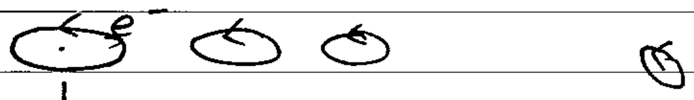
Magnetic fields in Matter

Magnetic properties of matters

All magnetic phenomena are due to electric charges in motion. The minimum unit for generating magnetic fields is the current loop. Microscopically, electrons and elementary particles carry magnetic dipoles either by spin or by circulating around neucleus.



$I\vec{A}=\vec{m}$

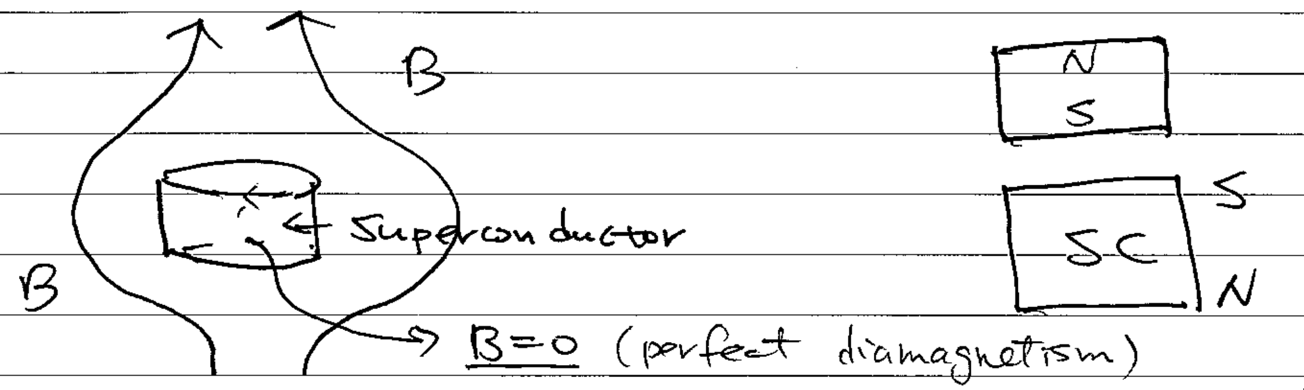
Similarly to electric dipoles, these magnetic dipoles will align under applied magnetic fields.

The materials are then said to be magnetically polarized. With magnetization $\vec{M}=\Sigma\vec{m_i}$

However, unlike electric dipoles, there are cases in which $\vec{M}$ is opposite to the $\vec{B}$ field. This is diamagnetic behavior.

~~These~~ Materials are <u>diamagnets</u>.

The most important diamagnets are the superconductors which show perfect diamagnetism below some temperature T_c.



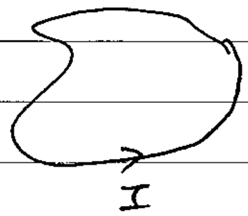
In addition, similiar to electric dipoles, in the absence of $\vec{B}$, if the magnetization of the material is zero ($B=0$) and becomes non-vanishing for $B \neq 0$, such materials are <u>paramagnets</u>

If materials can possess non-vanishing magnetization, they are called <u>ferromagnets</u>
(Fe, Co, Ni are three elements that are ferromagnets)
The mechanism that is behind ferromagnets is very complicated and is mainly due to dipole-dipole interactions. We shall introduce

~~its properties at the end.~~

Torque & force on magnetic dipole

Given a current loop, if $\vec{B}$ is uniform, then



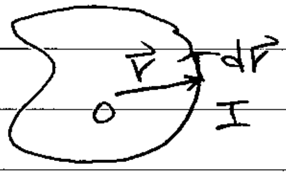
$$\vec{F} = \oint I\, d\vec{\ell} \times \vec{B}$$

$$= \underbrace{\left(\oint d\vec{\ell}\right)}_{0} \times I\vec{B} = 0$$

There is no net force.

However, there is a non-vanishing torque

$\because \vec{F} = 0$ The origin can be arbitrarily chosen.



$$\vec{\tau} = \oint \vec{r} \times d\vec{F} \qquad d\vec{F} = I\, d\vec{r} \times \vec{B}$$

$$= I \oint \vec{r} \times (d\vec{r} \times \vec{B}) \quad \text{— ①}$$

$$\vec{r} \times (d\vec{r} \times \vec{B}) = d\vec{r}\,(\vec{r}\cdot\vec{B}) - (\vec{r}\cdot d\vec{r})\,\vec{B}$$

Now, $d\vec{r}\,(\vec{r}\cdot\vec{B}) = \vec{r}\,(\vec{B}\cdot d\vec{r}) + (\vec{r} \times d\vec{r}) \times \vec{B}$ — ②

$$\therefore\ \vec{r} \times (d\vec{r} \times \vec{B}) = \vec{r}\,(\vec{B}\cdot d\vec{r}) - (\vec{r}\cdot d\vec{r})\,\vec{B} + (\vec{r} \times d\vec{r}) \times \vec{B} \quad \text{— ③}$$

$$\oint \vec{r}\cdot d\vec{r} = \oint \tfrac{1}{2}\, dr^2 = 0$$

If we choose $\vec{B} = B\hat{z}$,

$$\vec{r}\,(\vec{B}\cdot d\vec{r}) = \vec{r}\,B\,dz = B(x\,dz,\ y\,dz,\ z\,dz)$$

$$= B\left(d(zx) - z\,dx,\ d(yz) - z\,dy,\ dz^2 - z\,dz\right)$$

$$= B\,d(z\vec{r}) - Bz\,d\vec{r}$$

$$\therefore \oint \vec{r}\,(\vec{B}\cdot d\vec{r}) = \underbrace{B\oint d(z\vec{r})}_{=0} - \oint Bz\,d\vec{r}$$

$$= -\oint (\vec{B}\cdot\vec{r})\,d\vec{r}$$

Using ②, $\therefore$

$$\oint \vec{r}\,(\vec{B}\cdot d\vec{r}) = -\oint \vec{r}\,(\vec{B}\cdot d\vec{r}) - \oint (\vec{r}\times d\vec{r})\times\vec{B}$$

$$\oint \vec{r}\,(\vec{B}\times d\vec{r}) = \frac{-1}{2}\oint (\vec{r}\times d\vec{r})\times\vec{B}$$

$$\therefore \oint \vec{r}\times(d\vec{r}\times\vec{B}) \underset{\uparrow}{=} \frac{1}{2}\left(\oint \vec{r}\times d\vec{r}\right)\times\vec{B}$$

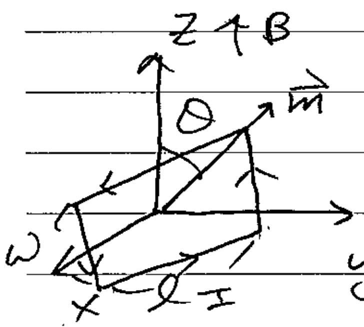
eq. ③

$$\vec{\tau} = I\left(\frac{1}{2}\oint \vec{r}\times d\vec{r}\right)\times\vec{B}$$

$$= \left(I\oint d\vec{a}\right)\times\vec{B} = \vec{m}\times\vec{B} \quad \cdots ④$$

Eq. ④ is valid even if the current loop does not lie on a plane.

Example. A special case of a current loop

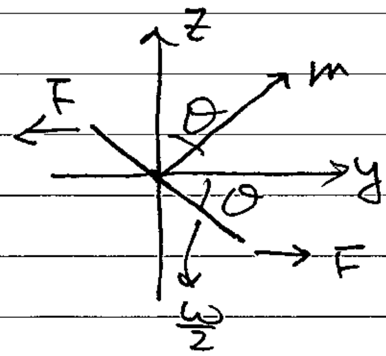


is the rectangular loop shown in the left figure

Currents on lateral side experience force ⊥ I

∴ They don't contribute torque

Only currents along l contribute torque.



$$F = lIB$$

$$\vec{\tau} = F \cdot \frac{\omega}{2} \sin\theta \, \hat{x} \times 2$$

$$= F\omega \sin\theta \, \hat{x} = \underbrace{Il\omega}_{m} B \sin\theta \, \hat{x}$$

$$= \vec{m} \times \vec{B}$$

Similar to the electric dipole, one applies an opposite torque to find the potential energy

$$U(\theta) - U(0) = \int_0^\theta dw = \int_0^\theta \tau \, d\theta$$

$$= \int_0^\theta mB \sin\theta \, d\theta = mB(1 - \cos\theta)$$

Choose $U(0) = -mB$, ∴ $U = -\vec{m} \cdot \vec{B}$ --- (5)

∴ When $\vec{B}$ is not uniform, the force acting on the magnetic dipole

$$\vec{F} = -\nabla U = \vec{\nabla}(\vec{m}\cdot\vec{B}) \quad \cdots \quad (6)$$

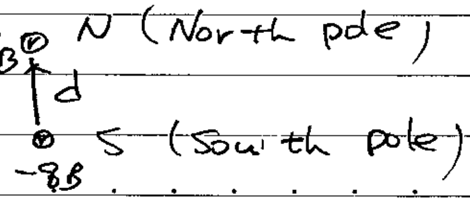
Eq. (6) is the magnetic force that acts on $\vec{m}$

<u>Gilbert model</u> when there is no external currents. see <u>6-6-1</u>

Comparing eqs. (4) & (5) to $\vec{\tau} = \vec{p}\times\vec{E}$ & $U = -\vec{p}\cdot\vec{E}$ for electric dipoles show the similarity of magnetic dipoles & electric dipoles.

