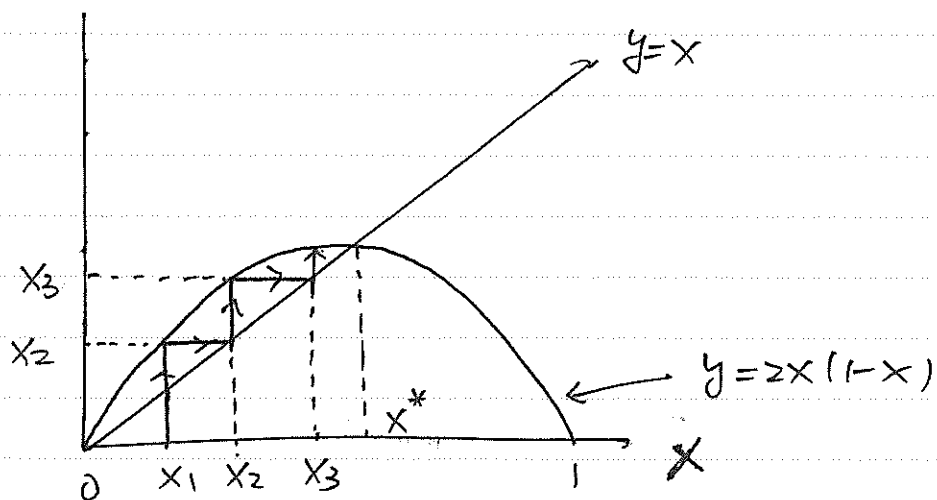
最單純的畫法
如下圖所示



但此法需將 x_2 由 y 轉到 x , 極不方便。

Logistic map 的作法是多畫一條 $y=x$ 的線則

$y=x_2$ 與 $y=x$ 之相交即為下一個的 x 值:



如此可得 x_1, x_2, x_3, \dots 整合一條 左右上下
行進的折線。

Fix points 由上圖可知 $\lim_{n \rightarrow \infty} X_n = X^*$

X^* 稱為 fix point.

$\because X_{n+1} = X_n \therefore$ fix points 必須滿足

$$X^* = 2X^*(1 - X^*) \quad \dots (40)$$

$$X^* = f(\alpha, X^*)$$

在 $\alpha = 2$ 之特例中, $X^* = 0$ 或 $\frac{1}{2}$

其中 $X^* = 0$ 不穩定, 為 unstable fixed point

而 $X^* = \frac{1}{2}$ 則為 stable fixed point.

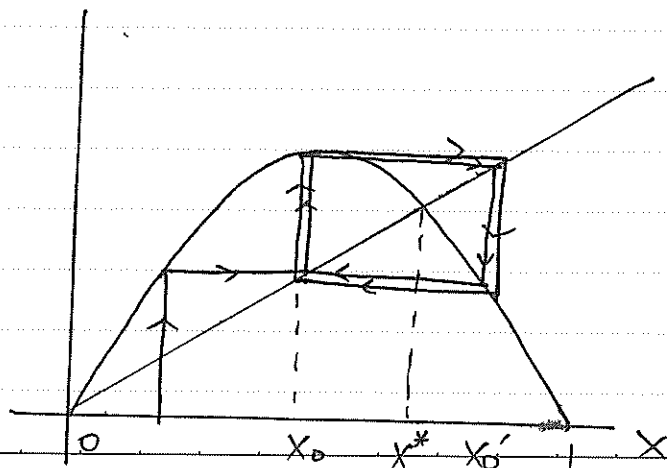
\therefore Iterative map 的微分方程式一樣可以有
平衡點之行爲

Bifurcation

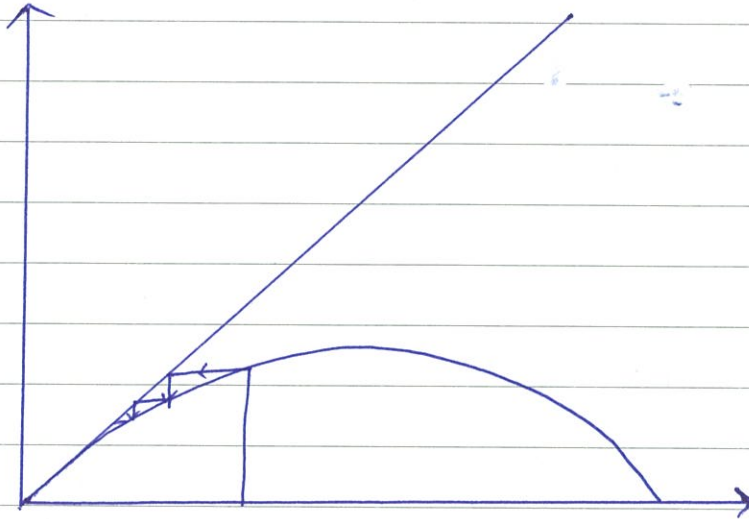
Fixed points 的出現與 α 值有很大的關係

大的 α 會有類似 limit cycle 的行爲, 如

$\alpha = 3.1$



More examples on map.



