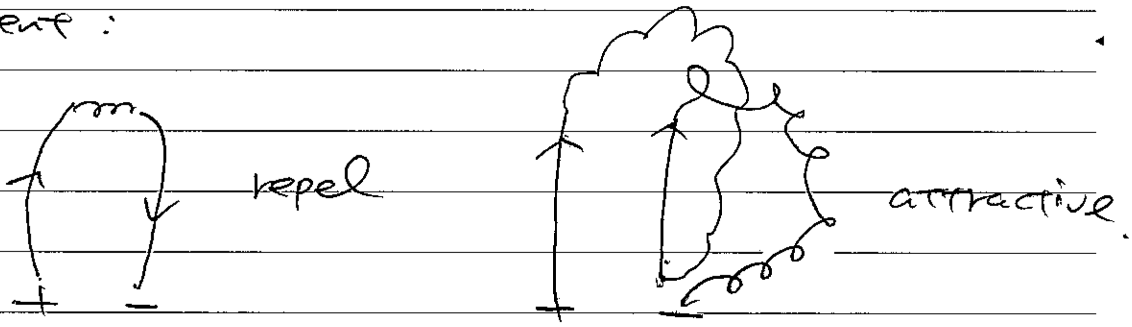
# Magnetostatics

## Lorenz force law & magnetic fields

Up to now, we have confined our attention to cases when all charges are static. In this case, $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$, $\vec{\nabla}\times\vec{E} = 0$.

Historically, it is found when two wires are connected to a battery with two different orientations, the forces between wires are different:

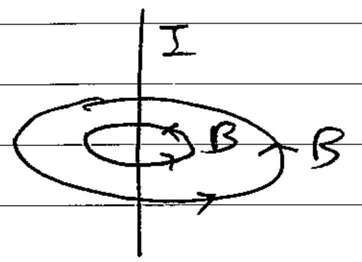



These forces are not due to charges as one can check it by using a charged test conductor to approach the wire. (the force is null).

Therefore, these forces are not static in nature? They are due to motion of charges. It's later revealed by using a compass that moving charges, compasses surrounding the

points in concentric circles. The connection
directions form

of these directions is the field line of magnetic field $\vec{B}$.




By measuring forces using magnets and direction of $\vec{B}$,

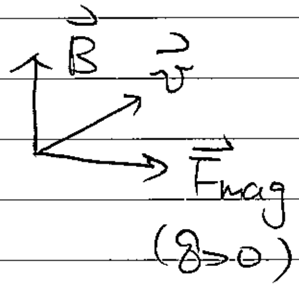
it is later realized that the force

acting on a charge $q$ with velocity $\vec{v}$

is given by

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad \cdots ①$$

The force $q\vec{v} \times \vec{B}$ $(= \vec{F}_{mag})$ is known as the Lorentz force.

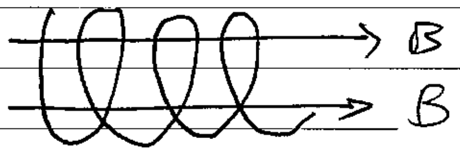



Note that $\vec{F}_{mag} \cdot \vec{v} = 0$ $\therefore$ Magnetic forces do not do work on charge particles!

$\therefore$ $\frac{1}{2}mv^2$ is kept fixed.

Specifically, $\vec{B}$ only changes the direction of $\vec{v}$ not the magnitude of $\vec{v}$ so that $\frac{1}{2}mv^2$ is fixed.

For a uniform $\vec{B}$ field, it's reflected in the trajectory of the charged particles: spiral & cyclotron motion

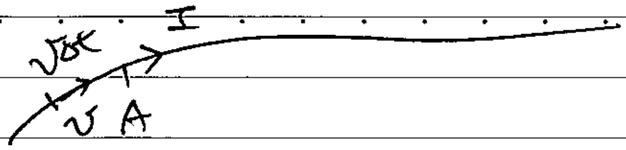



Current & Current density

To characterize the magnetic force quantatively, one needs to characterize the motion of charged particles.

Current = total charge that passes through a point / a surface per unit time.

For a wire, the current at a point = total charge that passes through a cross section at that point




Clearly, if the wire is a mathematical line with line charge density $\lambda$ and the speed is $v$ at the point A one has $I = \lambda v$ --- ②

Check: during $\Delta t$, all charges within $v\Delta t$ will pass A. $\therefore$ total charges that pass A $= \lambda \cdot (v\Delta t)$ $\therefore$ $I = \dfrac{\lambda v \Delta t}{\Delta t} = \lambda v$.

The direction of $v$ is tangential to the line. $\therefore$ $\vec{I} = \lambda \vec{v}$ is tangential to the line too! --- ③

Clearly, eq. ② implies that $\lambda > 0, v > 0$ & $\lambda < 0, v < 0$ are the same.

In other words, positive charges move right = negative charges move left. They give rise the same current. In practice, it is usually negatively charged electrons that move. In general, one can write

$$I = \lambda_+ v_+ + \lambda_- v_- \quad \text{---} \quad (4)$$

Both charges contribute.

The unit of current is ampere (A)

which is 1 coulomb/sec.

For a segment $dl$ on a wire, the charge on this segment $dq = \lambda\, dl$.

From eq. (1), the magnetic force that acts on this segment is

$$d\vec{F}_{mag} = dq\, \vec{v} \times \vec{B}$$

$$= \lambda\, \vec{v} \times \vec{B}\, dl$$

$\therefore$ $\vec{F}_{mag}$ (for the wire)

$$= \int (\lambda \vec{v} \times \vec{B})\, dl = \int \vec{I} \times \vec{B}\, dl \quad \text{--} \quad (5)$$

$\because$ $d\vec{l} \parallel \vec{I}$ $\therefore$ $\vec{I}\, dl = I\, d\vec{l}$

$$\therefore \vec{F}_{mag} = I \int d\vec{l} \times \vec{B} \quad \text{---} \quad (6)$$

is the magnetic force that acts on a wire

In eq.(6), we have assumed that $I$ is constant along the wire.

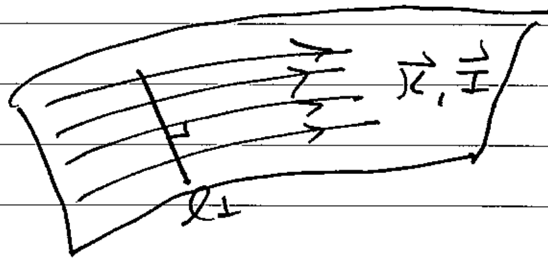
## Charge distribution & current density

In general, instead of being point charges, charges can be continuous distributed and move

In this case, one needs to define current density.

### Surface current density

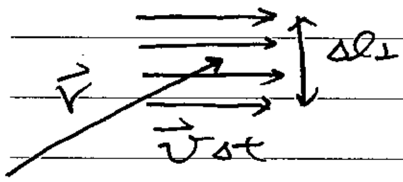
When charges are moving on a surface,




the surface current density

$$\vec{K} = \frac{d\vec{I}}{d\ell_\perp} \quad \cdots (7)$$

is current that flows across $\ell_\perp$ per length of $\ell_\perp$. Suppose $\sigma(\vec{r})$ is the surface charge density at $\vec{r}$. Consider $\Delta \ell_\perp$ near $\vec{r}$ as shown in the left figure. During $\Delta t$, all



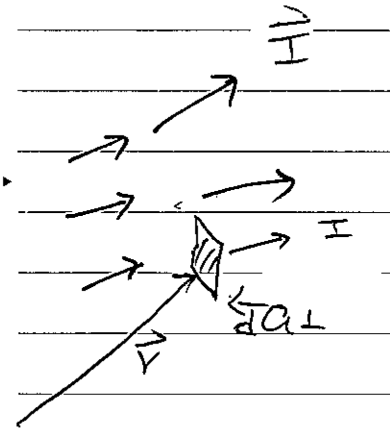

charges inside $v\Delta t \times \Delta \ell_\perp$ will flow across $\Delta \ell_\perp$. $\therefore \Delta q = \sigma v \Delta t \Delta \ell_\perp$ is the charge that flows across $\Delta l_\perp$

$$\therefore \Delta I = \frac{\Delta q}{\Delta t} = \sigma v \Delta l_\perp$$

$$\Delta \vec{I} = \sigma \vec{v} \Delta l_\perp \quad \therefore \quad \vec{K} = \frac{d\vec{I}}{dl_\perp} = \sigma \vec{v} \quad \text{--- (8)}$$

Hence surface charge current $\propto \sigma$ & $\vec{v}$!

Similarly, if the current is distributed on a volume, the volume current density




$\vec{J}(\vec{r}) \equiv$ current that flow across $a_\perp$ ($\perp \vec{I}$) per area

$$= \frac{d\vec{I}}{da_\perp} \quad \text{--- (9)}$$

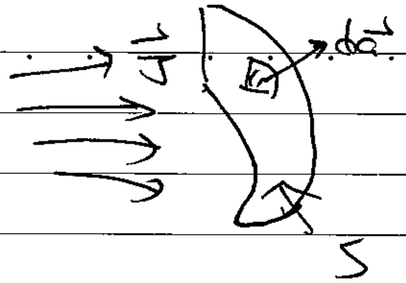
Following the same reasoning, one gets
$$\vec{J}(\vec{r}) = \rho(\vec{r})\vec{v} \quad \text{--- (10)}$$

where $\rho(\vec{r})$ is the charge density that moves with the velocity $\vec{v}$!

Eq (9) & (8) also imply that given $\vec{J}(\vec{r})$ & $\vec{K}(\vec{r})$, one obtains the currents across a curve $C$ or a surface $S$ are

$$I = \int_S \vec{J} \cdot d\vec{a}$$

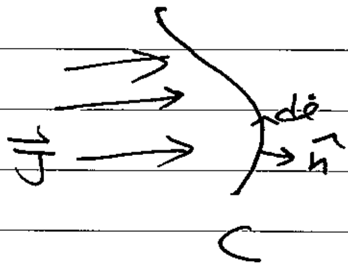
$$\left( = \int_S J \, da_\perp \right)$$



or

$$I = \int_C \vec{K} \cdot dl \, \hat{n}$$

$$\left( = \int_C K \, dl_\perp \right)$$



In addition,

$\because$ $dq = \rho \, d\tau$ $\quad dq \, \vec{v} = \rho \, d\tau \, \vec{v}$

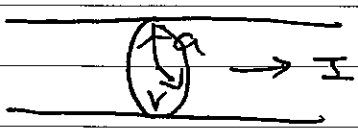
$$= \vec{J} \, d\tau$$

$\therefore$ the magnetic force (eq. 6) is more generally expressed as

$$\vec{F}_{mag} = \int dq \, \vec{v} \times \vec{B}$$

$$= \int \vec{J} \times \vec{B} \, d\tau \quad \text{--- (11)}$$

Example. A real conducting wire has a finite cross section, usually in the circular shape. The current is usually not very uniform.