It's thus tempting to think that there should exist magnetic monopoles (north & south poles) in nature.

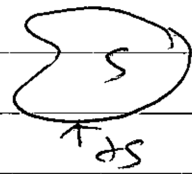
Even though the magnetic monopoles have not been found yet, the analogy leads to the <u>Gilbert model</u> of a magnetic dipole in which one tries to model $\vec{m}$ as $q_B\vec{d}$ due to $\pm q_B$ magnetic charges.

Gilbert model $\vec{m} =$ 

Most general force acting on a magnetic dipole.

In general, $\vec{F} = \int_V \vec{J}(\vec{r}) \times \vec{B}(\vec{r})\, d\tau$

$$= I \oint_{\partial S} d\vec{r} \times \vec{B}(\vec{r})$$



Using the identity

$$\oint_{\partial S} d\vec{r} \times \vec{B} = \int_S (d\vec{a} \times \vec{\nabla}) \times \vec{B}$$ (Homework 1 Ex 1(b))

$$\therefore \vec{F} = \left(\int_S I\, d\vec{a} \times \vec{\nabla} \right) \times \vec{B}$$

$$= (\vec{m} \times \vec{\nabla}) \times \vec{B} \quad \cdots \quad ⑥-1$$

⑥-1 is more general that eq. ⑥.

Now using $\nabla(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}\times(\vec{\nabla}\times\vec{B}) + (\vec{B}\cdot\vec{\nabla})\vec{A} + \vec{B}\times(\vec{\nabla}\times\vec{A})$

~~and~~ $\cancel{\nabla(\vec{A}\cdot\vec{B}) = \vec{A}(\vec{\nabla}\cdot\vec{B}) + (\vec{A}\times\vec{\nabla})\times\vec{B} + \vec{B}(\vec{\nabla}\cdot\vec{A}) + (\vec{B}\times\vec{\nabla})\vec{A}}$,

~~Substraction of two equations~~ yields

set $\vec{\nabla}\cdot\vec{A} = 0$, $\vec{\nabla}\times\vec{A}$, $(\vec{B}\times\vec{\nabla})\vec{A} = 0$ $\quad (\vec{A} = \text{const})$

$$\therefore (\vec{m}\times\vec{\nabla})\times\vec{B} = (\vec{m}\cdot\vec{\nabla})\vec{B} + \vec{m}\times(\vec{\nabla}\times\vec{B}) - \vec{m}(\vec{\nabla}\cdot\vec{B})$$

$$\therefore \vec{F} = (\vec{m}\cdot\vec{\nabla})\vec{B} + \vec{m}\times(\vec{\nabla}\times\vec{B}) - \vec{m}(\underbrace{\vec{\nabla}\cdot\vec{B}}_{\to 0})$$

$$= (\vec{m}\cdot\vec{\nabla})\vec{B} + \vec{m}\times(\vec{\nabla}\times\vec{B}) \quad \cdots ⑥-2$$

$\because \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}_{ext}$, $\therefore$ In addition to eq. ⑥, there is an extra force due to external current.

The resemblance of

$$\vec{B}_{dip} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]$$

and $$\vec{E}_{dip} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$

indicates that replacing $\frac{1}{\epsilon_0}$ by μ_0, $\vec{p}$ by $\vec{m}$

will get $\vec{B}$ from $\vec{E}_{dip}$.

In other words, one can assign the

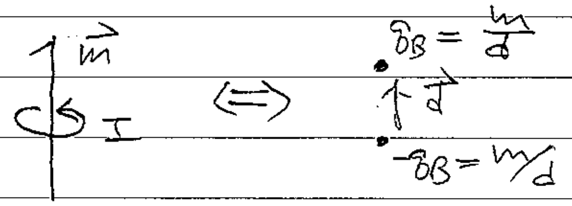
$\vec{B}$ field due to each q_B as

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q_B}{r^2}\hat{r} \quad \text{---(7)}$$

and introduce Φ_B such that $\vec{B} = -\nabla\Phi_B$, $\Phi_B = \frac{\mu_0 q_B}{4\pi r}$ --- (8)

Such approach is different from the (without using $\vec{A}$)

Ampère's current model



In some cases, Gilbert model may offer

quick solutions but there are cases

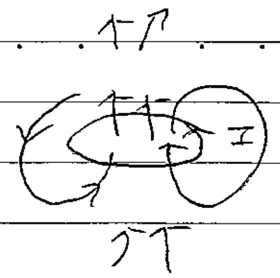
it gives incorrect results. Especially, for

fields inside the dipole ($r<d$), it gives

incorrect results (at $\pm q_B$, the Gilbert

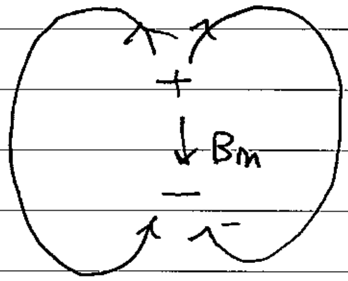
model yields $|\vec{B}| \to \infty$, but the real fields are

finite.)



Inside the current loop, $\vec{B}$ points to the same direction as $\vec{B}_{outside}$.

For the Gilbert model, however, $\vec{B}_m$ points to opposite direction to $\vec{B}_{out}$.



As a result, for the Gilbert model, a point dipole ($m = IA$ = fixed, $A \to 0$, $I \uparrow$)
(perfect dipole)

$$\vec{B}_G = \frac{\mu_0}{4\pi}\left[\frac{3\hat{r}(\vec{m}\cdot\hat{r})-\vec{m}}{r^3} - \underbrace{\frac{4\pi}{3}\vec{m}\,\delta(\vec{r})}_{\text{negative internal field}}\right] \quad \cdots (9)$$

(problem 3.48)

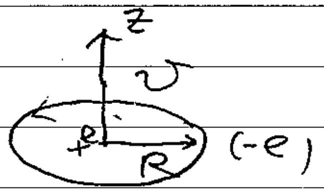
While for the current-loop model,

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3\hat{r}(\vec{m}\cdot\hat{r})-\vec{m}}{r^3} + \underbrace{\frac{8\pi}{3}\vec{m}\,\delta(\vec{r})}_{\text{positive internal field}}\right] \quad \cdots (10)$$

Diamagnetism of classical atomic orbits

In the classical model of atoms, electrons circulate around neucleus.

Consider an electron circulating around a neucleus of radius R.



The average current due to (-e) is

$$I = \frac{(-e)}{T} = -\frac{ev}{2\pi R}$$

$$T = \text{period} = \frac{2\pi R}{v}$$

∴ The orbital dipole moment

$$\vec{m} = I \cdot \pi R^2 \hat{z} = -\frac{1}{2} e v R \hat{z} \quad \cdots (11)$$

is determined by v & R.

In the absence of $\vec{B}$, v & R satisfies

$$\frac{1}{4\pi\epsilon_0} \frac{e^2}{R^2} = m_e \frac{v^2}{R} \quad \cdots (12)$$

In the presence of $\vec{B} = B\hat{z}$, an additional force

$(-e)\vec{v} \times \vec{B} = (-e) v B \hat{r}$ contributes the centripetal force. Let R fixed and v is changed to $\bar{v}$.

$$\therefore \frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} + e\bar{v}B = m_e\frac{\bar{v}^2}{R} \quad \cdots (13)$$

Comparing eqs. (12) & (13), we see that v increases

(13) − (12) ⇒ $e\bar{v}B = \frac{m_e}{R}(\bar{v}^2 - v^2)$

$$= \frac{m_e}{R}(\bar{v}+v)(\bar{v}-v) > 0 \quad \cdots (14)$$

$\therefore \bar{v} > v$

If $\Delta v = \bar{v} - v$ is small, one can approximate $\bar{v}+v$ by $\bar{v}$.

Therefore, eq. (14) implies. $\Delta v = \frac{eRB}{2m_e} \quad \cdots (15)$

This leads to

$$\Delta\vec{m} = -\frac{1}{2}e\Delta vR\hat{z}$$

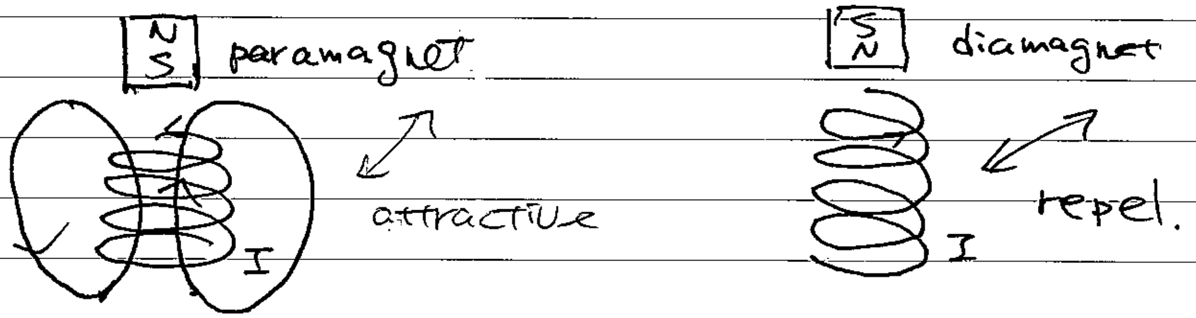
$$= -\frac{e^2R^2}{4m_e}\vec{B} \quad \cdots (16)$$

which is opposite to $\vec{B}$. and gives rise to diamagnetic behavior.

In the above derivation, the introducing of $\vec{B}$ speeds electrons up (for fixed R) so that the $\vec{B}'$ field generated by the electron increases to cancel the effect of $\vec{B}$. This is the so-called Lenz's law and is generally true without assuming that R is fixed.