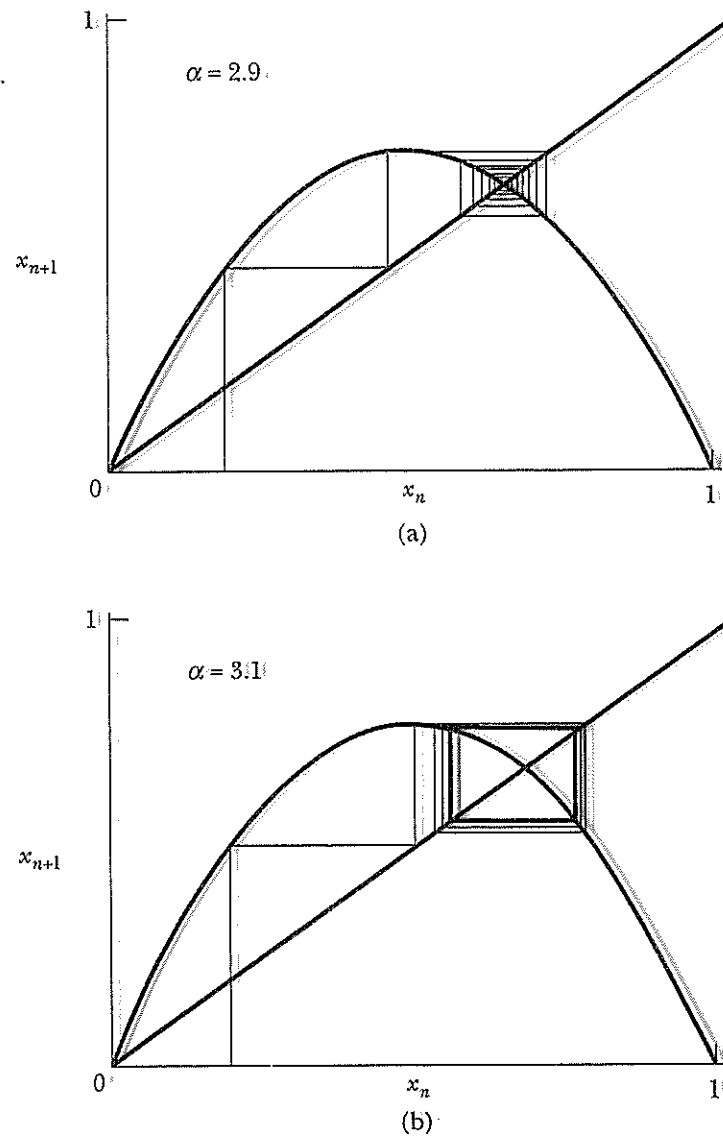
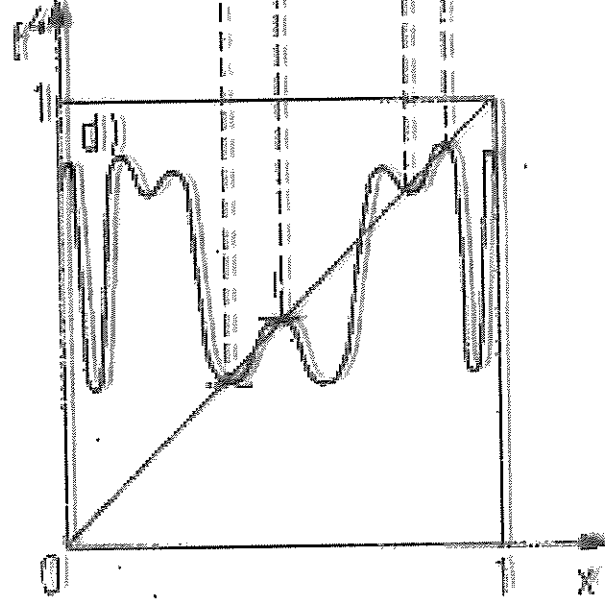
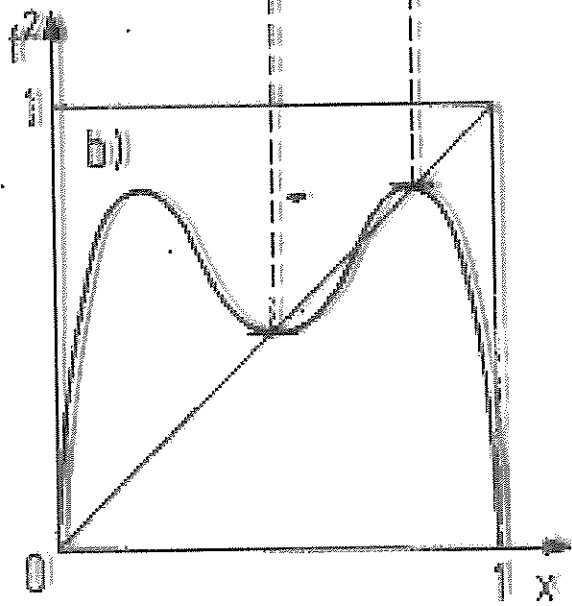
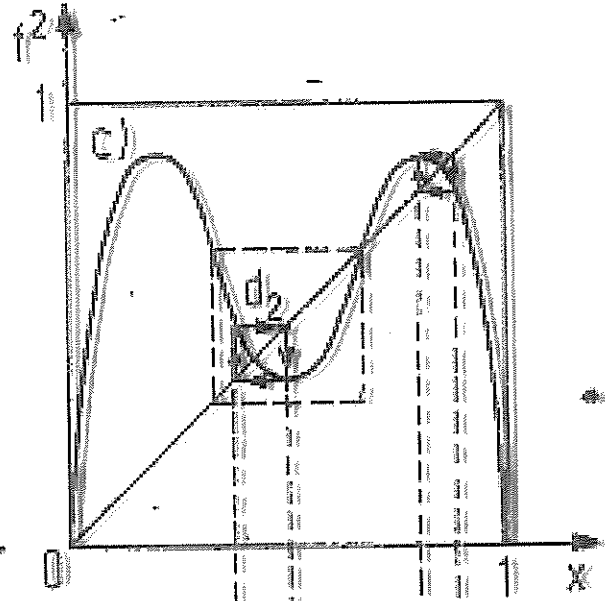