For uniform distribution,

$$J = \frac{I}{\pi a^2}$$

If $J = kr$,

$$I = \int J\, da = \int_0^a J \cdot 2\pi r\, dr$$

$$= 2\pi k \int_0^a r^2 dr$$

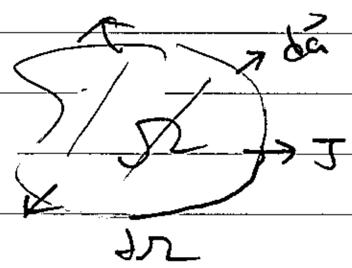
$$= \frac{2\pi}{3} k a^3$$

<u>Continuity equation</u>

Given charge distribution & current distribution,
$\rho$ $\vec{J}$

they are intimately related due to <u>local charge conservation</u>.

If charges are conserved, i.e., total charges are fixed, given a volume $\Omega$ with boundary $\partial\Omega$, one has




(i) Current flowing out of $\partial\Omega$
= charge decreases in $\Omega$

(ii) Current flowing in $\partial\Omega$
= charge increases in $\Omega$

$\because$ current flow across $\partial\Omega$

$$= \oint_{\partial\Omega} \vec{J} \cdot d\vec{a}$$

which is the change of charges in $\Omega$ per unit time

Q in $\Omega$

$$\therefore \oint_{\partial\Omega} \vec{J} \cdot d\vec{a} = - \frac{\Delta Q}{\underbrace{\Delta t}_{<0}}$$

represents decreasing!

$$\because \quad Q = \int_{\Omega} \rho(\vec{r}, t)\, d\tau$$

$$\therefore \frac{dQ}{dt} = \int_{\Omega} \frac{\partial \rho}{\partial t}\, d\tau$$

$$\therefore \oint_{\partial\Omega} \vec{J} \cdot d\vec{a} = -\int_{\Omega} \frac{\partial \rho}{\partial t}\, d\tau \quad \cdots \quad (12)$$

Now, according to the Stoke's theorem,

$$\oint_{\partial\Omega} \vec{J} \cdot d\vec{a} = \int_{\Omega} \vec{\nabla} \cdot \vec{J}\, d\tau$$

$\therefore$ Eq. (12) becomes $\int_{\Omega} \left( \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} \right) d\tau = 0$

Taking $\Omega \to 0$ at any point $\vec{r}$, one gets

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad --- (13)$$

Eq. (13) is the continuity equation, representing local charge conservation.

By "local", it means that when charges disappear at some position $\vec{r}$, they must escape via surfaces nearby $\vec{r}$.

## The Biot-Savart law

Just as static charges produce electric fields that are constants in time.

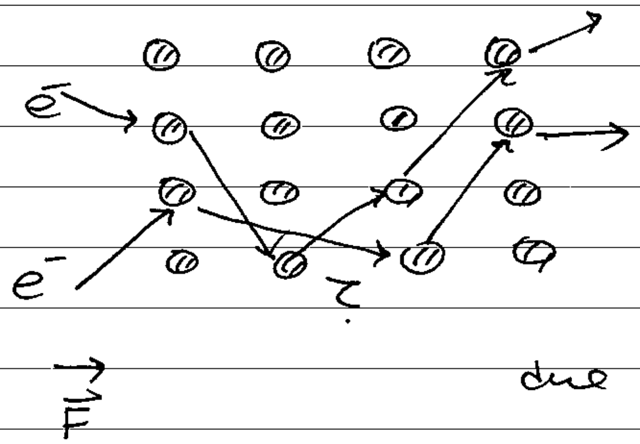
To produce magnetic fields that are static (independent of time), one needs steady currents that do not depends on time.

∴ Formally, electro/magnetostatics is the regime :

$$\frac{\partial \rho}{\partial t} = 0, \quad \frac{\partial \vec{J}}{\partial t} = 0 \quad -- (14)$$

## Ohm's law & current flow

To produce a current flow, one needs to push charges through forces $\vec{F}$.




As shown in the left figure, generally, charges can't be acceleratedly indefinitely due to collisions with the underlying lattices. After each collision, charges move in random directions. Between collisons, charges are accelerated. As a result of forces, there is a net flow along the direction of force and currenty density is proportional to force per unit charge $\vec{f}$

$$\vec{J} = \sigma \vec{f} \quad \cdots \quad (13)\text{-}1$$

where the proportional constant $\sigma$ is known as conductivity. $1/\sigma = \rho$ is used more often, generally termed as resitivity.

For insulators, essentially, $\sigma = 0$ ($\rho = \infty$)

For perfect Conductors, $\sigma = \infty$.

If the force is due to $\vec{E}$ & $\vec{B}$, one has

$$\vec{J} = \sigma ( \vec{E} + \vec{v} \times \vec{B} ) \quad \cdots (13)\text{-}2$$

In particular, when $\vec{B} = 0$ or the contribution of $\vec{B}$ can be neglected, one has

$$\vec{J} = \sigma \vec{E} \quad \cdots (13)\text{-}3$$

Which is known as Ohm's law.

## Drude model

For an ordinary accelaration, one has

$$l = \frac{1}{2} a t^2 \quad \text{if } v_0 = 0$$

$$\therefore t = \sqrt{\frac{2l}{a}}$$

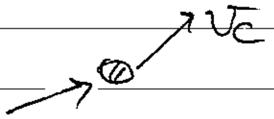
$$v_{average} = \frac{1}{2} a t = \sqrt{\frac{la}{2}}$$

$$\therefore J \propto v_{average} \propto \sqrt{a} \propto \sqrt{f}$$

Which can't reproduce eq. (13)-1 !

In real materials, electrons are accelerated and collide with ions.

After each collision, the velocity of electron points to random direction. Then the electron is being accelerated again.




∴ $\vec{v} = \vec{v_c} + \vec{a}t$ between two successive collisions.

↑ due to collision

∴ $v_{average} = a\tau$ (average of $v_c = 0$)

$\tau$ = mean free time = average duration between two successive collisions = independent f !

$a\tau$ is also called <u>drift velocity</u> to be distinguished from the real velocity of one electron.

∴ $v_{drift} = a\tau = \frac{qf}{m}\tau$

∵ $\vec{J} = \rho \vec{v}_{drift}$ ∴ $\vec{J} = \frac{nq^2\tau}{m}\vec{f}$

↑ charge density = $nq$, $n$ = number of charges per volumes.

∴ $\sigma = \frac{nq^2\tau}{m}$ --- (13)-4

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This is the Drude model (classical) of electric conduction. It's not an accurate formula but it captures the basic ingredients of conduction.

Electric field of steady current

For stationary charges, $\vec{J}=0$, we have $\vec{E}=0$ inside perfect conductors.

For perfect conductors, $\because \sigma=\infty$

$$\therefore \vec{E}=\vec{J}/\sigma=0 \quad (\because \vec{J} \text{ is always finite})$$

is still correct. inside the conductor.

$\therefore$ Usually, one treats conducting wires in electric circuits as equipotentials.

On the other hand, resistors are mad by poorly conducting materials. In that case, one can not set $\vec{E}=0$ in resistors.

However, for steady currents, one has

$$\vec{\nabla}\cdot\vec{E}=\frac{1}{\sigma}\vec{\nabla}\cdot\vec{J}=0 \quad \cdots (13)\text{-}5$$

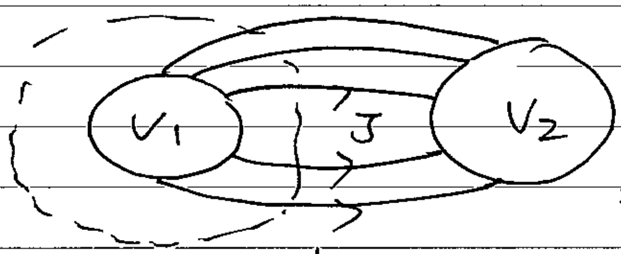
& $\vec{\nabla}\times\vec{E}=0$

Inside the ohmic conducting materials.

Hence one can set $\vec{E} = -\nabla V$

so that $\nabla^2 V = 0$ --- (13)-6

$\therefore$ One can still define electric potential $V$ inside an ohmic material when currents are steady.




S Ohmic material $\nabla^2 V = 0$

In this case, if currents flow from one potential $V_1$ to another potential $V_2$,

$$\Rightarrow V_2 - V_1 = -\int_1^2 \vec{E}\cdot d\vec{l}$$

$$= -\delta \int_1^2 \vec{J}\cdot d\vec{l}$$

--- (13)-7.

Generally, the total current flows out of $V_1$ to $V_2$, $I_{21} = \oint_S \vec{J}\cdot d\vec{a}$

double $I \Leftrightarrow$ double $\vec{J}$

$$\therefore \int_1^2 \vec{J}\cdot d\vec{l} \propto I_{21}$$

$\therefore V_1 - V_2 = I_{21}\cdot R$ --- (13)-8

i.e. $V = IR$ $(V = V_1 - V_2)$

The constant of proportionality $R$ is called <u>resistance</u>. It's unit = ohms $(\Omega)$ when $[I] = A$, $[V] = \text{Volt}$

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$V = IR$ is the more familiar form of Ohm's law.

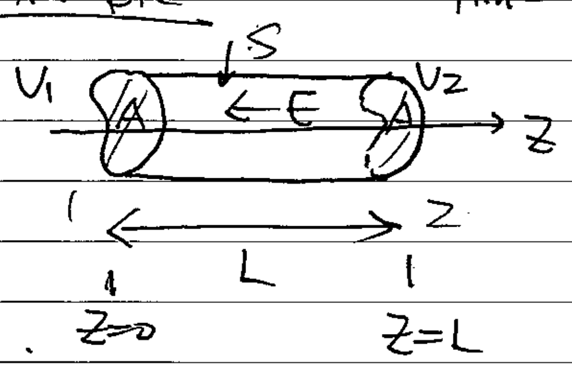
Unlike resistivity, resistance depends on geometry of arrangements of currents / voltage.

In eq. (13)-8, $V_1 > V_2$, charges are pushed from $V_1$ to $V_2$. The work done on unit charge $= V$, $\because I =$ charge flowing from $V_1$ to $V_2$ per unit time $= I$

$\therefore$ power delivered

$$P = IV = I^2 R = \frac{V^2}{R} \quad \cdots (13)\text{-}9$$

Example A cylinder with fixed cross-section A and length L is made by materials with conductivity $\sigma$. Find R.