In the real diamagnetism, it turns out to be a quantum phenomenon and there is no classical effect. The above derivation is to serve a heuristic arguement.

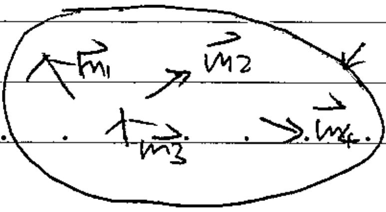
Ordinary materials are either paramagnets or diamagnets. For instance, water is diamagnetic. These materials experience different forces in close to magnets or solenoids with current:



The forces are weak but they can be observed.

Magnetization and field of a magnetized object

Similar to electric polarization, for a magnetic material, no matter what origin the magnetic dipole may arise, they usually point in different directions. One defines



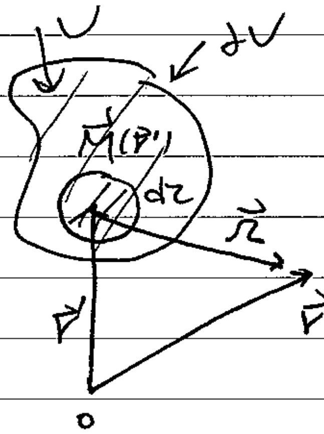
$d\tau$ the magnetization $\vec{M}$

as $$\vec{M}\, d\tau = \sum_{i \in d\tau} \vec{p}_i \quad \cdots \quad (17)$$

true without assuming that R is fixed.

The real diamagnetism originates from quantum effect and there is no classical effect. This is beyond the scope of this course. Diamagnetic effects are weak but they can still be observed. (e.g. water.)

<u>Magnetization and field of a magnetized object.</u>



Since for a given magnetic dipole $\vec{m}$ at $\vec{r}'$, the vector potential

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \quad \text{for } |\vec{r}-\vec{r}'| \gg r',$$

the field due to a magnetized object is given by

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \sum_i \frac{\vec{M}(\vec{r}_i)\times\hat{\mathscr{r}}_i}{\mathscr{r}^2} d\tau_i \quad \text{-- (18)}$$

Here $d\tau_i$ is a macroscopical volume but is still small enough so that one can replace $\sum_i$ in eq. (18) by an integral:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{M}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau' \quad --- \ (19)$$

when $|\vec{r}-\vec{r}'| \gg r'$.

Now, from electrostatics, one has

$$\nabla\frac{1}{|\vec{r}-\vec{r}'|} = -\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = -\nabla'\frac{1}{|\vec{r}-\vec{r}'|}$$

$$\therefore \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \nabla'\frac{1}{|\vec{r}-\vec{r}'|}$$

Hence $\vec{M}(\vec{r}')\times\dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \vec{M}(\vec{r}')\times\nabla'\dfrac{1}{\mathscr{r}}$

$$= -\vec{\nabla}'\times\left(\frac{\vec{M}(\vec{r}')}{\mathscr{r}}\right) + \frac{1}{\mathscr{r}}\left(\vec{\nabla}'\times\vec{M}(\vec{r}')\right)$$

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left\{\int \frac{1}{\mathscr{r}}\left(\vec{\nabla}'\times\vec{M}(\vec{r}')\right)d\tau' - \int \vec{\nabla}'\times\left(\frac{\vec{M}(\vec{r}')}{\mathscr{r}}\right)d\tau'\right\}$$

$$= \frac{\mu_0}{4\pi}\int \frac{\vec{\nabla}'\times\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\tau' + \frac{\mu_0}{4\pi}\oint \frac{1}{\mathscr{r}}\vec{M}(\vec{r}')\times d\vec{a}'$$

$$--- \ (20)$$

Therefore, instead of computing $\vec{A}(\vec{r})$ through eq.(19), one can compute $\vec{A}$ by replacing $\vec{M}(\vec{r}')$ by a volume current density

$$\vec{J}_b = \vec{\nabla}\times\vec{M} \quad --- \ (21)$$

and a surface current

$$\vec{K}_b = \vec{M} \times \hat{n} \quad \text{----} \quad (22)$$

So that

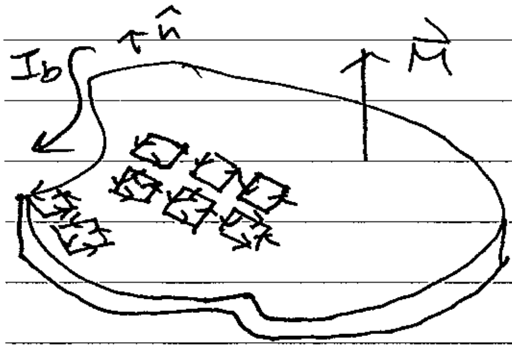
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}_b(\vec{r}')}{|\vec{r}-\vec{r}'|} d\tau' + \frac{\mu_0}{4\pi} \int_{\partial V} \frac{\vec{K}_b(\vec{r}')\, da'}{|\vec{r}-\vec{r}'|}$$

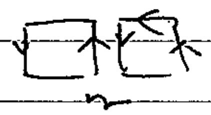
$\llcorner$ -- (23)

$\vec{J}_b$ & $\vec{K}_b$ are known as bound currents.

They play similar roles as those by bound charges, ρ_b & σ_b.

The physics picture behind $\vec{J}_b$ & $\vec{K}_b$ is to view $\vec{M}(\vec{r})$ as circulating loops as shown in the below. For uniform $\vec{M}$, currents of adjacent loops get cancelled

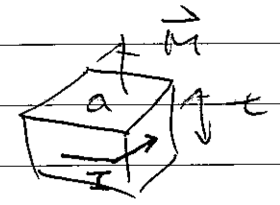




Cancelled!

As a result, only currents on surface don't get cancelled, resulting a bound surface current $\vec{I}_b \parallel \vec{M} \times \hat{n}$.

More precisely, if each small loop has the dimension

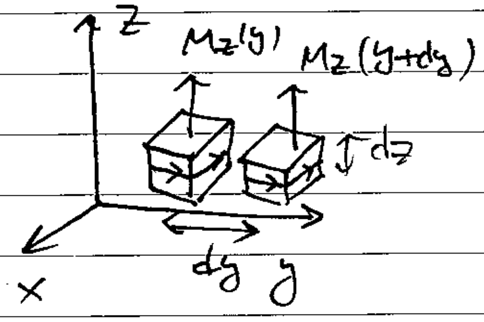


$$\vec{m} = \vec{M} a t = I \vec{a} \quad \therefore \quad I = Mt$$

$$K_b = I/t \Rightarrow \vec{K}_b = \vec{M} \times \hat{n}.$$

$\vec{K}_b$ ($\vec{I}_b$) is a genuine current and produces a magnetic field as other current does.

<u>If $\vec{M}$ is non-uniform</u>, currents in adjacent loops do not cancel each other.



Consider that M_z is non-uniform in y.

In between two areas separated by dy,

$$I_x = -M_z(y)dz + M_z(y+dy)dz$$

$$= \frac{\partial M_z}{\partial y} dy\,dz$$

$$\therefore (J_b)_x = \frac{I_x}{dy\,dz} = \frac{\partial M_z}{\partial y} \quad \cdots (23)$$

Similarly, for M_y being non-uniform in z directions, one has

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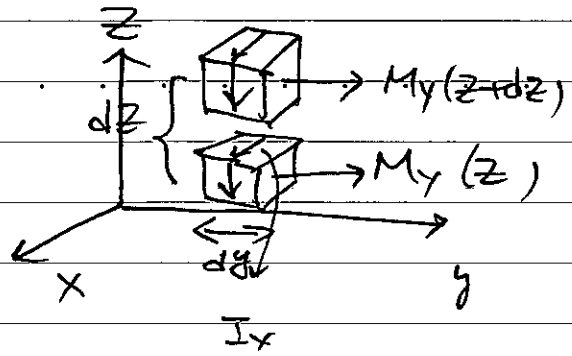


$$I_x = M_y(z)\,dy - M_y(z+dz)\,dy$$

$$= -\frac{\partial M_y}{\partial z}\,dy\,dz$$

$$\therefore (J_b)_x = -\frac{\partial M_y}{\partial z} \quad \cdots \quad (24)$$

Combing eqs. (23) & (24), one obtains

$$(\vec{\nabla}\times\vec{M})_x = (\vec{J}_b)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}.$$

In general, by considering other directions one gets $\vec{J}_b = (\vec{\nabla}\times\vec{M})$ as derived in eq. (21).

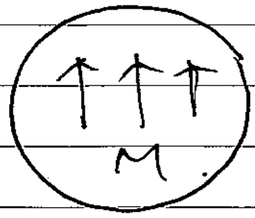
Note that $\vec{J}_b$ origins from circulating currents, it can't have divergence.

$$\therefore \vec{\nabla}\cdot\vec{J}_b = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{M}) = 0$$

is automatically satisfied in eq. (21).

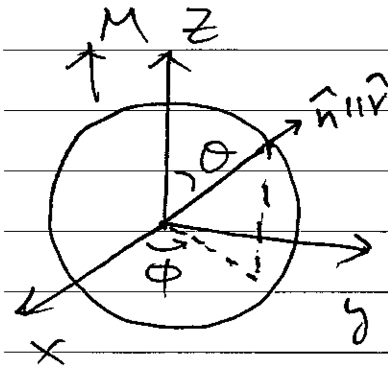
Example. A uniformly magnetized sphere.

Find $\vec{B}$.