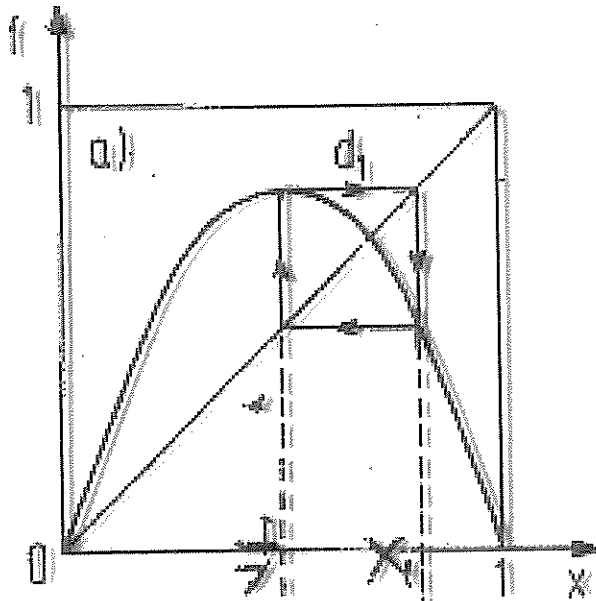
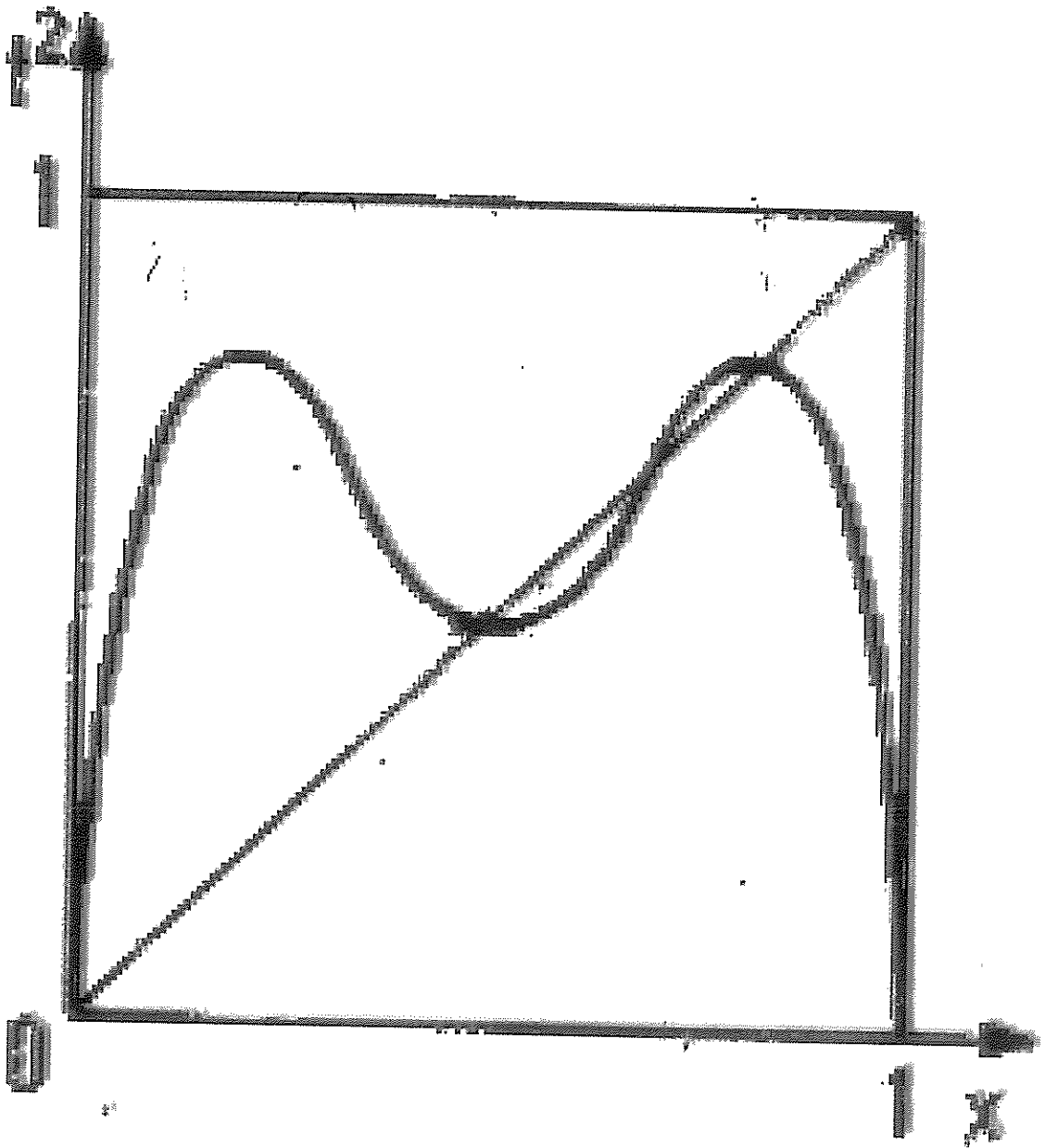
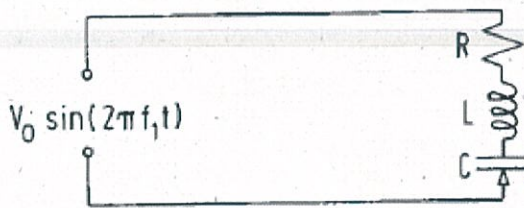