Solution: Applying a voltage difference $V$ between 1 & 2

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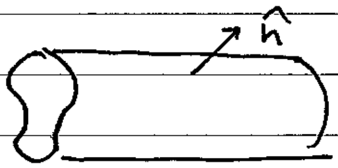
To find R, one needs to know $\vec{J}$ so that $I = \int \vec{J} \cdot d\vec{a}$ is known.

For this purpose, as we indicate, we need to solve

$$\nabla^2 V = 0 \quad (\text{eq } ⑬ - 6) \quad \text{--- (i)}$$

with $V(0) = V_1 \equiv 0$, $V(L) = V_2 = V$ --- (ii)

In addition, on lateral surfaces




$\vec{J} \cdot \hat{n} = 0$ (no currents flowing out)

$\because \vec{J} = \sigma \vec{E}$, $\therefore \vec{E} \cdot \hat{n} = 0$ --- (iii)

$\therefore$ This is the problem of solving the Laplace equation (i) with BCs specified by (ii) & (iii). By uniqueness, the solution is unique. One can guess the solution to be $V(z) = \frac{Vz}{L}$ which obviously satisfies (ii) & (iii)

$$\therefore \vec{E} = -\nabla V = -\frac{V}{L}\hat{z} = \text{constant}$$

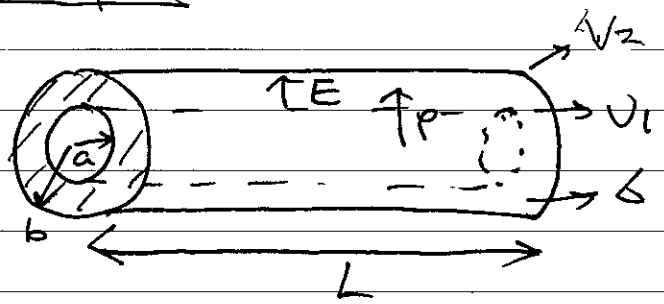
$$\therefore \vec{J} = \sigma\vec{E} = -\sigma\frac{V}{L}\hat{z} = \text{constant}$$

$$I = \int_A \vec{J}\cdot d\vec{a} = \sigma\frac{V}{L}A = \frac{V}{R}$$

$$\therefore R = \frac{L}{\sigma A}$$

or conductance $G = 1/R = \frac{\sigma A}{L}.$

Example



Find the resistance of a coaxial line with $a<\rho<b$ filled by a material of conductivity $\sigma$.
The current flows from inner surface to outer surface or vice versa.

Solution : Set the voltage at $\rho = a$ to $V_1$ and voltage at $\rho = b$ to $V_2$.

The constant voltages at $r=a$ & $b$ can be thought to be resulted from a line charge density $\lambda$ at $\rho = 0$

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In this case, $\vec{E} = \frac{\lambda}{2\pi\epsilon_0 \rho}\hat{\rho}$

$\therefore V = \frac{-\lambda}{2\pi\epsilon_0}\ln\rho + \text{constant}$

which satisfies $\nabla^2 V = 0$ for $a < \rho < b$.

~~(there is no charg in $a<\rho<b$)~~

$\therefore V_1 - V_2 = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right) = V$

$$I = \int \vec{J}\cdot d\vec{a} = \sigma\int\vec{E}\cdot d\vec{a}$$

$$= \sigma \cdot \frac{\lambda}{2\pi\epsilon_0\rho}\cdot L\cdot 2\pi\rho$$

$$= \frac{\sigma\lambda L}{\epsilon_0}$$

$$\therefore \frac{V}{I} = \left(\ln\frac{b}{a}\right)\frac{1}{2\pi\sigma L}$$

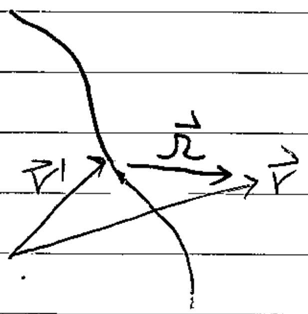
$$\therefore R = \frac{\ln b/a}{2\pi\sigma L}$$

Eq. (14) implies $\vec{\nabla}\cdot\vec{J}=0$ --- (15)

In real situations, there is no electro/magnetostatics for all time. However, the steady state still represents good approximation to situations when fluctuations in time are kept far away.

<u>Magnetic field</u> of <u>a steady current</u>

For a steady line current, it's found by Biot and Savart, the magnetic field is proportional to $\int \frac{\vec{I}\,d\vec{l}\times\hat{\imath}}{\imath^2}$.



$$\therefore \vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\int\frac{\vec{I}\times\hat{\imath}\,dl}{\imath^2} \quad \text{--- (16)}$$

where $\mu_0/4\pi$ is the proportional constant. The unit of $\vec{B}$ is determined by eq. (1). Once the unit of charge is determined, unit of $\vec{B}$ is determined and

In the MKS unit,

$$[B] = \frac{\text{Newton}}{\text{Coulomb} \times \text{M/sec}} = \frac{[E]\cdot \text{sec}}{\text{M}}$$

$$= \frac{[E]\cdot \text{m}\cdot\text{sec}}{\text{m}^2} = \frac{\text{volt}-\text{sec}}{\text{m}^2} = \frac{\text{weber}}{\text{meter}^2}$$

$$= \text{Telsa} = 10^4 \text{ gauss.}$$

1 Ampere = 1 Coulomb/sec

$$\therefore 1\ \text{Tesla} = \frac{N}{A\cdot m}$$

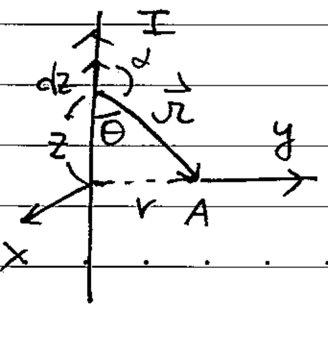
$$\mu_0 = 4\pi \times 10^{-7} \frac{N}{A^2}$$

Eq.(16) is the Biot-Savart law.
For general current distributions, eq.(16) is replaced by:

surface current: $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, da'$ --(17)

Volume current: $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$ --(18)

Example: Find $\vec{B}$ at A which is at a distance of r from a long straight wire carrying current I




Consider a segment at z,

$$|\vec{\mathscr{r}}| = \sqrt{z^2 + r^2}$$

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$$I\vec{dl} \times \hat{\mathscr{r}} = I\, d\vec{l} \times \hat{\mathscr{r}} = I(dz\,\hat{z}) \times \hat{\mathscr{r}}$$

$$= I \sin\alpha\, dz\, (-\hat{x})$$

$$= I \sin\theta\, dz\, (-\hat{x}) = I \frac{r}{\sqrt{r^2+z^2}}\, dz\, (-\hat{x})$$

$$\therefore\ d\vec{B} = (-\hat{x}) \frac{\mu_0}{4\pi} \frac{Ir}{(r^2+z^2)^{3/2}}\, dz$$

$$\therefore\ \vec{B} = (-\hat{x}) \frac{\mu_0 I r}{4\pi} \int_{-\infty}^{\infty} \frac{dz}{(r^2+z^2)^{3/2}}$$

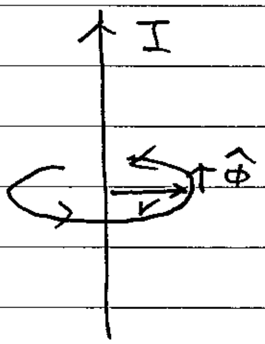
set $z = r\tan\theta$, $dz = r\sec^2\theta\, d\theta$

$$\therefore\ \vec{B} = (-\hat{x}) \frac{\mu_0 I r}{4\pi} \frac{r}{r^3} \underbrace{\int_{-\pi/2}^{\pi/2} \frac{\sec^2\theta}{\sec^3\theta}\, d\theta}$$

$$\int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta = \sin\theta \Big|_{-\pi/2}^{\pi/2} = 2$$

$$\therefore\ \vec{B} = \frac{\mu_0 I}{2\pi r} (-\hat{x})$$

Generally, one has

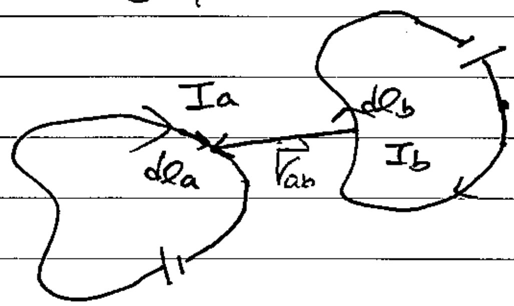



$$\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi} \quad \text{--- (19)}$$

Ampere's right-hand rule

## Forces on wires carrying steady currents

Combining the Biot-Savart law & eqs. (5) & (11) (Lorentz force), one can find that there is a force between two wires carrying $I_a$ & $I_b$.
currents of




The magnetic field generated by $d\vec{\ell}_b$ ($I_b$) at $d\vec{\ell}_a$ is

$$d\vec{B}_{ab} = \frac{\mu_0}{4\pi} I_b \frac{d\vec{\ell}_b \times \hat{r}_{ab}}{r_{ab}^2}$$

$\therefore$ $d\vec{F}_{ab}$ (acting on $d\vec{\ell}_a$)

$$= I_a d\vec{\ell}_a \times d\vec{B}_{ab}$$

$$= \frac{\mu_0}{4\pi} I_a I_b \frac{d\vec{\ell}_a \times (d\vec{\ell}_b \times \hat{r}_{ab})}{r_{ab}^2}$$

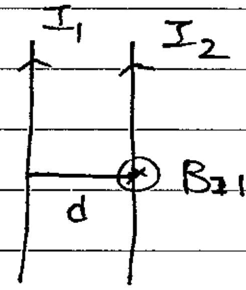
$\therefore$ Force that acts on wire a due to wire b is

$$\vec{F}_{ab} = \frac{\mu_0}{4\pi} I_a I_b \oint_a \oint_b \frac{d\vec{\ell}_a \times (d\vec{\ell}_b \times \hat{r}_{ab})}{r_{ab}^2} \quad \text{--- (20)}$$

Similarly, force exerted by current $I_a$ on current $I_b$ is

$$\vec{F}_{ba} = \frac{\mu_0}{4\pi} I_a I_b \oint_a \oint_b \frac{d\vec{l}_b \times (d\vec{l}_a \times \hat{r}_{ba})}{r_{ab}^2} \quad \text{--- (21)}$$