Solution. $\vec{J}_b = \vec{\nabla} \times \vec{M} = 0$ ($\because \vec{M} = \text{const.}$)

$\therefore$ Only $\vec{K}_b = \vec{M} \times \hat{n} = M \sin\theta\, \hat{\phi}$ needs



to be considered

The field due to $\vec{K}_b$ is thus exactly the same as the field due to a rotating surface charge density σ that we solve before.

In that case, the surface current

$$\vec{K} = \sigma \vec{v} = \sigma v \hat{\phi} = \sigma \omega R \sin\theta \hat{\phi}$$

$\therefore M = \sigma \omega R$

Since $\vec{B} = \frac{2}{3}\mu_0 \underbrace{\sigma R \vec{\omega}}_{\vec{M}}$ for $r<R$,

$\therefore \vec{B} = \frac{2}{3}\mu_0 \vec{M}$ for $r<R$ — (25)

For $r>R$, $\vec{A} = \frac{\mu_0 R^4 \omega \sigma}{3} \frac{\sin\theta}{r^2} \hat{\phi}$

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Comparing with

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2} .$$

For $r > R$, the $\vec{B}$ field is the same as a magnetic dipole with dipole moment

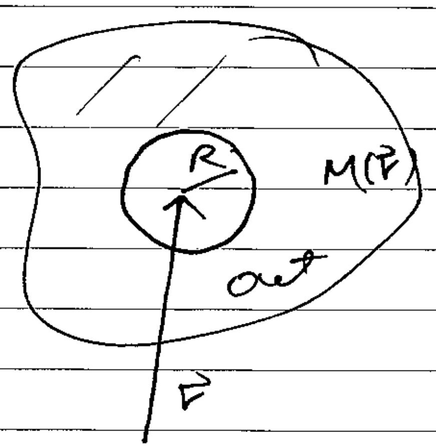
$$\vec{m} = \frac{4\pi}{3} R^4 \omega \sigma \ \hat{z}$$

$$= \frac{4\pi}{3} R^3 \vec{M} \quad --- \quad (26)$$

<u>Macroscopic $\vec{B}$ field in Matter</u>

Similar to electric polarization, for $\vec{B}$ fields, there are also microscopic $\vec{B}$ fields which <u>varies a lot inside the materials</u>

Eq. (14) may seem to fail when one takes $\vec{r}$ in the material.



One defines $\vec{B}$ as the macroscopic field. For this purpose, we consider a sphere of radius R centered at $\vec{r}$. R is much larger than molecular scales, but is still small macroscopically.

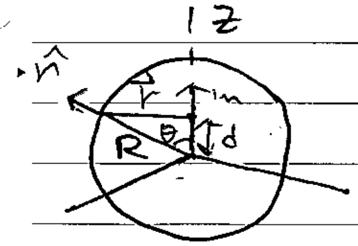
$$\therefore \vec{B} = \vec{B}_{out} + \vec{B}_{in} \quad \text{---} \quad (27)$$

$\vec{B}_{out} = \vec{\nabla} \times \vec{A}_{out}$ with

$$\vec{A}_{out} = \frac{\mu_0}{4\pi} \int_{|\vec{r}'-\vec{r}|>R} \frac{\vec{M}(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau' \quad \text{---} \quad (28)$$

$$\vec{B}_{in} = \frac{1}{\frac{4\pi}{3}R^3} \int_{r \in \text{sphere}} \vec{B}_{micro}(\vec{r})\, d\tau \quad \text{---} \quad (29)$$

= average of $\vec{B}_{micro}$ due to magnetic dipoles inside the sphere.



Now, $\because \vec{B}_{micro} = \vec{\nabla} \times \vec{A}_{micro}$

Using the identity

$$\int_V \vec{\nabla} \times \vec{A}\, d\tau = \oint_{\partial V} d\vec{a} \times \vec{A} = \oint_{\partial V} \hat{n} \times \vec{A}\, da$$

(Consider $\int \vec{\nabla} \cdot (\vec{C} \times \vec{A}) d\tau = \int da\, \hat{n} \cdot \vec{C} \times \vec{A}$ for any constant vector $\vec{C}$)

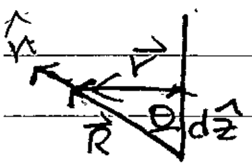
$$\therefore \int \vec{B}_{micro}\, d\tau = \int_{r=R} \hat{n} \times \vec{A}_{micro}\, da \quad \text{---} \quad (30)$$

We first consider a magnetic dipole located at $d\hat{z}$ with $\vec{m} = m\hat{z}$.

$$\therefore \vec{A}_{micro} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3}$$

$$\hat{n}\times\vec{A}_{micro} = \frac{\mu_0 m}{4\pi}\frac{\hat{n}\times(\hat{z}\times\vec{r})}{r^3}$$

$$= \frac{\mu_0 m}{4\pi}\frac{1}{r^3}\left[\hat{z}(\hat{n}\cdot\vec{r}) - (\hat{n}\cdot\hat{z})\vec{r}\right] \quad \text{--- (31)}$$



Now, $\vec{r} = \vec{R} - d\hat{z}$

$\therefore$ The integral $\int \frac{(\hat{n}\cdot\hat{z})\vec{r}}{r^3}da$

$$= \int \frac{\cos\theta(\vec{R}-d\hat{z})}{r^3}da$$

$$= \int \frac{\hat{n}R\cos\theta - d\cos\theta\hat{z}}{r^3}da$$

For a fixed θ, $\int \frac{\hat{n}da}{r^3} = \int \frac{\cos\theta\hat{z}da}{r^3}$

as $\underbrace{\hat{n}_x \,\&\, \hat{n}_y}_{\text{integrations of}}$ vanish by symmetries.

$$\therefore \int \frac{d\cos\theta\hat{z}}{r^3}da = \int \frac{d\hat{n}da}{r^3}$$

$$\int \frac{\hat{n}R\cos\theta - d\cos\theta\hat{z}}{r^3}da = \int \frac{\hat{n}\left[(\vec{R}-\hat{z}d)\cdot\hat{z}\right]}{r^3}da$$

$$= \int \frac{\hat{n}(\vec{r}\cdot\hat{z})}{r^3}da$$

Hence

$\oint \hat{n} \times \vec{A}_{micro}\, da$

$= \frac{\mu_0 m}{4\pi} \oint \frac{1}{r^3} \left[\hat{z}(\hat{n}\cdot\vec{r}) - \hat{n}(\vec{r}\cdot\hat{z}) \right] da$

$= \frac{\mu_0}{4\pi} m \oint \frac{1}{r^2} (\hat{n}\times\hat{z}) \times \hat{r}\, da$ -- (32)

Eq. (32) is precisely the magnetic field at the point $d\hat{z}$ due to a surface current $\vec{K} = -m\hat{n}\times\hat{z}$

$= (m\hat{z})\times\hat{n}$

From the example we solved, this gives

$$\oint \hat{n} \times \vec{A}_{micro}\, da = \frac{2}{3}\mu_0 \vec{m} \quad \text{--- (33)}$$

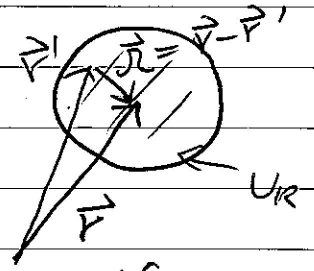
Using the principle of superposition, when there are dipoles $\vec{m_1}, \vec{m_2}, \ldots$ inside the sphere, $\int \vec{B}_{micro}\, d\tau = \frac{2}{3}\mu_0 \sum_i \vec{m_i}$ -- (34)

Hence $\vec{B}_{in} = \frac{2}{3}\mu_0 \frac{\sum \vec{m_i}}{\frac{4\pi}{3}R^3} = \frac{2}{3}\mu_0 \vec{M}$ --- (35)

which is precisely the average $\vec{B}$ field

(see 6-20-1 for more general proof)

For general current distribution $\vec{J}(\vec{r})$.



$$\vec{B}_{in} = \frac{2}{3}\mu_0 \vec{M}(\vec{r})$$

$$= -\frac{\mu_0}{3}\frac{1}{\underbrace{\frac{4\pi}{3}R^3}_{V_R}}\int_{V_R}\vec{J}(\vec{r})\times\vec{r}\,d\tau$$

$$\left(\vec{m} = \frac{I}{2}\oint \vec{r}'\times d\vec{r}' = \frac{-1}{2}\int d\tau\, \vec{J}\times\vec{r}\right)$$

proof:

$$\vec{B}_{in} = \frac{1}{V_R}\int_{V_R}\vec{B}_{micro}(\vec{r})\,d\tau$$

$$\vec{B}_{micro}(\vec{r}) = \int_{V_R'} \frac{\vec{J}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

$$\therefore \vec{B}_{in} = \int_{V_R'} d\tau' \frac{\mu_0}{4\pi V_R}\int_{V_R}\frac{\vec{J}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,d\tau$$

$$= \int_{V_R'} d\tau' \frac{\vec{J}(\vec{r}')}{V_R}\times\frac{\mu_0}{4\pi}\int_{V_R}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\,d\tau$$

$\because$ $\frac{1}{4\pi\epsilon_0}\int_{V_R}\frac{\rho(\vec{r})(\vec{r}'-\vec{r})}{|\vec{r}'-\vec{r}|^3}\,d\tau$ = electric field at $\vec{r}'$

due to $\rho(r)$ in V_R.