FIGURE 4-22 Logistic equation map for α values of 2.9 and 3.1, indicating stable populations in (a) and multiple possible solutions for $\alpha > 3.0$ in (b).



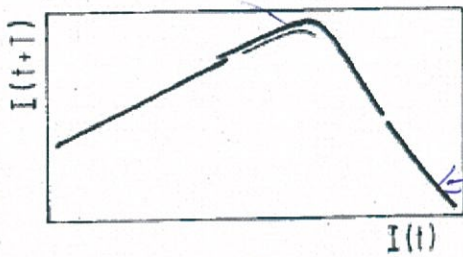




A)

varactor diode

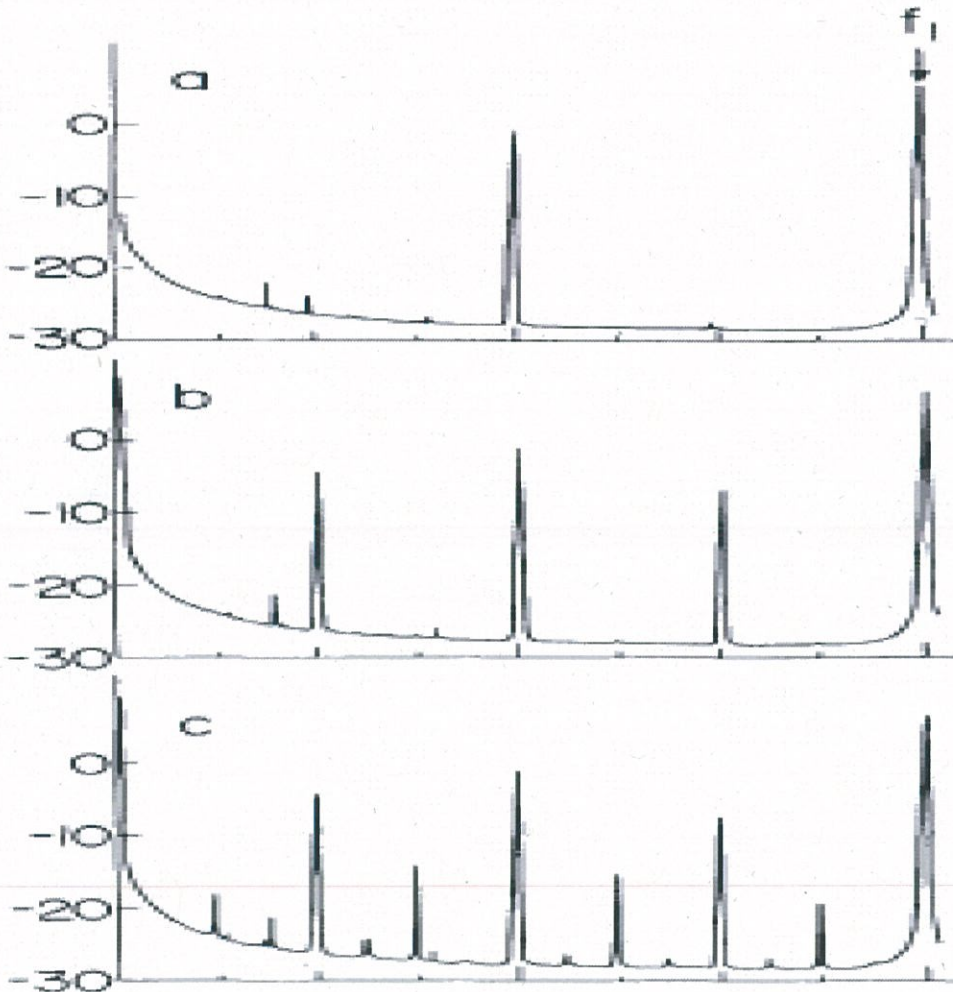
$$C \approx \frac{C_0}{(1 + V/a_0)^{0.43}}$$



B)

$I(t_0 + nT)$

$$T = 1/f$$



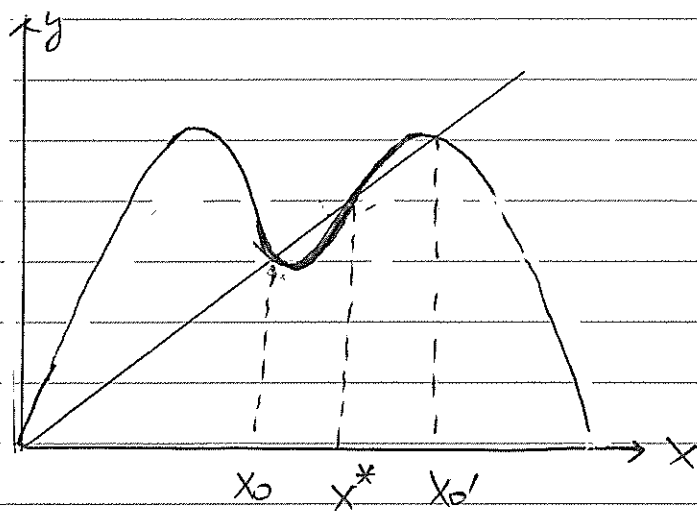
P.S. Linsay, Phys. Rev. Lett. 47, 1349, 1981

在最後階段， x 在 x_0, x_0' 之間反覆

跳躍 $x_0 \rightarrow x_0' = f(x_0) \rightarrow x_0 = f(x_0') = f^2(x_0) \rightarrow x_0' = f(x_0) = f^2(x_0')$

\therefore 所對應的 x_0 及 x_0' 為 $f^2(x)$ 之 fixed points:

(原先 f 之 fix point 仍為 $f^2(x)$ 之 fix point)



其中 x^* 成為 unstable fixed point.