Example




$$B_{21} = \frac{\mu_0 I_1}{2\pi d}$$

$$\therefore F_{21} = I_2 B_{21} \int dl_2$$

$$\therefore \frac{F_{21}}{L_2} = I_2 B_{21} = \frac{\mu_0 I_1 I_2}{2\pi d}$$

Similarly, $F_{12} = I_1 B_{12} \int dl_1$

$$\frac{F_{12}}{L_1} = I_1 B_{12} = \frac{\mu_0 I_1 I_2}{2\pi d} = \frac{F_{21}}{L_2}$$

$F_{12}$ & $F_{21}$ were used to define units of current / thus charges

Note that for any two current segments $d\vec{l}_a$ & $d\vec{l}_b$,

$$d\vec{F}_{ab} = \frac{\mu_0}{4\pi} I_a I_b \frac{d\vec{l}_a \times (d\vec{l}_b \times \hat{r}_{ab})}{r_{ab}^2}$$

$$d\vec{F}_{ba} = \frac{\mu_0}{4\pi} I_a I_b \frac{d\vec{l}_b \times (d\vec{l}_a \times \hat{r}_{ba})}{r_{ab}^2}$$

roles of a & b are not symmetric :

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$\frac{d\vec{l}_a\times(d\vec{l}_b\times\hat{r}_{ab})}{r_{ab}^2} = \frac{1}{r_{ab}^2}\left[d\vec{l}_b(d\vec{l}_a\cdot\hat{r}_{ab}) - \hat{r}_{ab}(d\vec{l}_a\cdot d\vec{l}_b)\right]$$

$$\frac{d\vec{l}_b\times(d\vec{l}_a\times\hat{r}_{ba})}{r_{ab}^2} = \frac{1}{r_{ab}^2}\left[d\vec{l}_a(d\vec{l}_b\cdot\hat{r}_{ba}) - \hat{r}_{ba}(d\vec{l}_a\cdot d\vec{l}_b)\right]$$

$$\therefore \frac{d\vec{l}_a\times(d\vec{l}_b\times\hat{r}_{ab})}{r_{ab}^2} + \frac{d\vec{l}_b\times(d\vec{l}_a\times\hat{r}_{ba})}{r_{ab}^2} \quad (\text{using } \hat{r}_{ab}+\hat{r}_{ba}=0)$$

$$= \frac{1}{r_{ab}^2}\left[d\vec{l}_b(d\vec{l}_a\cdot\hat{r}_{ab}) - d\vec{l}_a(d\vec{l}_b\cdot\hat{r}_{ab})\right] \neq 0 \quad \text{--- (22)}$$

$$\therefore d\vec{F}_{ab} + d\vec{F}_{ba} \neq 0 \text{ in general. --- (23)}$$

That is, forces between any two segments don't obey Newton's 3rd law.

However, for each term in eq. (22),

$$\because \frac{d\vec{l}_a\cdot\hat{r}_{ab}}{r_{ab}^2} = -d\vec{r}_a\cdot\nabla_a\frac{1}{r_{ab}}$$

$$\left(\frac{\partial}{\partial x_a}\frac{1}{r_{ab}}, \frac{\partial}{\partial y_a}\frac{1}{r_{ab}}, \frac{\partial}{\partial z_a}\frac{1}{r_{ab}}\right)$$

$$= -\underbrace{d_a}_{\text{change along } a}\frac{1}{r_{ab}}$$

$$\therefore \oint_a\oint_b \frac{d\vec{l}_b(d\vec{l}_a\cdot\hat{r}_{ab})}{r_{ab}^2}$$

$$= -\oint_b\oint_a d_a\frac{1}{r_{ab}} = 0.$$

Similarly $\frac{d\vec{l}_b\cdot\hat{r}_{ab}}{r_{ab}^2} = d\vec{r}_b\cdot\nabla_b \frac{1}{r_{ab}} = d_b\frac{1}{r_{ab}}$

$$\oint_b\oint_a \frac{d\vec{l}_a(d\vec{l}_b\cdot\hat{r}_{ab})}{r_{ab}^2} = 0$$

$$\therefore\ \vec{F}_{ab}+\vec{F}_{ba}=0 \quad --- (24)$$

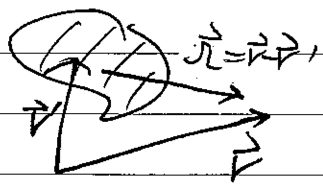
Total forces between wires obey Newton's 3rd law.

<u>Divergence of $\vec{B}$</u>

From eq. (18), $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$

& $\vec\nabla\cdot\vec{J}(\vec{r}')=0$, one can evaluate $\vec\nabla\cdot\vec{B}$.

$$\vec\nabla\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \vec\nabla\cdot\left(\frac{\vec{J}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right) d\tau'$$




$\because\ \vec\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec\nabla\times\vec{A}) - \vec{A}\cdot(\vec\nabla\times\vec{B})$

$\vec\nabla\times\vec{J}(\vec{r}') = 0$ ( $\vec\nabla$ acts on $\vec{r}$.)

$$\therefore\ \vec\nabla\cdot\left(\frac{\vec{J}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = -\vec{J}(\vec{r}')\cdot\vec\nabla\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\vec\nabla\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \vec\nabla\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \left[\vec\nabla\times(\vec{r}-\vec{r}')\right]\frac{1}{|\vec{r}-\vec{r}'|^3} + \left(\nabla\frac{1}{|\vec{r}-\vec{r}'|^3}\right)\times(\vec{r}-\vec{r}')$$

$\left(\because \nabla\times(f\vec{A}) = \nabla f\times\vec{A} + f\nabla\times\vec{A}\right)$

$$\vec{\nabla}\times(\vec{r}-\vec{r}\,') = \vec{\nabla}\times\vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

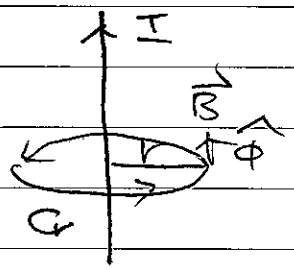
$$\nabla\frac{1}{r^3} = \frac{d}{dr}\left(\frac{1}{r^3}\right)\hat{r}$$

$$\therefore \nabla\frac{1}{|\vec{r}-\vec{r}\,'|^3} \parallel (\vec{r}-\vec{r}\,') \quad \therefore \nabla\frac{1}{|\vec{r}-\vec{r}\,'|^3}\times(\vec{r}-\vec{r}\,') = 0$$

Hence. $\vec{\nabla}\cdot\vec{B} = 0$ ...... (25)

That is, there is no magnetic charges.
in contrast to static $\vec{E}$ field.

## Ampere's law & curl of $\vec{B}$

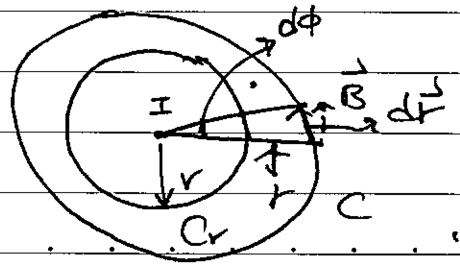



For a straight wire carrying I, one has $\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$

On the circle of radius r, it implies

$$\oint_{C_r}\vec{B}\cdot d\vec{r} = \int_0^{2\pi}\frac{\mu_0 I}{2\pi r}\, r\, d\phi$$

$$= \mu_0 I$$




For a general curve C,

$\because d\vec{r} = dr\,\hat{r} + r\,d\phi\,\hat{\phi}$

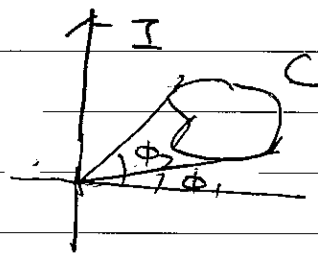
$$\therefore \quad \vec{B}\cdot d\vec{r} = \frac{\mu_0}{2\pi}\frac{I}{r}\hat{\phi}\cdot d\vec{r}$$

$$= \frac{\mu_0}{2\pi}\frac{I}{r}\, r\, d\phi$$

$$\therefore \oint_C \vec{B}\cdot d\vec{r} = \frac{\mu_0 I}{2\pi}\int_0^{2\pi} d\phi = \mu_0 I \quad \text{-- (26)}$$

is true for general close curve.

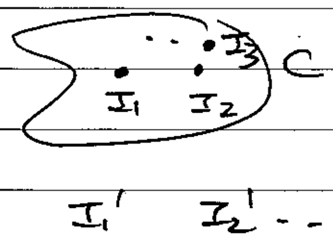
If $C$ doesn't enclose $I$, one gets




$$\oint_C \vec{B}\cdot d\vec{r} = \frac{\mu_0 I}{2\pi}\left(\int_{\phi_1}^{\phi_2} d\phi + \int_{\phi_2}^{\phi_1} d\phi\right) = 0$$

-- (27)

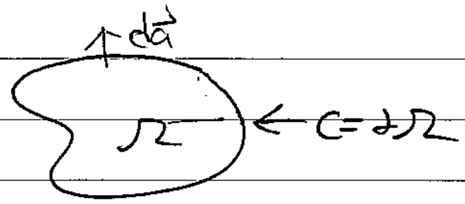
Using the principle of superposition, the above eqs. (26) & (27) imply




$$\oint_C \vec{B}\cdot d\vec{r} = \mu_0(I_1 + I_2 + \cdots)$$

$$= \mu_0 I_{enclosed} \quad \text{--- (28)}$$

If we denote $C$ as the boundary of $\Omega$,




$I_{enclosed}$

$= \underbrace{\text{current}}_{\text{total}}$ passing through

$\Omega$

$= \int_{\Omega} \vec{J} \cdot d\vec{a}$

Where the direction of $d\vec{a}$ is determined by Ampere's right-hand rule.

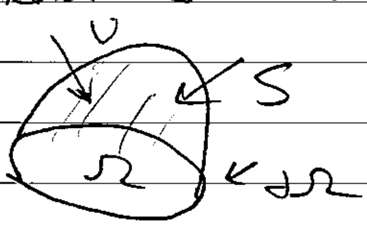
Eq. (28) becomes

$$\oint_{\partial\Omega} \vec{B} \cdot d\vec{r} = \mu_0 \int_{\Omega} \vec{J} \cdot d\vec{a}$$

(counterclockwise)

$$= \int_{S} \mu_0 \vec{J} \cdot d\vec{a} \quad \cdots \quad (29)$$

where $S$ = any surface bounded by $\partial\Omega$




$\because \vec{\nabla} \cdot \vec{J} = 0$

$$\int_{\Omega} \vec{J} \cdot (-d\vec{a}) + \int_{S} \vec{J} \cdot d\vec{a}$$

$$= \int_{V} \vec{\nabla} \cdot \vec{J} \, d\tau = 0$$

$\therefore \int_{\Omega} \vec{J} \cdot d\vec{a} = \int_S \vec{J} \cdot d\vec{a}$

Which is the consequence of current conservation.