For constant ρ, $\vec{E} = \frac{\rho}{3\epsilon_0}\vec{r}'$

$$\therefore \frac{1}{4\pi}\int_{V_R}\frac{\vec{r}'-\vec{r}}{|\vec{r}'-\vec{r}|^3}\,d\tau = \frac{1}{3}\vec{r}'$$

$$\therefore \vec{B}_{in} = -\int_{V_R'} d\tau'\, \vec{J}(\vec{r}')\times\frac{\mu_0}{3V_R}\vec{r}' = -\frac{\mu_0}{3}\frac{1}{V_R}\int_{V_R}\vec{J}(\vec{r})\times\vec{r}\,d\tau$$

$$= \frac{2}{3}\mu_0\vec{M}$$

$$\left(= \frac{\mu_0}{4\pi R^3}\times 2\vec{m}\right)$$

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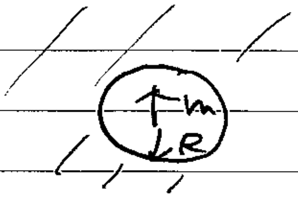
Relation between average field & local field in linear media.



Similar to Clausius-Mossotti formula, consider a magnetic dipole $\vec{m}$

$$\vec{m} = \beta \vec{B}_{out}$$

$$\vec{B} = \vec{B}_{in} + \vec{B}_{out} \qquad \vec{B}_{in} = \frac{\mu_0}{2\pi R^3}\vec{m}$$

$$\therefore \vec{B} = \left(\frac{\mu_0}{2\pi R^3}\beta + 1\right)\vec{B}_{out}$$

$$= (1 + 2\mu_0 \beta N)\vec{B}_{out} \qquad N = \frac{1}{\frac{4\pi}{3}R^3}$$

$$\therefore \vec{M} = N\vec{m} = N\beta\vec{B}_{out} = \frac{N\beta}{1+2\mu_0\beta N}\vec{B}$$

caculated by assuming that eq. (28) is correct inside the sphere.

Therefore, we conclude that the averaged macroscopic field $\vec{B}$ can be computed in the following way:

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{M}(\vec{r}') \times \hat{\imath}}{\imath^2} \, d\tau' \quad \cdots (36)$$

no matter what position $\vec{r}$ may take.

The auxiliary field H

(Some books called H : magnetic field intensity and B : magnetic induction. These terms are rather confusing. We shall maintain B as the magnetic field and term H as auxiliary field)

In the presence of $\vec{M}$, one has

$$\vec{J}_b = \vec{\nabla} \times \vec{M}.$$

In addition to $\vec{J}_b$, there may also be free charges that move in the system.

These are called <u>free current</u>, $\vec{J_f}$.

$\therefore$ Total $\vec{J} = \vec{J_f} + \vec{J_b}$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} = \mu_0 (\vec{J_f} + \vec{J_b})$$

$$= \mu_0 (\vec{J_f} + \vec{\nabla} \times \vec{M})$$

Moving $\vec{M}$ to the left, we get

$$\vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J_f} \quad \text{--- (37)}$$

$\therefore$ Similar to $\vec{D}$, one defines

$$\vec{H} \equiv \frac{1}{\mu_0} \vec{B} - \vec{M} \quad \text{--- (38)}$$

so that its curl is solely determined by $\vec{J_f}$

$$\vec{\nabla} \times \vec{H} = \vec{J_f} \quad \text{--- (39)}$$

or integral form $\oint \vec{H} \cdot d\vec{\ell} = I_f$ --- (40)

$\therefore [H] = [I]/\text{length} = \text{Ampere}/\text{meter}$

Hence $\vec{H}$ is closely related to external current. It is a more useful quantity than $\vec{D}$ as in the laboratory, one usually sets up external current to generate magnetic fields.

In that case, the current that one reads determines H.

Similar to $\vec{D}$, $\vec{\nabla}\times H$ does not determine $\vec{H}$ uniquely as one also needs to know $\vec{\nabla}\cdot\vec{H}$.

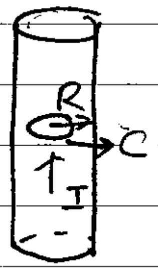
But $\vec{\nabla}\cdot\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\cdot\vec{B} - \vec{\nabla}\cdot\vec{M} = -\vec{\nabla}\cdot\vec{M}$ --- (41)

is determined by $\vec{M}$.

However, if there is a symmetry, eq. (40) ($i.e.$ $\vec{\nabla}\times\vec{H} = \vec{J_f}$) is sufficient to find H, and hence $\vec{B}$.

Example.

Find H.



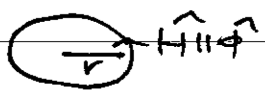
long copper rod

Solution: for $r \le R$

$$\oint_c \vec{H}\cdot d\vec{\ell} = I_{enclose} = I\frac{\pi r^2}{\pi R^2}$$

C = circle of radius r

By symmetry, $\vec{H} \parallel \hat{\phi}$



$$\therefore H\cdot 2\pi r = I\frac{\pi r^2}{\pi R^2}$$

$$\therefore \vec{H} = \frac{I}{2\pi R^2} r\hat{\phi} \quad \text{for } r \le R$$

For $r > R$, $I_{enclose} = I$

$$\therefore \vec{H} = \frac{I}{2\pi r}\hat{\phi}$$

$\because M = 0$ for $r > R$

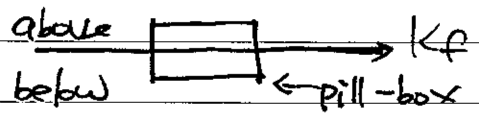
$$\therefore \vec{B} = \mu_0 \vec{H} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$

But $\vec{M}$ is not known for $r < R$, $\therefore \vec{B}$ is not determined.

Boundary conditions

As mentioned in eq. (41), $\nabla \times \vec{H} = \vec{J}_f$ doesn't determine H unless there is a special symmetry. In general, $\nabla \cdot \vec{M} \neq 0$. $\therefore \vec{\nabla} \cdot \vec{H} \neq 0$. and is determined by $-\nabla \cdot \vec{M}$.

$\nabla \cdot \vec{M} \neq 0$ must occur accross a boundary between different materials. Therefore, one needs boundary conditions.



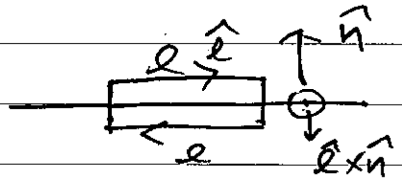
Using $\nabla \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}$ & Gauss's law in the pill-box shown in the left figure, one gets

$$H^{\perp}_{above} - H^{\perp}_{below} = -(M^{\perp}_{above} - M^{\perp}_{below}) \quad (42)$$

If $\vec{K}_f$ is the surface current,

$\vec{\nabla}\times\vec{H} = \vec{J}_f$ implies

$$\vec{H}^{\parallel}_{above} - \vec{H}^{\parallel}_{below} = \vec{K}_f \times \hat{n} \quad \cdots \quad (43)$$



$$\left[\because\ l\left(\vec{H}^{\parallel}_{above} - \vec{H}^{\parallel}_{below}\right)\cdot\hat{l} = -\hat{l}\times\hat{n}\cdot\vec{K}_f\, l = \hat{l}\cdot\vec{K}_f\times\hat{n}\, l\right]$$

In terms of $\vec{B} = \mu_0(\vec{H}+\vec{M})$, one has

eq. (42) $\Rightarrow$ $B^{\perp}_{above} = B^{\perp}_{below}$

(43) $\Rightarrow$ $\vec{B}^{\parallel}_{above} - \vec{B}^{\parallel}_{below}$

$$= \mu_0 \vec{K}_f\times\hat{n} + \mu_0 \underbrace{\vec{M}^{\parallel}_{above}}_{\vec{M}_{above}\times\hat{n}} - \mu_0 \underbrace{\vec{M}^{\parallel}_{below}}_{\vec{M}_{below}\times\hat{n}}$$

$$= \mu_0 (\vec{K}_f + \underbrace{\vec{K}_b}_{\text{bound current}})\times\hat{n}$$

$$= \mu_0 \vec{K}\times\hat{n}$$

Consistent with the boundary conditions obtained in chapt. 5.

Linear media

For paramagnetic & diamagnetic materials, when $\vec{B}=0$, $\vec{M}=0$. Furthermore, for most substances, $\vec{M} \propto \vec{B}$. $\therefore \vec{M} \propto \vec{H}$.

For dielectrics, one defines susceptibility χ_e as $\vec{P}=\epsilon_0 \chi_e \vec{E}$.

Here instead of defining $\vec{M} = \frac{1}{\mu_0} \chi_m \vec{B}$ (×), one defines

$$\vec{M}=\chi_m \vec{H} \quad --- \quad (44)$$

as $\vec{H}$ is frequently used. χ_m = magnetic susceptibility.

$\because [\vec{H}]=[\vec{M}]$, $\therefore \chi_m$ is dimensionless.

Typical values of $\chi_m \sim 10^{-5}-10^{-6}$.

Materials that obey eq. (44) are called linear media.

From eq. (44), $\vec{B} = \mu_0(\vec{H}+\vec{M}) = \mu_0(1+\chi_m)\vec{H}$

$$= \mu \vec{H} \quad -- \quad (45)$$

where $\mu = \mu_0(1+\chi_m) \equiv$ permeability of the material.
(μ_0 = permeability of free space)

For linear media, eq. (45) implies

$$\vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot\frac{\vec{B}}{\mu} = \frac{1}{\mu}\vec{\nabla}\cdot\vec{B} = 0 \quad \text{--- (46)}$$

in the bulk area of the media. However, $\vec{\nabla}\cdot\vec{H}$ is not known across the boundary where μ changes. Therefore, similar to the strategy of solve $\vec{D}/\vec{E}$ in dielectrics, the strategy to solve $\vec{H}/\vec{B}$ is to solve $\vec{H}$ in 1 & 2

μ_1: $\vec{\nabla}\times\vec{H}_1 = \vec{J}_{f1}$, $\vec{\nabla}\cdot\vec{H}_1 = 0$

μ_2: $\vec{\nabla}\times\vec{H}_2 = \vec{J}_{f2}$, $\vec{\nabla}\cdot\vec{H}_2 = 0$

separately and join $\vec{H}_1$ & $\vec{H}_2$ by using the boundary conditions eqs. (42) & (43).