由於由 x_0 回到 x_0 需經過 f 2 次，所對應的現象類似 periodic doubling.

我們可以將 stable 的 fixed points 對 α 作圖，可得 Fig. 4-23.

當 α 繼續增加時，有更多的 fixed points

在 $f^4(x)$, ... 發生。

由圖 4-23 可知，在 $\alpha=3.0$ 時，系統開始有 2 個穩定點 x_0 ，此稱為 bifurcation，由於其形狀。

像一個叉子，所以稱為 pitchfork bifurcation.

Logistic equation 之分析 最方便是使用 較值

子法。使用 mathematica 可求出發生 bifurcation

之 $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \dots$

($\alpha_1=3.0$), $\alpha_2 = 4\sqrt{6}$ (4-cycle), ...

$$\lim_{n \rightarrow \infty} \alpha_n = 3.5699456$$

另外 $\delta_n \equiv \frac{\Delta \alpha_n}{\Delta \alpha_{n+1}} \rightarrow 4.669202$ (Feigenbaum's δ)

即 $\alpha_n - \alpha_\infty = \text{Constant} \cdot \delta^{-n}$, (δ number)

Feigenbaum 發現 δ 是一個普遍 (universal) 值

只要是透過 periodic doubling (periodic doubling route)

達到最後 chaos ($n \rightarrow \infty$) 之過程，其 δ 一樣。

這個 universality 不僅在 1D，在 2D 被

檢驗是對的！

1D: Feigenbaum $x_{n+1} = f(x_n)$, 只要 f has a single quadratic maximum 即可

* Chaos identification.

Chaos 並沒有一個嚴謹大家都同意的定義。

一般而言 chaos theory 是指研究 complex dynamic non-linear

system 的不可預測性 (unpredictability).

特別是含起始條件之 sensitive dependence 的現象

以下 iterative equation 為例

$$\text{若 } X_{n+1} = \alpha X_n (1 - X_n^2) \quad \alpha = 2.5$$

$$\text{取 } X_1 = 0.7000000000$$

$$\text{及 } X_1' = 0.7000000001$$

在 $n = 39$ 時 (見 Fig 4-20) X_n 及 X_n' 開始產生差別, \therefore 此程式有 chaotic 行為。

這種放大的行為, 又稱為 butterfly effect

— 意指蝴蝶果翅膀之拍動^會雖然只在局部上

造成空氣小小的流動, 但可以被放大以致

於讓我們無法預測天氣。

Lyapunov exponents

一個有效測量對 initial condition 之 sensitive

dependence 是蘇聯數學家 A.M. Lyapunov

所先定義的 Lyapunov characteristic exponent

如果 initial conditions: $x_0, x_0 + \epsilon$

$$\text{在 } n \text{ step } \Delta_n = X_n' - X_n \approx \epsilon e^{n\lambda}, \quad n \rightarrow \infty$$

例 $\lambda = \text{Lyapunov exponent}$

$\therefore \lambda < 0$ two orbits eventually converge

$\lambda > 0$ nearby trajectories diverge, chaotic

对 1D map 有 $\frac{1}{2}$, $x_{n+1} = f(x_n)$

$$\Delta_n = f^{(n)}(x_0 + \varepsilon) - f^{(n)}(x_0) \rightarrow \varepsilon e^{n\lambda}$$

$$\therefore n\lambda = \ln \frac{f^{(n)}(x_0 + \varepsilon) - f^{(n)}(x_0)}{\varepsilon}$$

$$\therefore \lambda = \frac{1}{n} \ln \frac{f^{(n)}(x_0 + \varepsilon) - f^{(n)}(x_0)}{\varepsilon}$$

$$\xrightarrow{\varepsilon \rightarrow 0} \frac{1}{n} \ln \left| \frac{d f^{(n)}(x)}{dx} \right|_{x=x_0}, \quad \because f^{(n)}(x_0) = f(f(f \dots f(x_0)))$$

$$\begin{aligned} \therefore \frac{d f^{(n)}(x)}{dx} &= \frac{df(x_{n-1})}{dx_{n-1}} \frac{dx_{n-1}}{dx_0} \\ &= \frac{df(x_{n-1})}{dx_{n-1}} \frac{df(x_{n-2})}{dx_{n-2}} \frac{dx_{n-2}}{dx_0} \\ &= \frac{df}{dx_{n-1}} \frac{df}{dx_{n-2}} \dots \frac{df}{dx_1} \frac{df}{dx_0} \end{aligned}$$

$$\therefore \lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| \frac{df}{dx_i} \right| \quad (41)$$

入的計算一般要借助數值方法。

對 logistic equation 而言, Fig. 4-25

為計算的結果 (利用 Nestlist 及 Drop

指令, mathematica 可以輕鬆的計算入, 見

之後的附錄) 其中 $\lambda > 0$ 處為 chaos 隱含處。
3-64

Fig. 4-26 為 forced plane pendulum 在 $F=0.4$

及 $F=0.6$ 之 Lyapunov exponents. 其中 $F=0.6$

有 $\lambda > 0$ 之情形, 由 Fig 4-19 元: 是 chaotic
顯示
motion 一致。

例

在 iterative equation 中, 某些特

定的點, 入值可以精確的求出

* $x_{n+1} = f(x_n)$ 的 $x_{n+1} = x_n$ 之交集 x_0

此時 $x_0 = f(x_0)$ 為 fixed point

$\therefore x_{n+1} - x_0 = f(x_n) - f(x_0) = (x_n - x_0) f'(x_0)$ if $x_n \rightarrow x_0$