$\because \oint_{\partial\Omega} \vec{B} \cdot d\vec{r} = \int_{\Omega} \vec{\nabla} \times \vec{B} \cdot d\vec{a}$ by Stoke's theorem

$$= \int_S \vec{\nabla} \times \vec{B} \cdot d\vec{a}$$

$$(\because \nabla \cdot (\vec{\nabla} \times \vec{B}) = 0)$$

$\therefore$ Eq. 29 becomes

$$\int_S \vec{\nabla} \times \vec{B} \cdot d\vec{a} = \mu_0 \int_S \vec{J} \cdot d\vec{a} \quad \cdots \ 30$$

for any surface $S$.

Taking $S$ as a small area $d\vec{a}$ around a given point $\vec{r}$, eq. 30 implies

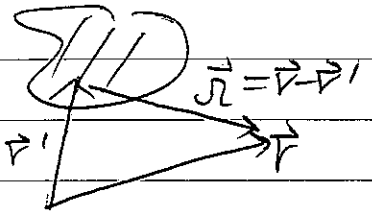
$$\vec{\nabla} \times \vec{B} \cdot d\vec{a} = \mu_0 \vec{J} \cdot d\vec{a}$$

for any $d\vec{a}$, $\therefore \ \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \cdots \ 31$

This is the Ampere's law in differential form. Eq. 29 is the integral form.

Direct check:

From $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$,




$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\left(\frac{\vec{J}(\vec{r}\,')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

Using $\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A})$

& realizing $\vec{\nabla}\cdot\vec{J}(\vec{r}\,')=0$ ($\vec{J}(\vec{r}\,')$ is independent of $\vec{r}$)

$$(\vec{A}\cdot\nabla)\vec{J}(\vec{r}\,') = 0$$

$$\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 4\pi\delta^3(\vec{\mathscr{r}}) \quad \left[\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho \qquad \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathscr{r}}}{\mathscr{r}^2},\ \rho = \delta(\vec{\mathscr{r}})\right]$$

we have

$$\vec{\nabla}\times\left(\frac{\vec{J}(\vec{r}\,')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \vec{J}(\vec{r}\,')\underbrace{\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}}_{4\pi\delta^3(\vec{\mathscr{r}})} - (\vec{J}(\vec{r}\,')\cdot\vec{\nabla})\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\therefore\ \vec{\nabla}\times\vec{B} = \mu_0\int \vec{J}(\vec{r}\,')\,\delta^3(\vec{r}-\vec{r}\,')\,d\tau' - \frac{\mu_0}{4\pi}\int(\vec{J}(\vec{r}\,')\cdot\vec{\nabla})\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau' \quad \cdots (32)$$

Now, $\because \dfrac{\partial}{\partial x_i} \dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \dfrac{\partial}{\partial x_i} \dfrac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$

$$\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \left( \frac{x-x'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}, \frac{y-y'}{\sqrt{\;''}}, \frac{z-z'}{\sqrt{\;''}} \right)$$

in which $\underset{\underset{\Delta x_i}{\shortparallel}}{x_i - x_i'}$ appears together

$$\therefore \frac{\partial}{\partial x_i} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\partial}{\partial x_i} f(\Delta x_1, \Delta x_2, \Delta x_3)$$

$$= \frac{\partial}{\partial \Delta x_i} f(\Delta x_1, \Delta x_2, \Delta x_3) \underbrace{\frac{\partial \Delta x_i}{\partial x_i}}_{1}$$

$$= \frac{\partial}{\partial \Delta x_i} f(\Delta x_1, \Delta x_2, \Delta x_3) \left(-\frac{\partial \Delta x_i}{\partial x_i'}\right)$$

$$= -\frac{\partial f}{\partial x_i'}$$

$$\therefore \frac{\partial}{\partial x_i} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = -\frac{\partial}{\partial x_i'} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\therefore (\vec{J}(\vec{r}') \cdot \vec{\nabla}) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = -(\vec{J}(\vec{r}') \cdot \vec{\nabla}') \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$= -\vec{\nabla}' \cdot \left[ \vec{J}(\vec{r}') \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right] + \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \cancelto{0}{\vec{\nabla}' \cdot \vec{J}'}$$

$$\therefore \int_V (\vec{J}(\vec{r}') \cdot \vec{\nabla}) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau' = -\int_V \vec{\nabla}' \cdot \left( \vec{J}(\vec{r}') \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau'$$

$$= -\int_S \vec{J}(\vec{r}') \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} d\vec{a}$$ by taking V large

$V/S$

$\vec{J}=0$ on $S$ $\therefore$ $\int_V (\vec{J}\cdot\vec{\nabla})\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau' = 0$

$\therefore$ Eq. (32) becomes

$$\vec{\nabla}\times\vec{B} = \mu_0 \int \vec{J}(\vec{r}')\,\delta^3(\vec{r}-\vec{r}')\,d\tau'$$

$$= \mu_0 \vec{J}(\vec{r})$$

confirming eq. (31), the Ampere's law.

The Ampere's law plays a role in magnetostatics that is similar to the role played by the Gauss's law.

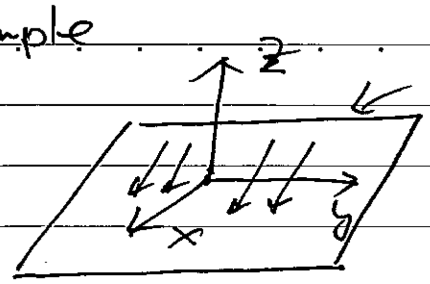
We have the following useful comparison:

Electrostatics : Coulomb law $\rightarrow$ Gauss

Magnetostatics : Biot-Savart $\rightarrow$ Ampere

Hence for current distributions with particular symmetries, one can use the integral form of the Ampere's law to find $\vec{B}$ just as we use the Gauss's law to find $\vec{E}$ for symmetric charge distributions.

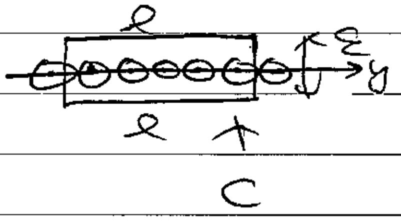
Example




uniform surface current $\vec{K} = K\hat{x}$.

Find $\vec{B}$.

Solution: Construct a Amperian loop as a rectangular loop.




By symmetry, for $z>0$

$\vec{B} \parallel \hat{y}$, $\vec{B} = B\hat{y}$, $B<0$

$z<0$. $B>0$, $|B(z)| = |B(-z)|$

$$\therefore \oint_C \vec{B}\cdot d\vec{l} = Bl + Bl = \mu_0 I(\text{in } l)$$

$$= \mu_0 K \cdot l$$

$$\therefore 2B\cancel{l} = \mu_0 K \cancel{l}, \quad B = \frac{\mu_0}{2}K$$

$$\therefore \vec{B} = \frac{\mu_0}{2}K\hat{y}, \quad z<0$$

$$= -\frac{\mu_0}{2}K\hat{y}, \quad z>0$$

Example A toroidal coil consists of a donut-like ring around which a long wire is wrapped.

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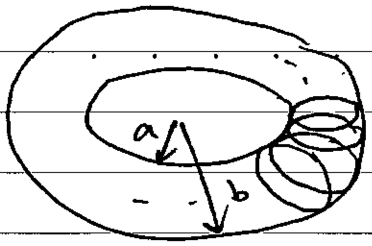
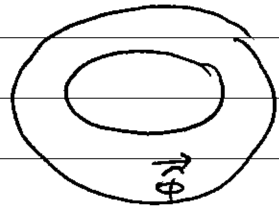



Assuming that the winding is uniform & tight enough so that each turn can be viewed as a close loop.
Shape of cross section is fixed.

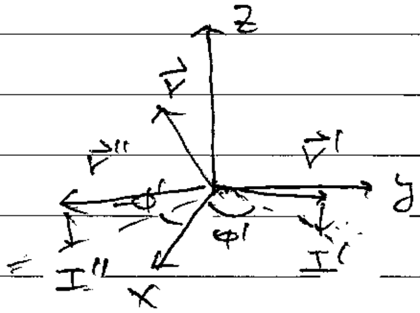
Find $\vec{B}$ inside.

Solution: Regardless the shape of cross sections, $\vec{B}$ is along $\hat{\phi}$ direction through symmetry consideration:




$\vec{r}$ lies on $xz$ plane

$\vec{r} = (x, 0, z)$




$\vec{r}' = (\rho'\cos\phi', \rho'\sin\phi', z')$

$\vec{r}'' = (\rho'\cos\phi', -\rho'\sin\phi', z')$

$\vec{\mathscr{r}}' = \vec{r} - \vec{r}'$

$= (x - \rho'\cos\phi', -\rho'\sin\phi', z - z')$

$\vec{\mathscr{r}}'' = \vec{r} - \vec{r}''$

$= (x - \rho'\cos\phi', \rho'\sin\phi', z - z')$

Current lies in $z$ & $\rho$ plane

$\therefore$ There is no $\phi$ component.

$\vec{I}' = I_\rho\hat{\rho} + I_z\hat{z}$ , $\vec{I}'' = I_\rho\hat{\rho}' + I_z\hat{z}'$

$\hat{\rho} = (\cos\phi', \sin\phi', 0)$ $\hat{\rho}' = (\cos\phi', -\sin\phi', 0)$

$\therefore \vec{I}' = (I_\rho \cos\phi', I_\rho \sin\phi', I_z)$

$\vec{I}'' = (I_\rho \cos\phi', -I_\rho \sin\phi', I_z)$

$$\vec{I}' \times \vec{\mathcal{R}}' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ I_\rho \cos\phi' & I_\rho \sin\phi' & I_z \\ x-\rho'\cos\phi' & -\rho'\sin\phi' & z-z' \end{vmatrix}$$

$$= \left(\sin\phi'\left(I_\rho(z-z')+\rho' I_z\right),\ I_z(x-\rho'\cos\phi') - I_\rho\cos\phi'(z-z'),\ -I_\rho \sin\phi' x\right)$$

$\vec{I}'' \times \vec{\mathcal{R}}'' = \vec{I}' \times \vec{\mathcal{R}}' \ (\phi' \to -\phi')$

$\therefore$ x & z components get cancelled.