L. (47)

Note that in homogeneous linear media,

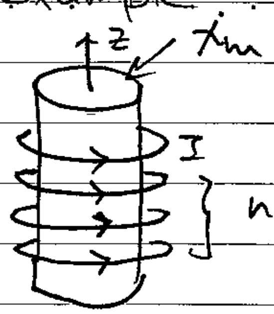
$$\because\ \vec{J}_b = \vec{\nabla}\times\vec{M} = \vec{\nabla}\times(\chi_m\vec{H}) = \chi_m\vec{\nabla}\times\vec{H}$$
$$= \chi_m\vec{J}_f$$

$\therefore$ If $\vec{J}_f = 0$, there is no volume bound current $\vec{J}_b$ too!

However, there will be surface bound current

$$\vec{K}_b = \vec{M}\times\hat{n} = \chi_m\vec{H}\times\hat{n}.$$

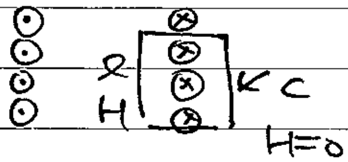
Example



Find $\vec{B}$ inside.

Solution : $\oint_C \vec{H}\cdot d\vec{\ell} = I_f$

Take c as follows and realize $\vec{H} \parallel \hat{z}$ by symmetry.



One gets

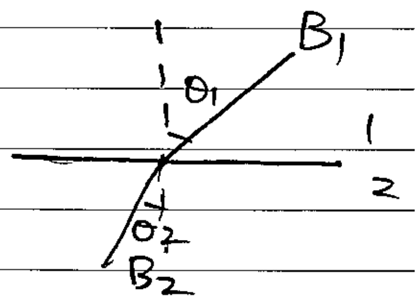
$$H\ell = n \cdot \ell \cdot I$$

$$\therefore \vec{H} = nI\hat{z} = \frac{1}{\mu}\vec{B}$$

$$\therefore \vec{B} = \mu_0(1+\chi_m)nI\hat{z}.$$

<u>Boundary conditions</u> of eq. (42)/(43)

in terms of μ_1 & μ_2 when $K_f = 0$



Eq. (42) $\Rightarrow$ $B^{\perp}_{above} = B^{\perp}_{below}$

$$(\mu_1 H^{\perp}_{above} = \mu_2 H^{\perp}_{below})$$

$\nabla\times\vec{H} = 0$ $\quad \vec{H}^{\parallel}_{above} = \vec{H}^{\parallel}_{below}$

$$\frac{1}{\mu_1}B^{\parallel}_{above} = \frac{1}{\mu_2}B^{\parallel}_{below}$$

$$\therefore \tan\theta_1 = \frac{B^{\parallel}_{above}}{B^{\perp}_{above}} = \frac{\mu_1}{\mu_2}\frac{B^{\parallel}_{below}}{B^{\perp}_{below}} = \frac{\mu_1}{\mu_2}\tan\theta_2.$$

Magnetic field computation

As shown in the above, in general, if $\vec{J}_f \neq 0$, one needs to solve

$$\vec{\nabla}\times\vec{H} = \vec{J}_f \quad \text{--- (48)}$$

$$\vec{\nabla}\cdot\vec{H} = 0 \quad \text{--- (49)}$$

in different linear media region and then match $\vec{H}$ across the boundaries.

In general, this is done by writting

$$\vec{H} = \frac{1}{\mu}\vec{B} = \frac{1}{\mu}\nabla\times\vec{A}$$

and the Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$

One needs to solve

$$\nabla^2\vec{A} = -\mu\vec{J}_f \quad \text{in}$$

different regions. and match $\vec{A}$ across the boundary:

$$\vec{A}_{above} = \vec{A}_{below} \quad \text{--- (50)}$$

$$\frac{1}{\mu_1}(\vec{\nabla}\times\vec{A}_{above})^{\parallel} - \frac{1}{\mu_2}(\vec{\nabla}\times\vec{A}_{below})^{\parallel} = \vec{K}_f\times\hat{n} \quad \text{--- (51)}$$

In case of $\vec{J}_f = 0$, there is a further simplification as

$$\vec{\nabla}\times\vec{H} = 0$$

$\therefore$ $\vec{H}$ must be a gradient of some function

$$\vec{H} = -\nabla \Phi_H \quad \text{--- (5-2)}$$

$$(\vec{B} = -\mu \nabla \Phi_H)$$

$\vec{\nabla} \cdot \vec{H} = 0$ then reduces to

$$\nabla^2 \Phi_H = 0 \quad \text{--- (5-3)}$$

Which is simpler than $\nabla^2 \vec{A} = -\mu \vec{J}_f$

and the whole problem can be solved in a similar way as that was done in dielectrics. except that the boundary conditions are

$$\Phi_H(\text{above}) = \Phi_H(\text{below}) \quad \text{--- (5-4)}$$

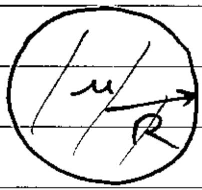
$$\mu_{above} \frac{\partial \Phi_H}{\partial n}(\text{above}) = \mu_{below} \frac{\partial \Phi_H}{\partial n}(\text{below})$$

$$(B^{\perp}_{above} = B^{\perp}_{below}) \quad \text{--- (5-5)}$$

Example

$\uparrow B_0 \quad \uparrow B_0 \quad \uparrow B_0$

Find $\vec{B}$ for $r < R$ & $r > R$



Solution : $\because r \to \infty \quad \vec{B} \to B_0 \hat{z}$

$\therefore \vec{H} \to \frac{B_0}{\mu_0} \hat{z}$

$\uparrow \quad \uparrow \quad \uparrow$
$B_0 \quad B_0 \quad B_0$

$$\Phi_H \to -\frac{B_0}{\mu_0} z = -\frac{B_0}{\mu_0} r \cos\theta$$

General solution to $\nabla^2 \Phi_H = 0$

is still

$$\Phi_H = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta).$$

For $r < R$ $\quad B_l = 0$

$$\Phi_H^{in} = \sum_{l=0}^{\infty} A_l r^l \, P_l(\cos\theta)$$

$r > R$

$$\Phi_H^{out} = -\frac{1}{\mu_0} B_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

Eq. (54) $\Rightarrow$ $\quad \Phi_H^{out}(R,\theta) = \Phi_H^{in}(R,\theta)$ --- (i)

Eq. (55) $\Rightarrow$ $\quad \mu_0 \frac{\partial \Phi_H^{out}}{\partial r}(R,\theta) = \mu \frac{\partial \Phi_H^{in}}{\partial r}(R,\theta)$ --- (ii)

(i)

$l = 1$ $\quad A_1 R = -\frac{1}{\mu_0} B_0 R + B_1/R^2$ --- (iii)

$l \neq 1$ $\quad B_l = A_l R^{2l+1}$ --- (iv)

(ii) $\quad \mu_0 \left[\frac{1}{\mu_0} B_0 \cos\theta + \sum_{l=0}^{\infty} (l+1) \frac{B_l}{R^{l+2}} P_l(\cos\theta)\right]$

$$= -\sum_{l=0}^{\infty} \mu A_l R^l \, P_l(\cos\theta)$$

$\therefore$ $l=1$, $\quad B_0 + 2\mu_0 \frac{B_1}{R^3} + \mu A_1 = 0$ --(v)

$l \neq 1$ $\quad \mu_0 (l+1) B_l = -\mu A_l R^{2l+2}$ —(vi)

(iv) & (vi) $\Rightarrow$ $\quad A_l = B_l = 0 \quad l \neq 1$

(iii) & (v) $\Rightarrow$ $\quad A_1 = -\frac{3B_0}{2\mu_0 + \mu}$, $\quad B_1 = R^3\left[A_1 + \frac{B_0}{\mu_0}\right]$

$$= \frac{\mu - \mu_0}{\mu_0(2\mu_0 + \mu)} B_0 R^3$$

$\therefore$ For $r<R$

$$\Phi_H = -\frac{3B_0}{2\mu_0+\mu} r\cos\theta = -\frac{3B_0}{2\mu_0+\mu} z$$

$$\vec{B} = -\mu\nabla\Phi_H = \frac{3\mu B_0}{2\mu_0+\mu}\hat{z}$$

$$= \frac{1+\chi_m}{1+\chi_m/3}\vec{B_0}$$

$r>R$

$$\Phi_H = -\frac{1}{\mu_0}B_0 r\cos\theta + \frac{1}{r^2}\frac{\mu-\mu_0}{\mu_0(2\mu_0+\mu)}B_0R^3\cos\theta$$

$$= -\frac{B_0}{\mu_0}z + \frac{1}{r^2}\vec{u}\cdot\hat{z}$$

$$\vec{u} = B_0R^3\frac{\chi_m}{\chi_m+3}\hat{z}$$

$$\therefore \vec{B} = -\mu_0\nabla\Phi_H = \vec{B_0} + \frac{\mu_0}{r^3}\left(3\vec{u}(\vec{u}\cdot\hat{r}) - \vec{u}\right)$$

(Similar to electric dipole)

Company to a magnetic dipole

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left(3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right)$$

The sphere acts as a magnetic dipole

$$\vec{m} = 4\pi R^3\frac{\chi_m}{\chi_m+3}\vec{B_0}$$

Scalar potential theory of magnetization

When $J_f = 0$, $\vec{H} = -\nabla \Phi_H$ (eq. 52) is actually the mathematical realization of the Gilbert model.