\therefore 若 $|\frac{df}{dx}| < 1$ 則為 stable fixed point

反之若 $|\frac{df}{dx}| > 1$ 則 \therefore unstable. " " " "

由 (4) 式可知 $n \rightarrow \infty$ 时 $\left| \frac{df}{dx_n} \right| < 1$, $\ln \left| \frac{df}{dx_n} \right| < 0$

对
stable
fixed points 而言

$$\therefore \lambda < 0$$

Super stable: $\left| \frac{df}{dx} \right| = 0$ 如 $\alpha = 2$, $x = \frac{1}{2}$

$$\text{此时 } \frac{df}{dx} = 2 - 4x = 0$$

$$\therefore \ln \left| \frac{df}{dx} \right| = -\infty \quad \lambda = -\infty$$

bifurcation: 发生在某 x_0 由 stable 转变为

unstable.

$$\therefore \left| \frac{df}{dx_0} \right| \text{ 由 } < 1 \text{ 转变为 } > 1$$

$$\therefore \text{在 bifurcation 点上 } \left| \frac{df}{dx_0} \right| = 1$$

$$\therefore \text{此时 } \left(\frac{df}{dx_0} \right) = 1, \quad \ln \left| \frac{df}{dx_0} \right| = 0$$

$$\therefore \lambda \rightarrow 0 \text{ for bifurcation points.}$$

Bifurcation points for logistic map

* $f(x) = x \iff |f'(x)| = 1 \Rightarrow$ fixed point becomes unstable

$$x = \alpha x(1-x)$$

$$\therefore x = 0 \text{ or } x = 1 - \frac{1}{\alpha}$$

$$f'(x) = \alpha(1-2x)$$

$$|f'(0)| = \alpha \quad \alpha \geq 1 \text{ unstable}$$

$$|f'(1 - \frac{1}{\alpha})| = |\alpha - 2\alpha + 2| = |2 - \alpha|, \quad \alpha \geq 3 \text{ unstable}$$

* $f^2(x) = f(\alpha x(1-x))$

$$= \alpha x(1-x) [1 - \alpha x(1-x)]$$

$$= \alpha x(1-x) (1 - \alpha x + \alpha x^2)$$

$$f_2(x) = f^2(x) = x$$

$$\Rightarrow x = 0 \text{ or } \frac{1}{\alpha} = (1-x)(1 - \alpha x + \alpha x^2)$$