Only y component survives. For $\vec{r}$ in xz plane, $\hat{y} \parallel \hat{\phi}$. $\therefore$ $\vec{B} \parallel \hat{\phi}$.

By taking a circle of radius $r$ as the Amperian loop, $\oint \vec{B} \cdot d\vec{r} = 2\pi r B = \mu_0 I_{enclose}$

$= \mu_0 \times N \times I$

$N =$ total # of turns $\therefore \vec{B} = \dfrac{\mu_0 N I}{2\pi r} \hat{\phi}$ for $a < r < b$

$= 0$ outside

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Magnetostatics vs electrostatics

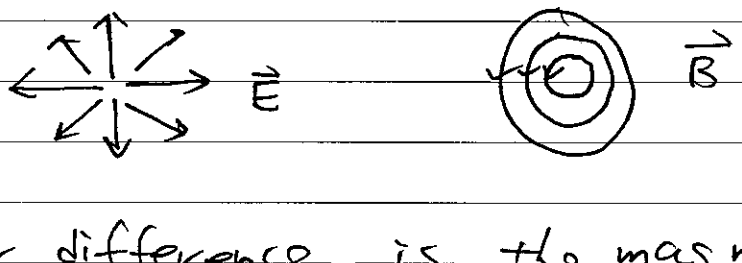
$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0}\rho$ (Gauss's law) $\vec{\nabla} \cdot \vec{B} = 0$

$\nabla \times \vec{E} = 0$ $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ (Ampere's law)

$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ Lorentz force

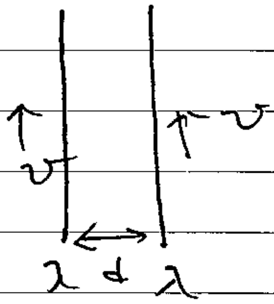
Clearly, the big difference is that there is no magnetic charges (now called magnetic monopole.).

All $\vec{B}$ fields have to be generated by currents $\vec{J}$, which satisfies $\vec{\nabla} \cdot \vec{J} = 0$ in steady state. There is no divergence in $\vec{B}$ force lines of $\vec{B}$ can not end at a point.




Another difference is the magnitudes of $\vec{E}$ & $\vec{B}$: $E \gg B$ in most cases when the velocity of particles $\ll c$ (speed of light.)

Example : two lines with line charge density $\lambda$. Charges move with speed $v$.



$$E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{d}$$

$\therefore$ Electrostatic force per unit length $f_E = \dfrac{1}{2\pi\epsilon_0}\dfrac{\lambda^2}{d}$

On the other hand, $B = \dfrac{\mu_0 I}{2\pi d}$ $\quad I = \lambda v$

$$\therefore\ f_B = \frac{\mu_0 I^2}{2\pi d} = \frac{\mu_0 v^2}{2\pi}\frac{\lambda}{d}$$

$f_E = f_B$ only when $\mu_0 v^2 = \dfrac{1}{\epsilon_0}$

$$\text{i.e. } v^2 = \frac{1}{\mu_0\epsilon_0} = \frac{1}{8.85\times10^{-12}\times4\pi\times10^{-7}}$$

$$= 9\times10^{16}\,(m/s)^2 = c^2$$

$\therefore$ For $v \ll c$, $f_B \ll f_E$.

## Magnetic vector potential

The divergenceless of $\vec{B}$

$$\vec{\nabla}\cdot\vec{B} = 0$$

implies that one can write

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad \cdots (33)$$

This is the theorem 2 in sec 1.6.2.

To show it, we will construct a $\vec{A}$ such that $\vec{\nabla}\times\vec{A}=\vec{B}$

with $\vec{\nabla}\cdot\vec{B}=\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=0$

Solution: we shall take

$$A_x=0$$

$$A_y=\int_0^x B_z(x',y,z)\,dx'$$

$$A_z=-\int_0^x B_y(x',y,z)\,dx'+\int_0^y B_x(0,y',z)\,dy'$$

verify: $(\vec{\nabla}\times\vec{A})_y=\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}$

$$=-\frac{\partial A_z}{\partial x}=B_y(x,y,z)$$

$$(\vec{\nabla}\times\vec{A})_z=\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}$$

$$=\frac{\partial A_y}{\partial x}=B_z(x,y,z)$$

$$(\vec{\nabla}\times\vec{A})_x=\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}$$

$$=-\int_0^x\frac{\partial B_y(x',y,z)}{\partial y}dx'+B_x(0,y,z)$$

$$-\int_0^x\frac{\partial B_z(x',y,z)}{\partial z}dx'$$

$\because\ \vec{\nabla}\cdot\vec{B}=0\ \therefore\ \frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=-\frac{\partial B_x(x',y,z)}{\partial x'}$

$$\therefore (\vec{\nabla}\times\vec{A})_x = \int_0^x \frac{\partial B_x(x',y,z)}{\partial x'}dx' + B_x(0,y,z)$$

$$= B_x(x,y,z) - \cancel{B_x(0,y,z)} + \cancel{B_x(0,y,z)}$$

$\therefore (\vec{\nabla}\times\vec{A}) = \vec{B}$ is satisfied!

$\vec{A}$ is known as vector potential

Just as $\vec{E} = -\nabla V$, $\vec{B} = \nabla\times\vec{A}$, there is no absolute meaning of $V$ & $\vec{A}$.

One can shift $V \to V + \text{const}$ so that $\vec{E}$ is the same.

Similarly, $\vec{A} \to \vec{A} + \nabla\chi$, $\because \nabla\times\nabla\chi = 0$ --(34)

$\therefore \vec{\nabla}\times(\vec{A}+\nabla\chi) = \vec{B}$ gives the same $\vec{B}$

$V$ & $\vec{A}$ are known as gauge potentials (or gauge fields).

<u>Coulomb gauge</u> & <u>finding $\vec{A}$</u>

How do we determine $\vec{A}$?

Just as the electrostatic case, where $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ determines $V$: $\nabla^2 V = -\rho/\varepsilon_0$ with the replacement

$\vec{E} = -\nabla U$, here we substituting

$\vec{B} = \nabla \times \vec{A}$ into the Ampere's law

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

$\therefore \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J}$

$\because \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$

$\therefore$ We obtain

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} \quad \cdots (35)$$

As we have freedom to add $\nabla \chi$ to $\vec{A}$ without changing $\vec{B}$, we can choose $\vec{\nabla} \cdot \vec{A} = 0 \quad \cdots (36)$

Eq. (36) is known as the Coulomb gauge (condition).

This is always possible: suppose originally $\vec{A}_0$ doesn't satisfy $\vec{\nabla} \cdot \vec{A}_0 = 0$

One can define $\vec{A} = \vec{A}_0 + \nabla \chi$ and choose $\chi$ such that $\nabla^2 \chi = -\nabla \cdot \vec{A}_0 \quad \cdots (37)$

As a result, $\vec{\nabla} \cdot \vec{A} = 0$ is satisfied.

Eq. (37) is an electrostatic problem with

$\vec{\nabla}\cdot\vec{A}_0 \Leftrightarrow \rho/\epsilon_0$ and $\chi \leftrightarrow V$

$\therefore$ One can always find $\chi$ as ~~for~~ a given charge-distribution $(\vec{\nabla}\cdot\vec{A}_0)$, there is always a corresponding electric potential $\chi$:

If $\vec{\nabla}\cdot\vec{A}_0 \to 0$ as $r\to\infty$, clearly

$$\chi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')}{|\vec{r}-\vec{r}'|}d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\epsilon_0\vec{\nabla}'\cdot\vec{A}_0(\vec{r}')}{|\vec{r}-\vec{r}'|}d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\vec{\nabla}'\cdot\vec{A}_0(\vec{r}')}{|\vec{r}-\vec{r}'|}d\tau'$$

Therefore, the Coulomb condition eq. (36) can be always satisfied.

Hence eq. (35) becomes

$$\nabla^2\vec{A} = -\mu_0\vec{J} \quad \cdots \quad (38)$$

Which is similar to the Poisson equation

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho$$

except that there are 3 equations for x, y & z components separately! $\nabla^2 A_x = -\mu_0 J_x$
$\nabla^2 A_y = -\mu_0 J_y$
$\nabla^2 A_z = -\mu_0 J_z$

Assuming $\vec{J} \to 0$ as $r \to \infty$, we get

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} d\tau' \quad \cdots \text{(39)}$$

Therefore, once $\vec{J}$ is known, one gets $\vec{A}(\vec{r})$.

Remark : $\vec{A}$ plays the role to $\vec{B}$ in the same as $V$ to $\vec{E}$.

However, there is a difference in meaning : $\vec{B}$ does not do work on charged particles, $\therefore$ $\vec{A}$ is not associated with energy directly.

If the currents are confined to surface & lines, eq (39) needs to be replaced by

surface : $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}')}{|\vec{r}-\vec{r}'|} da' \quad \cdots \text{(40)}$

line : $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I}(\vec{r}')\, dl'}{|\vec{r}-\vec{r}'|} \quad \cdots \text{(41)}$

No.
Date.


Check: From eq. (38) one can check

$$\vec{B} = \nabla \times \vec{A} = \frac{\mu_0}{4\pi} \int \nabla \times \left( \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \right) dz'$$

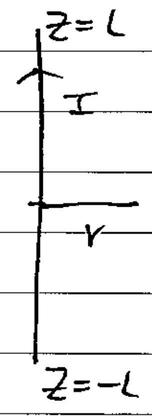
$$\because \nabla \times \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} = \left( \nabla \frac{1}{|\vec{r}-\vec{r}\,'|} \right) \times \vec{J}(\vec{r}\,')$$

$$= -\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \times \vec{J}(\vec{r}\,')$$

$$\therefore \vec{B} = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}\,') \times \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} dz'$$

reproduces the Biot-Savart law.

Example $\vec{A}$ mimics the direction of current.




A long wire of length L carrying I, find $\vec{A}$ due to I at a distance r from the wire, in the limit $L \gg r$.