In this case, $J_f = 0$, there are only magnetization in the system. Hence we shall called $\vec{H} = \vec{H}_M = -\nabla \Phi_M$

To find the general expression of Φ_M, we first consider a point magnetic dipole $\vec{m}$ at $\vec{r}_0$

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times (\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}$$

Now, $\because \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} = -\nabla \frac{1}{|\vec{r}-\vec{r}_0|}$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{\mu_0}{4\pi} \vec{\nabla} \times \left(\frac{\vec{m} \times \vec{\mathscr{r}}}{\mathscr{r}^3} \right)$$

$$= \frac{\mu_0}{4\pi} \left[\vec{m}\, \vec{\nabla} \cdot \frac{\vec{\mathscr{r}}}{\mathscr{r}^3} - (\vec{m} \cdot \vec{\nabla}) \frac{\vec{\mathscr{r}}}{r^3} \right]$$

$$= \frac{\mu_0}{4\pi} \left[-\vec{m}\, \underbrace{\nabla^2 \frac{1}{|\vec{r}-\vec{r}_0|}}_{-4\pi \delta^3(\vec{r}-\vec{r}_0)} - (\vec{m} \cdot \vec{\nabla}) \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} \right]$$

$$\therefore \vec{B} = \mu_0 \left[\vec{m}\, \delta^3(\vec{r}-\vec{r}_0) - \nabla \left[\frac{1}{4\pi} \frac{\vec{m} \cdot (\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3} \right] \right] \quad \text{---} (55)\text{-}1$$

Clearly, in comparison to the

constitutive relation

$$\vec{B} = \mu_0 (\vec{M} + \vec{H}) \ , \quad \text{One identifies}$$

$$\vec{H}_M = -\nabla \frac{1}{4\pi} \frac{\vec{m}\cdot(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3} = -\nabla \Psi_M .$$

$$\therefore \Psi_M = \frac{1}{4\pi} \frac{\vec{m}\cdot(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}$$ which is the

exact correspondance in magnetic field

to the electric potential $\frac{1}{4\pi}\frac{\vec{p}\cdot\vec{r}}{r^3}$ for

an electric dipole.

Integrating eq. (5J), we obtain

$$\vec{B}_M(\vec{r}) = \mu_0 \int d\tau' \, \vec{M}(\vec{r}') \, \delta^3(\vec{r}-\vec{r}')$$

reminding $\vec{B}$ due to $\vec{M}$ only

$$- \nabla \frac{\mu_0}{4\pi} \int d\tau' \, \vec{M}(\vec{r}') \cdot \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

$$= \mu_0 [\vec{M}(\vec{r}) \underbrace{- \nabla \Psi_M}_{\vec{H}_M}] \quad --- \quad (5J)-2$$

with $$\Psi_M = \frac{1}{4\pi} \int \frac{\vec{M}(\vec{r}')\cdot(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} d\tau \quad --- \quad (5J)-3$$

Similar to the derivation for electric dipoles, (5J)-2

leads to $$\Psi_M = \frac{1}{4\pi} \int_V d\tau' \frac{\rho_M(\vec{r}')}{|\vec{r}-\vec{r}'|} + \frac{1}{4\pi} \int_S da' \frac{\sigma_M(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

with $\rho_M = -\nabla\cdot\vec{M}$, $\sigma_M = \vec{M}\cdot\hat{n}$ --- (5J)-4

$\therefore$ Consiston with eq. (4), one gets

$$\vec{\nabla}\cdot\vec{H}_M = -\nabla^2\varphi_M = \rho_M = -\vec{\nabla}\cdot\vec{M}$$

ρ_M & σ_M are fictious magnetic volume charge density and surface charge density.

Demagnetization field

Just like the electric dipole, inside an electric dipole, the electric field generated by $\vec{P}$ tends to point in opposite direction of $\vec{P}$!

$+$ $P\uparrow \downarrow E$ $-$

$\therefore$ $\vec{H}_M$ is also tending to point in $(-\vec{M})$.

Often this is expressed as

$$\vec{H}_M = -\overleftrightarrow{N}\cdot\vec{M} \quad \text{--- (5)-5}$$

$\overleftrightarrow{N}$ is called demagnetization tensor & $\vec{H}_M$ is the demagnetization field.

It can be shown by calculating $-\nabla^2\varphi_M$, $\mathrm{tr}\,\overleftrightarrow{N} = 1$.

Hence for a magnetized sphere, $\overleftrightarrow{N} = \frac{1}{3}\overleftrightarrow{I}$

$\therefore$ $\vec{H}_M = \frac{-1}{3}\vec{M}$

Check: Uniform polarized $\vec{P}$, $\sigma_b = \vec{P}\cdot\hat{n}$, $\vec{E}_{in} = -\frac{\vec{P}}{3\epsilon_0}$

Removing ϵ_0, $\therefore$ $\sigma_M = \vec{M}\cdot\hat{n}$, $\vec{H}_M = -\vec{M}/3$

$\vec{B} = \mu_0(\vec{M} + \vec{H}_M) = \frac{2}{3}\mu_0\vec{M}$

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Ex 6.15 $\nabla^2 \Phi_M = 0$

$$\Phi_M^{in} = \sum A_\ell r^\ell P_\ell(\cos\theta) \quad r<R$$

$$\Phi_M^{out} = \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \quad r>R$$

$$\Phi_M^{in}(R,\theta) = \Phi_M^{out}(R,\theta)$$

$$B_\ell = R^{2\ell+1} A_\ell$$

$$-\frac{\partial \Phi^{out}}{\partial r} + \frac{\partial \Phi^{in}}{\partial r}\bigg|_{r=R} = \vec{M}\cdot\hat{r} = M\cos\theta$$

$$\sum_\ell (\ell+1)\frac{B_\ell}{R^{\ell+2}} P_\ell(\cos\theta) + \sum_\ell \ell A_\ell R^{\ell-1} P_\ell(\cos\theta)$$

$$= M\cos\theta$$

$\therefore \ell \neq 1$ $\quad B_\ell = -A_\ell \frac{\ell}{\ell+1} R^{2\ell+1}$

$$\therefore A_\ell = B_\ell = 0$$

$\ell = 1$ $\quad A_1 + 2\frac{B_1}{R^3} = M$

$\therefore \quad B_1 = R^3 A_1 \qquad \therefore 3A_1 = M. \quad A_1 = \frac{M}{3}$

$B_1 = \frac{R^3}{3} M$

$$\therefore \Phi_M^{in} = \frac{M}{3} r\cos\theta = \frac{M}{3} z$$

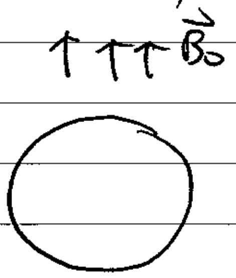
$$\vec{H}_M = -\nabla\Phi_M = -\frac{M}{3}\hat{z} = -\frac{1}{3}\vec{M}$$

$$\vec{B} = \mu_0(\vec{M} + \vec{H}_M) = \frac{2}{3}\mu_0\vec{M}$$

$\Phi_{out} = \frac{MR^3}{3}\frac{1}{r^2}\cos\theta \qquad \vec{H}_M = -\nabla\Phi_M = \frac{R^3}{3}\left[\frac{3(\hat{r}\cdot\vec{M})\hat{r} - \vec{M}}{r^3}\right]$

$$\vec{B} = \mu_0 \vec{H}_M \quad \text{for} \quad r>R$$

From the above result, one can identify $\vec{M}$ induced by external $\vec{B_0}$,



$$\vec{B}_{in} = \vec{B_0} + \vec{B}_M$$

$$\vec{H}_M = -\frac{1}{3}\vec{M}, \quad \vec{B}_M = \frac{2}{3}\mu_0 \vec{M} \quad r<R$$

$$\therefore \vec{B}_{in} = \vec{B_0} + \frac{2}{3}\mu_0 \vec{M}$$

$$\vec{H}_{in} = \vec{H_0} + \vec{H}_M = \vec{H_0} - \frac{1}{3}\vec{M}$$

$$\vec{B}_{in} = \mu \vec{H}_{in} \Rightarrow \vec{B_0} + \frac{2}{3}\mu_0 \vec{M}$$

$$= \mu \underbrace{\vec{H_0}}_{= \frac{1}{\mu_0}\vec{B_0}} - \frac{1}{3}\mu \vec{M}$$

$\therefore$

$$\vec{M} = \frac{3}{\mu_0} \frac{\mu - \mu_0}{\mu + 2\mu_0} \vec{B_0}$$

$\therefore$ total $\vec{m} = \frac{4\pi}{3} R^3 \vec{M}$

$$= \frac{4\pi R^3}{\mu_0} \frac{\mu - \mu_0}{\mu + 2\mu_0} \vec{B_0}$$

$$= \frac{4\pi R^3}{\mu_0} \frac{\chi_m}{\chi_m + 3} \vec{B_0}$$

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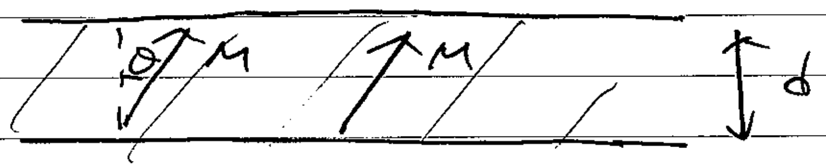
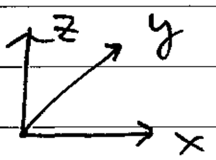
Potential theory v.s. vector theory of $\vec{M}$

Example

Infinite slab of matter with uniform

$$\vec{M} = M\cos\theta\,\hat{z} + M\sin\theta\,\hat{x}$$



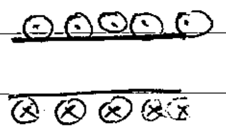


Find $\vec{B}$ inside and outside slab.