利用 Mathematica 解 $f^2(x) = x$

$$\& f_2(x) = -1$$

$$\text{可得 } \alpha = 1 + \sqrt{6}$$

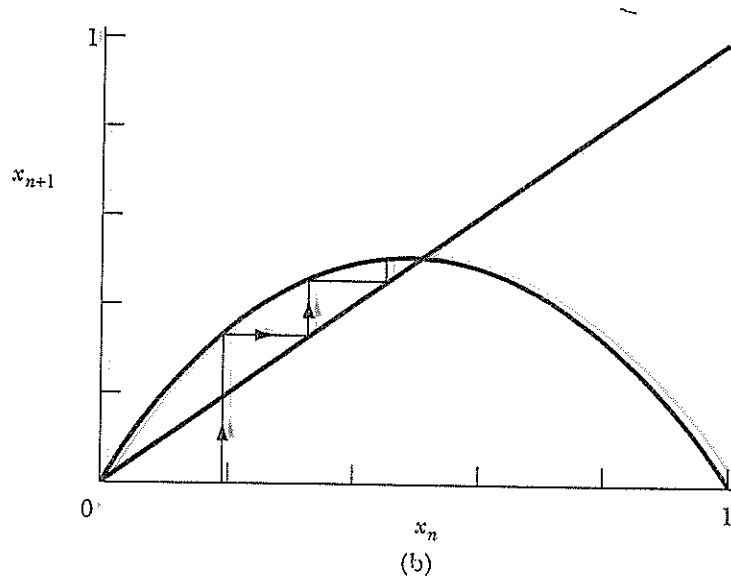
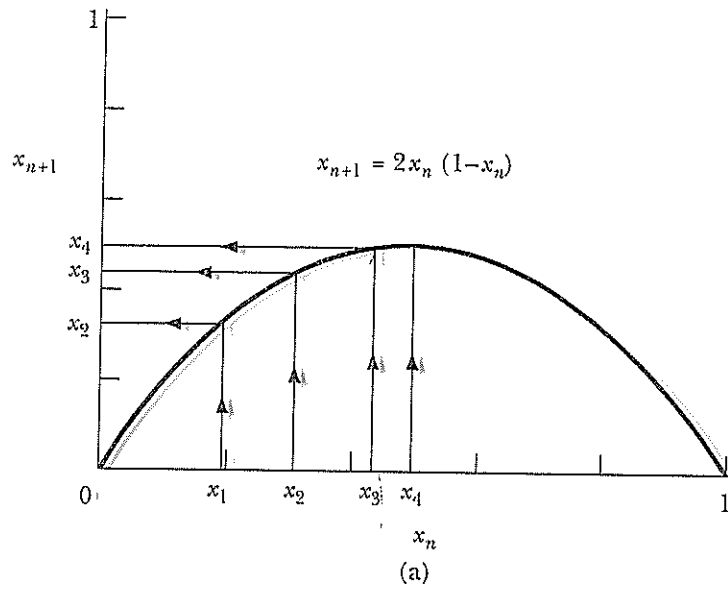
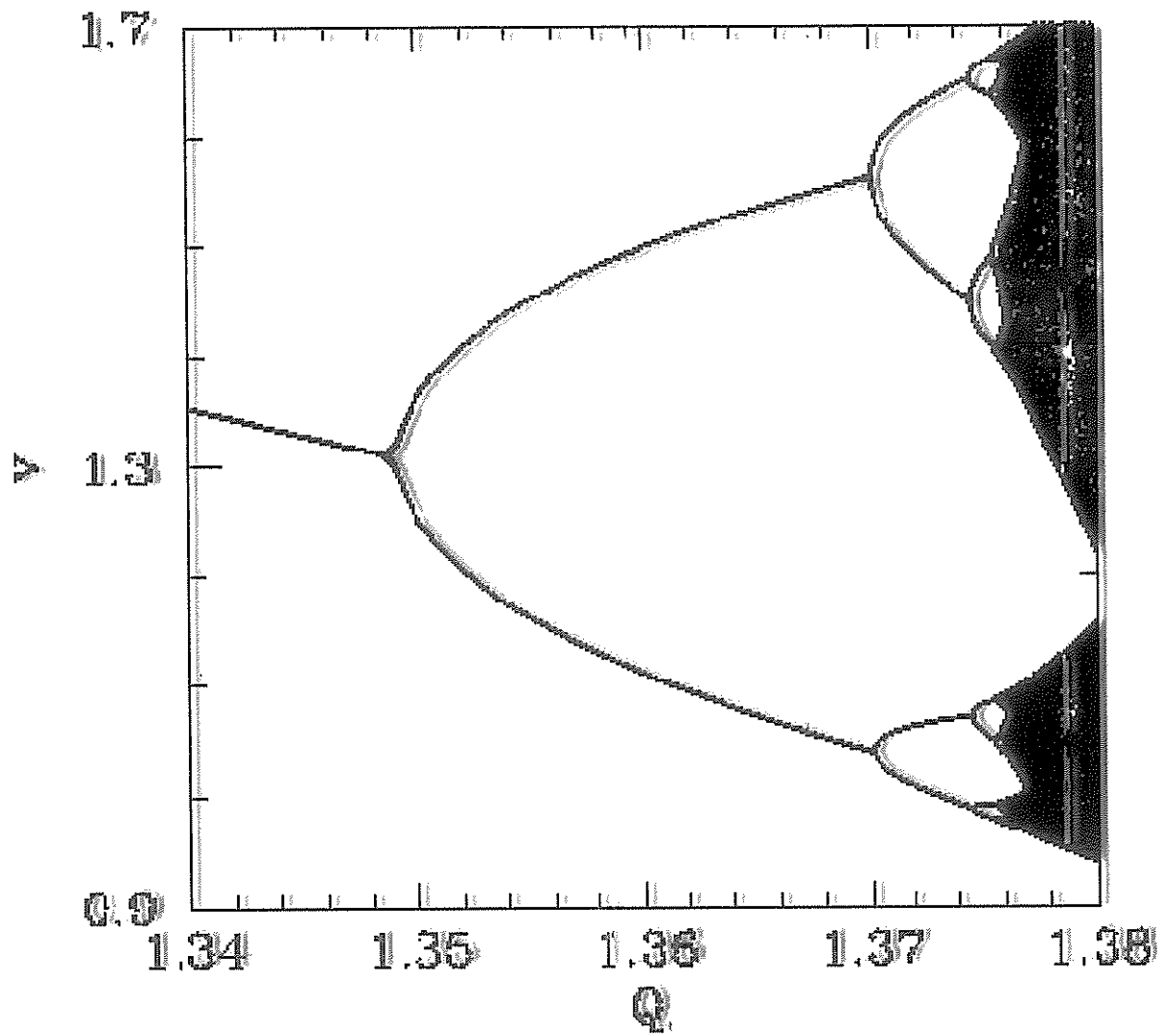


FIGURE 4.21 Techniques for producing a map of the logistics equation.