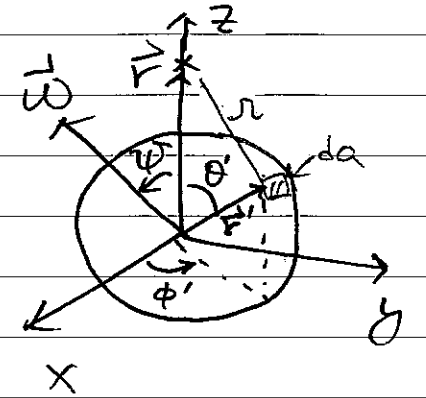
Solution: $\because I d\vec{r} \parallel \hat{z}$, $\therefore \vec{A} = A_z \hat{z}$

with $$A_z = \int_{-L}^{L} \frac{\mu_0 I \, dz}{4\pi (r^2+z^2)^{1/2}}$$

$$= \frac{\mu_0 I}{4\pi} \ln \left[ z + (r^2+z^2)^{\frac{1}{2}} \right]_{z=-L}^{z=L}$$

$$= \frac{\mu_0 I}{4\pi} \ln \left[ \frac{(1+r^2/L^2)^{1/2}+1}{(1+r^2/L^2)^{1/2}-1} \right] \xrightarrow[r \ll L]{} \frac{\mu_0 I}{4\pi} \ln \left( 1 + \frac{4L^2}{r^2} \right)$$

Example : A spherical shell of radius $R$, carrying a uniform surface charge $\sigma$, is spinning at an angular velocity $\vec{\omega}$. Find the vector potential $\vec{A}$ at $\vec{r}$.




Solution: It's convenient to put $\vec{r} \parallel \hat{z}$.

$\vec{\omega}$ in the $xz$ plane with an angle $\psi$ relative to $z$ axis. (see the above figure)

Now, the surface current at $\vec{r}\,'$

$$\vec{K} = \sigma \vec{v} \quad \text{with} \quad \vec{v} = \vec{\omega} \times \vec{r}\,'$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega \sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$= R\omega\left(-\sin\theta'\sin\phi'\cos\psi,\ -\sin\psi\cos\theta' + \cos\psi\sin\theta'\cos\phi',\ \sin\theta'\sin\phi'\sin\psi\right) \quad \text{--- (42)}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}\,')\, da'}{\mathscr{r}}$$

with $\mathscr{r} = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$, $da' = R^2 \sin\theta'\, d\theta'\, d\phi'$

$$\because \int_0^{2\pi} \sin\phi'\, d\phi' = 0 = \int_0^{2\pi} \cos\phi'\, d\phi'$$

$\therefore$ Only $y$ component in eq. (42) survives

$$\therefore \vec{A}(\vec{r}) = -\frac{\mu_0}{4\pi} R^3 \omega \sigma \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \frac{\cos\theta' \sin\psi}{\mathscr{r}} \hat{y}$$

$$= -\frac{1}{2}(\mu_0 R^3 \sigma \omega \sin\psi) \int_{-1}^{1} \frac{x\,dx}{\sqrt{R^2+r^2-2Rrx}} \qquad x = \cos\theta'$$

$$\because \int \frac{x\,dx}{\sqrt{R^2+r^2-2Rrx}} = \int x \frac{d\sqrt{R^2+r^2-2Rrx}}{-Rr}$$

$$= -\frac{x}{Rr}\sqrt{R^2+r^2-2Rrx} + \int \sqrt{R^2+r^2-2Rrx} \frac{dx}{Rr}$$

$$= -\frac{x}{Rr}\sqrt{R^2+r^2-2Rrx} - \frac{1}{2Rr}\frac{1}{Rr}\cdot\frac{2}{3}(R^2+r^2-2Rrx)^{3/2}$$

$$= -\sqrt{R^2+r^2-2Rrx}\cdot\frac{R^2+r^2+Rrx}{3R^2r^2}$$

$$\therefore I = \int_{-1}^{1} \frac{x\,dx}{\sqrt{R^2+r^2-2Rrx}} = -\sqrt{R^2+r^2-2Rrx}\,\frac{r^2+R^2+Rrx}{3R^2r^2}\Bigg|_{x=-1}^{x=1}$$

$$= -\frac{1}{3R^2r^2}\left[(R^2+r^2+Rr)|R-r| - (R^2+r^2-Rr)(R+r)\right]$$

$R>r, \qquad I = \frac{2r}{3R^2}$

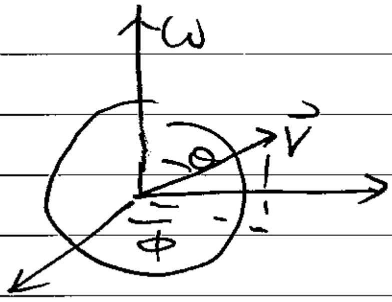
$R<r, \qquad I = \frac{2R}{3r^2}$

Together with $\vec{\omega}\times\vec{r} = -\omega r \sin\psi\,\hat{y}$, we get

$$\vec{A}(\vec{r}) = \begin{cases} \frac{\mu_0 R\sigma}{3}(\vec{\omega}\times\vec{r}) & r<R \\ \frac{\mu_0 R^4\sigma}{3r^3}(\vec{\omega}\times\vec{r}) & r>R \end{cases}$$

If we now use $\vec{\omega}$ as z-axis, $\vec{r}$

is represented by $(r, \theta, \phi)$




$$\vec{A}(r,\theta,\phi) = \begin{cases} \dfrac{\mu_0 R \omega \sigma}{3} r\sin\theta\,\hat{\phi} & r \le R \\ \dfrac{\mu_0 R^4 \omega \sigma}{3} \dfrac{\sin\theta}{r^2}\hat{\phi}, & r \geq R \end{cases}$$

$$\therefore \vec{B} = \vec{\nabla}\times\vec{A} = \frac{2\mu_0 R\omega\sigma}{3}(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$

$$= \frac{2}{3}\mu_0 \sigma R\omega\hat{z}$$

$$= \frac{2}{3}\mu_0 \sigma R\vec{\omega} \quad \text{for } r < R$$

(= constant)

$$= \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & \frac{\mu_0 R\omega\sigma}{3} r\sin\theta \times \underbrace{r\sin\theta}_{h_\phi} \end{vmatrix}$$

Example. Find the vector potential of an infinite solenoid with n turns per unit length, radius R and current I

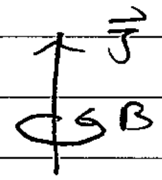

Solution: It's not convenient to use eq. (41) as the current extends to $\infty$.

Instead, $\because$ $\hat{B} = \mu_0 n I \hat{z}$, one considers

Amperian loop : circles of radius $r$

$$\because \overset{(i)}{\oint_{C_r}} \vec{A} \cdot d\vec{\ell} = \int \underset{\underset{\vec{\nabla}\times\vec{A}}{\parallel}}{\vec{B}} \cdot d\vec{a} = \Phi_B = \mu_0 n I \cdot \pi r^2 \quad r \leq R$$

(ii) $\vec{\nabla}\times\vec{A} = \vec{B}$ is similar to $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$




$\therefore \vec{A} \parallel \hat{\phi}$, $\vec{A} = A_\phi \hat{\phi}$

Hence $\oint_{C_r} \vec{A} \cdot d\vec{\ell} = 2\pi r A_\phi = \mu_0 n I \pi r^2$

$$A_\phi = \frac{\mu_0 n I}{2} r$$

$$\vec{A} = \frac{\mu_0 n I}{2} r \hat{\phi} \quad \text{for } r \leq R$$

For $r > R$, $\Phi_B = \mu_0 n I \cdot \pi R^2$

$$\therefore \vec{A} = \frac{\mu_0 n I}{2} \frac{R^2}{r} \hat{\phi} \quad r > R$$

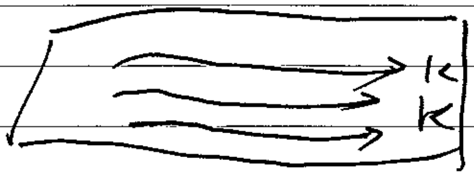
Boundary Conditions

Similar to electrostatic problems,

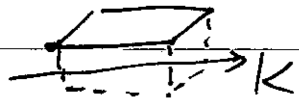
$$\left.\begin{array}{l} \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \\ \vec{\nabla}\times\vec{E} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} E_\perp^{above} - E_\perp^{below} = \frac{\sigma}{\varepsilon_0} \\ E_\parallel^{above} = E_\parallel^{below} \end{array}$$

When there are surface currents $\vec{K}$,

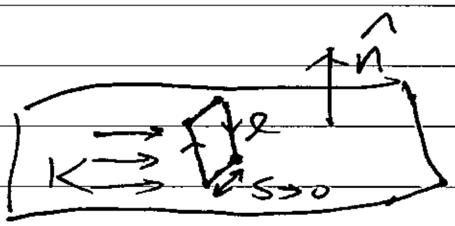
$\because \vec{\nabla}\cdot\vec{B}=0 \quad \therefore \quad B_\perp^{above} = B_\perp^{below}$ --- (43)




This is proved by taking the same pillar box




However, $\because \vec{\nabla}\times\vec{B}=\mu_0\vec{J} \neq 0 \therefore B_\parallel$ is not continuous across a surface current.




$l \perp K$

$\vec{l} \parallel \vec{K}\times\hat{n}$

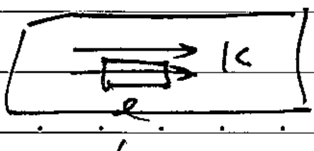
By taking an Amperian loop perpendicular to $\vec{K}$, one gets

$$\because \oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed}$$
$$= \mu_0 K l$$

$$(B^{\parallel}_{above} - B^{\parallel}_{below})\, l = \mu_0 K l$$

$$\therefore B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K \quad \text{--- (44)}$$

However, if the Amperian loop is parallel to $\vec{K}$




$\because I_{enclosed} = 0$

$\therefore B^{\parallel\prime}_{above} = B^{\parallel\prime}_{below}$ --- (45)

$l' \parallel K$,

Combining eqs. (44) & (45), one gets

$$\vec{B}_{\parallel}^{above} - \vec{B}_{\parallel}^{below} = \mu_0 \vec{K} \times \hat{n}$$

Together with eq. (43), we have

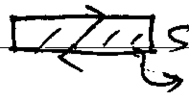
$$\vec{B}_{above} - \vec{B}_{below} = \mu_0 \vec{K} \times \hat{n} \quad \text{--- (46)}$$

In terms of $\vec{A}$, we resort to

$$\vec{\nabla} \cdot \vec{A} = 0$$
$$\vec{\nabla} \times \vec{A} = \vec{B}$$

$\vec{\nabla} \cdot \vec{A} = 0$ implies $A^{\perp}_{above} = A^{\perp}_{below}$

$\vec{\nabla} \times \vec{A} = \vec{B}$ implies $\oint \vec{A} \cdot d\vec{l} = \int \vec{B} \cdot d\vec{a}$



s. $\int \vec{B} \cdot d\vec{a} \to 0$ as $s \to 0$

For finite $\vec{B}$, $\int \vec{B} \cdot d\vec{a} \to 0$ as $s \to 0$

$\therefore \quad A^{\parallel}_{above} = A^{\parallel}_{below}$

$$\therefore \quad \vec{A}_{above} = \vec{A}_{below} \quad \text{--- (47)}$$

However, due to $\nabla^2 \vec{A} = -\mu_0 \vec{J}$,

in comparison to $\nabla^2 V = -\rho/\varepsilon_0$ where

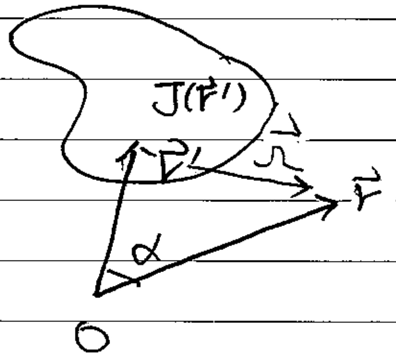
$$\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\epsilon_0}$$

$$\therefore \quad \frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K} \quad \cdots (48)$$

has to be satisfied if there is a surface current.

## Multipole expansion of $\vec{A}$

The similarity of $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} d\tau'$ and $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} d\tau'$ indicates that one can also perform multipole expansion for $\vec{A}$ if $\vec{J}(\vec{r}')$ is confined to finite region.




The expansion uses

$$\frac{1}{\mathscr{r}} = \frac{1}{(r^2 + r'^2 - 2rr'\cos\alpha)^{1/2}}$$

$$= \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\alpha) \quad \cdots (49)$$

Hence $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\alpha)\, \vec{J}(\vec{r}')\, dz'$

$\leftarrow \cdots (50)$

is the multipole expansion.

In practice, currents are <u>confined to wires</u>. We shall set $\vec{J}(\vec{r}')\, dz' = I\, d\vec{r}'$

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \underbrace{\oint}_{\text{over wire}} (r')^n P_n(\cos\alpha)\, d\vec{r}'$$

$$= \frac{\mu_0 I}{4\pi} \Big[ \frac{1}{r} \oint d\vec{r}' + \frac{1}{r^2} \oint r' \cos\alpha\, d\vec{r}' + \frac{1}{r^3} \oint (r')^2 \left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) d\vec{r}' + \cdots \Big]$$

$\leftarrow \cdots (51)$

Eq. (51) is the multipole expansion for currents flowing on a wire.

$\therefore$ The magnetic monopole term

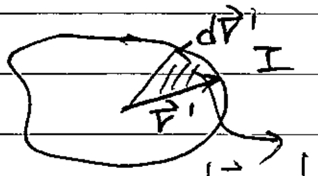
$\Rightarrow n = 0$ $\quad \frac{\mu_0}{4\pi} \frac{I}{r} \underbrace{\oint d\vec{r}'}_{= 0} = 0$

Which reflects $\vec{\nabla} \cdot \vec{B} = 0$, there is no magnetic monopole.

$n=1$, magnetic dipole.

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi}\frac{I}{r^2}\oint r'\cos\alpha\, d\vec{r}'$$

$$\because\ r'\cos\alpha = \hat{r}\cdot\vec{r}'$$



$$\hat{r}\times(\vec{r}'\times d\vec{r}') = \vec{r}'(\hat{r}\cdot d\vec{r}') - d\vec{r}'(\hat{r}\cdot\vec{r}')$$

$d\vec{a} = \frac{1}{2}\vec{r}'\times d\vec{r}'$ :

$$\therefore\ d\vec{r}'(\hat{r}\cdot\vec{r}') = \vec{r}'(\hat{r}\cdot d\vec{r}') - \hat{r}\times(\vec{r}'\times d\vec{r}') \quad \text{-- (52)}$$

Now $\vec{r}'(\hat{r}\cdot d\vec{r}') = \vec{r}'\, d(\hat{r}\cdot\vec{r}') = d[\vec{r}'(\hat{r}\cdot\vec{r}')] - (\hat{r}\cdot\vec{r}')\, d\vec{r}'$

$\uparrow$ $\hat{r}$ = fixed

$$\therefore\ \oint \vec{r}'(\hat{r}\cdot d\vec{r}') = \underbrace{\oint d(\vec{r}'(\hat{r}\cdot\vec{r}'))}_{0} - \oint \hat{r}\cdot\vec{r}'\, d\vec{r}'$$

$$= -\oint(\hat{r}\cdot\vec{r}')\, d\vec{r}'$$

$\therefore$ $\oint$ (52) yields

$$\oint d\vec{r}'(\hat{r}\cdot\vec{r}') = -\oint(\hat{r}\cdot\vec{r}')\, d\vec{r}' - 2\int \hat{r}\times d\vec{a}$$

$$\therefore\ \oint d\vec{r}'(\hat{r}\cdot\vec{r}') = -\int \hat{r}\times d\vec{a} = \int d\vec{a}\times\hat{r}$$

$$\therefore\ \vec{A}_{dip} = \frac{\mu_0}{4\pi}\frac{1}{r^2}\left(I\oint d\vec{a}\right)\times\hat{r} \equiv \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

where $\vec{m} = I\int d\vec{a} = \frac{I}{2}\oint \vec{r}'\times d\vec{r}' = I\vec{a}$

is the magnetic dipole moment ; $\vec{a}$: area of current loop.

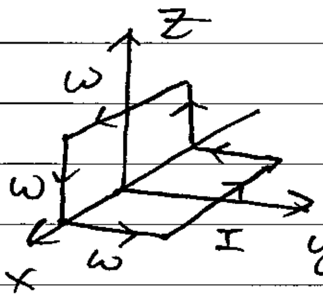
$\therefore$ $A_{dip}$ decays as $\frac{1}{r^2}$ as $r \to \infty$.

Note that $\vec{a}$ = vector area of current loop.

For a flat loop, $\vec{a}$ = ordinary area enclosed by the wire with direction assigned by the usual right-hand rule.

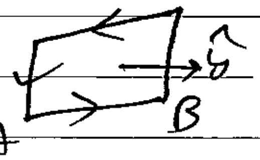
In general, if the current loop is not flat, $\vec{a}$ is not a simple area.

<u>Example</u>

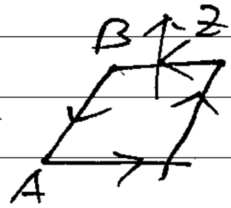



A current wire of "bookend-shape" shown in the left figure. It carries a current I. Find the magnetic dipole moment.

Solution. The loop can be viewed as a summation of




&




with currents in $\overline{AB}$ being cancelled.

$\therefore \ \vec{m} = Iw^2\hat{y} + Iw^2\hat{z}$

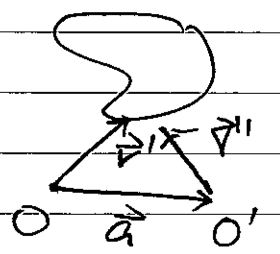
$|\vec{m}| = \sqrt{2}\, Iw^2$.

## Origin dependence

Just as electric dipole, when the monopole $Q$ vanishes, $\vec{p}$ doesn't depend on the origin of the coordinate.

Similarly, $\because$ there is no magnetic monopole, $\vec{m}$ does not depend on the origin of coordinate.

One may check this explicitly:




For O', $\vec{m}' = \frac{I}{2}\oint \vec{r}'' \times d\vec{r}''$

$\because \vec{r}'' = \vec{r}' - \vec{a}$

$d\vec{r}'' = d\vec{r}'$

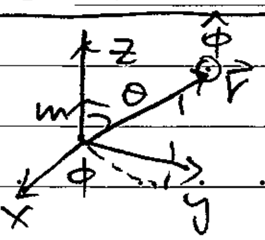
$$\therefore \vec{m}' = \frac{I}{2}\oint (\vec{r}' - \vec{a}) \times d\vec{r}'$$

$$= \frac{I}{2}\oint \vec{r}' \times d\vec{r}' - \frac{I}{2}\vec{a} \times \underbrace{\oint d\vec{r}'}_{0}$$

$$= \vec{m} \ !$$

## Fields due to a dipole

Align $\vec{m}$ to z-axis.




$\hat{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$\vec{m} \times \hat{r} = m\hat{\phi}$

$$\therefore \quad \vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$$

$$= A_\phi \hat{\phi}, \qquad A_\phi = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}$$

$$\vec{B}_{dip}(\vec{r}) = \vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & r\sin\theta A_\phi \end{vmatrix}$$

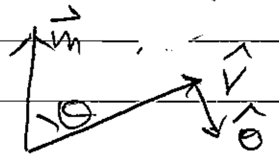
$$= \frac{1}{r^2\sin\theta}\left[-\frac{\partial}{\partial r}(r\sin\theta A_\phi)\, r\hat{\theta} + \frac{\partial}{\partial\theta}(r\sin\theta A_\phi)\,\hat{r}\right]$$

$$= \frac{\mu_0 m}{4\pi}\left[\frac{1}{r^3}2\cos\theta\,\hat{r} + \frac{1}{r^3}\sin\theta\,\hat{\theta}\right]$$

$$= \frac{\mu_0 m}{4\pi}\frac{1}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right] \quad \text{-- (53)}$$

• which has the identical form as that $\vec{E}_{dip}$!

Note that $\because \; \vec{m} = \underbrace{(\vec{m}\cdot\hat{r})}_{m\cos\theta}\hat{r} - m\sin\theta\,\hat{\theta}$




$$\therefore \; m\sin\theta\,\hat{\theta} = -\vec{m} + (\vec{m}\cdot\hat{r})\hat{r}$$

Eq. (53) can be rewritten as

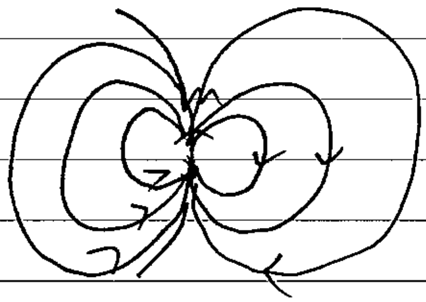
$$\vec{B}_{dip} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[2(\vec{m}\cdot\hat{r})\hat{r} - \vec{m} + (\vec{m}\cdot\hat{r})\hat{r}\right]$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right] \quad \text{-- (54)}$$

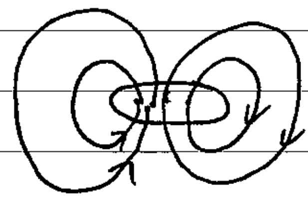
which is similar to

$$\vec{E}_{dip} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\vec{p} - \vec{p}\right]$$

The field of dipole is also similar



point dipole



Physical dipole