Driven Pendulum: Transition to chaos



$$F=1.5, \omega=2/3, Q=1/c, v(0)=0, \theta(0)=0$$

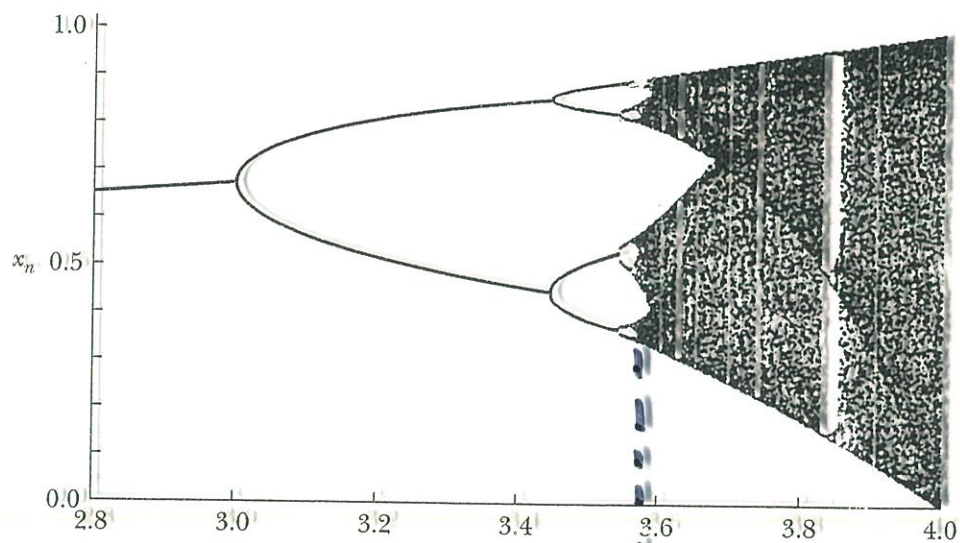
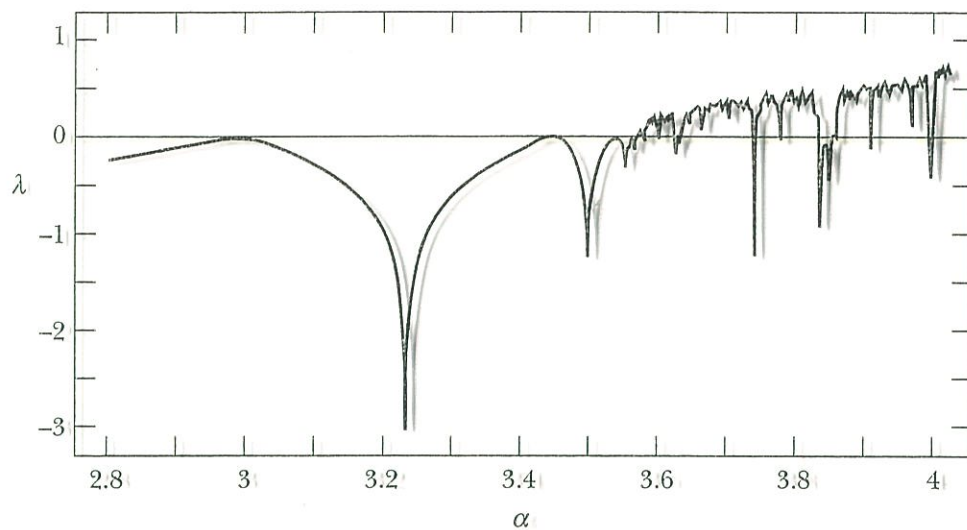


FIGURE 4-23 Bifurcation diagram for the logistic equation map.

3.5699456

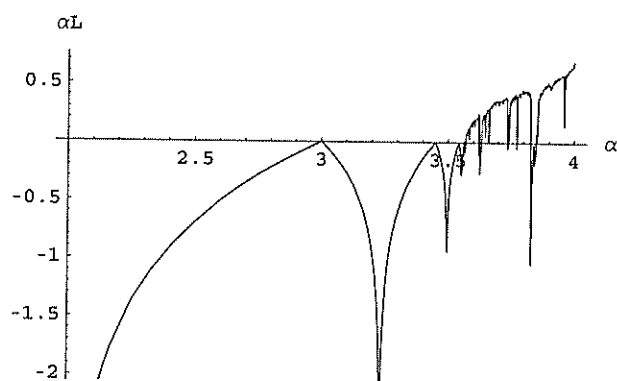
Mathematica 的 Lyapunov exponent

The Lyapunov exponent is denoted by lya , which is obtained by generating a list of x_i using `NestList`. The average of the elements of the list can be calculated using `Apply[Plus, list]` and dividing by `Length[list]`

```
f[x_] :=  $\alpha$  * x (1 - x);
```

```
lya[l_, xinit_, n_, ndrop_] := ( $\alpha$  = l; xlist = Drop[NestList[f, xinit, n], ndrop + 1];
  Apply[Plus, Log[Abs[f'[xlist]]]] / Length[xlist])
```

```
Plot[lya[ $\alpha$ , 0.7, 5000, 2000], { $\alpha$ , 2.0, 4.0},
  AxesOrigin -> {2.0, 0}, AxesLabel -> {" $\alpha$ ", " $\alpha L$ "}];
```



4 possible criteria for chaotic motion

- * Time dependence of the signal looks chaotic
- * Power spectrum exhibits broadband noise at low frequencies
- * Autocorrelation function decays rapidly
- * Poincaré map shows space-filling points

Universality (Feigenbaum) & routes to chaos

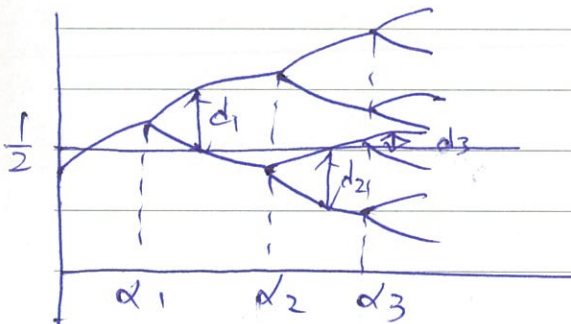
$x_{n+1} = f(x_n)$, only one x_{max} such that

$$0 \leq x_n \leq 1$$

$$f'(x_{max}) = 0$$

$$f''(x_{max}) < 0$$

$$d_n = d_{\infty} - \text{const. } \delta^{-n}$$



$$\frac{d_n}{d_{n-1}} \rightarrow -\delta$$

d, δ are the same!

This is known as the bifurcation route to chaos.

There are two alternative routes:

- * The intermittency route to chaos
- * Ruelle-Takens-Newhouse Route to chaos (strange attractor)