Solution 1.

$$\vec{K} = \vec{M}\times\hat{n}, \quad \hat{n} = \hat{z} \text{ for } z=d \text{ surface}$$
$$= -\hat{z} \text{ for } z=0 \text{ ''}$$

$$\therefore \vec{K} = \mp M\sin\theta\,\hat{y}$$



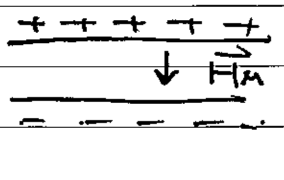
For a singlet sheet current density $\vec{K} = -M\sin\theta\,\hat{y}$

$$\vec{B} = \mp\frac{1}{2}M\sin\theta\,\mu_0\,\hat{x}$$

$$\therefore \vec{B} = \mu_0 M\sin\theta\,\hat{x} \quad \text{inside slab}$$
$$= 0 \quad \text{outside slab}$$

Solution 2. $\vec{B} = \mu_0(\vec{M} + \vec{H}_M)$

$$\vec{H}_M = -\nabla\Psi_M, \quad \sigma_M = \vec{M}\cdot\hat{n} = \pm M\cos\theta$$



Analogy to electrostatic,

$$\vec{H}_M = -\frac{\sigma_M}{\underset{\sim 1}{\epsilon_0}}\hat{z} = -M\cos\theta\,\hat{z} \quad \text{inside}$$
$$= 0 \quad \text{outside}$$

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$\therefore \quad \vec{M} + \vec{H}_M = M \sin\theta \, \hat{x}$ inside

$= 0$ outside

$\vec{B} = \mu_0 (\vec{M} + \vec{H}_M) = \mu_0 M \sin\theta \, \hat{x}$. inside

$= 0$ outside

Paramagnetism

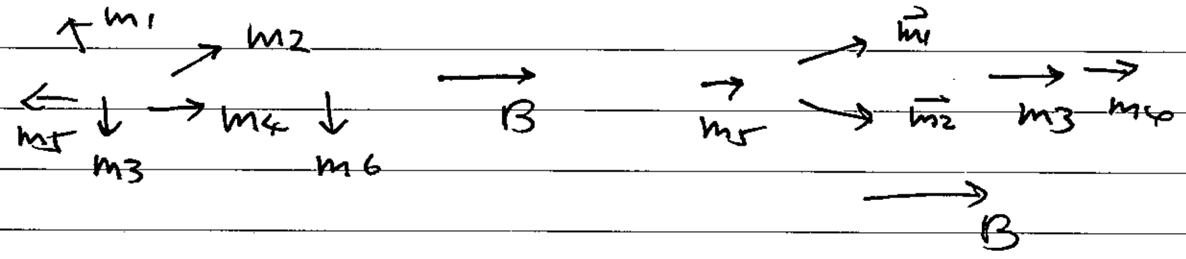
A material is paramagnetic if $\vec{M}=0$ when $\vec{B}=0$.

This may occur in two different ways

(i) each molecule/atom has no magnetic moment

the applied magnetic field generates $\vec{m_i}$ for each atom/molecule as well

(ii) each molecule/atom has magnetic momentum. the application of magnetic fields align those moments



This is similar to the situation described by the Langevin equation for electric dipoles

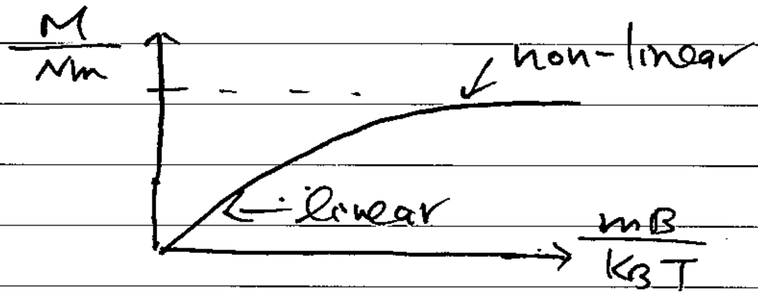
$$P \propto e^{-\frac{p_i E}{k_B T}\cos\theta}$$

$$P = N p\left[\coth\frac{pE}{k_B T} - \frac{k_B T}{pE}\right]$$

$\therefore$ In this case, one has

$$M = Nm\left(\coth\frac{mB}{K_BT} - \frac{K_BT}{mB}\right) \quad \text{---(16)}$$

Note that here B should be the local field that acts on each magnetic dipole.



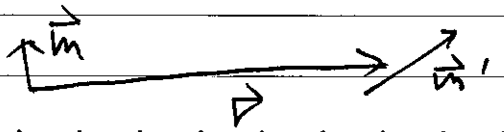
$\therefore$ Only when $B \to \infty$ or $T \to 0$, $M \to mN$

For $B=0$, M always vanishes.

The behavior is due to the competion between effect of temperature and alignment of B field.

In real materials, a magnetic dipole also interact with other magnetic dipole.

For instance, a magnetic dipole generates a magnetic field $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}(3\hat{r}(\vec{m}\cdot\hat{r}) - \vec{m})$.



If there is another magnetic dipole $\vec{m}'$ at $\vec{r}$, the interaction energy between $\vec{B}(\vec{r})$ & $\vec{m}'$

is $U = -\vec{m}' \cdot \vec{B}(\vec{r})$

This is the dipole-dipole interaction

$$U = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[\vec{m} \cdot \vec{m}' - 3(\vec{m} \cdot \hat{r})(\vec{m}' \cdot \hat{r}) \right] \quad \text{---} \quad (17)$$

In real materials, there are other kinds of interaction between magnetic dipoles. Some of interactions will tend to align dipole together, which as we shall discuss is known as ferromagentism.

$$\left. \begin{array}{c} \uparrow\uparrow\uparrow \\ \uparrow\uparrow\uparrow \\ \uparrow\uparrow\uparrow \end{array} \right\} \text{ due to interaction.}$$

Therefore, there are 3 factors in determine alignment of $\vec{m}_i$:

(i) temperature
(ii) B
(iii) interaction

Temperature is the only factor that tends to destroy the alignment of dipoles at large T.

Therefore, when the result of ($B=0 \Rightarrow M=0$) depends on temperatures, only when T is large enough, $B=0$ leads to $M=0$

In this case, T in eq. (56) is replace by $T-T_c$ and is valid for $T>T_c$. Here T_c is some temperature known as critical temperature.

Non-linearity

The relation between M & B in the Langevin eq. (eq. (58)) is non-linear.

Using eqs. (44) & (45), $\vec{M} = \frac{\chi_m}{\mu}\vec{B}$, it implies $\frac{\chi_m}{\mu}$ depends on B as well.

Only when $B \to 0$, $\because \coth x \to \frac{1}{x} + \frac{x}{3}$

$$M = N m \frac{m}{3k_BT} B$$

$$\therefore \frac{\chi_m}{\mu}(B\to 0) = \frac{N}{3k_BT}.$$

Hence, for large magnetic fields, materials are usually non-linear.

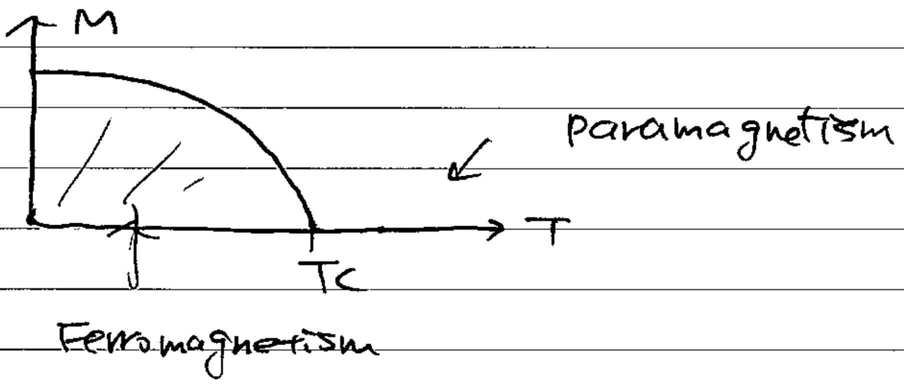
Ferromagnetism

As we mentioned, there exists dipole-dipole interactions in nature. In addition to magnetic dipoles due to orbital motion of charges, there are also dipoles due to <u>spins of particles</u>. These dipoles originate from quantum mechanics and we shall not discuss them here. However, they yield spin-spin interaction for neighbouring spins. For some materials, the interaction tends to lock dipoles in the same directions. The locking is so strong that even in $B=0$, $M \neq 0$. Such a state is called ferromagnetic state.

$\uparrow \uparrow \uparrow \uparrow$
$\uparrow \uparrow \uparrow \uparrow$
$\uparrow \uparrow \uparrow \uparrow$

The phenomenon is called <u>spontaneous magnetization</u>.
There are known 3 elements (Fe, Co, Ni) that are ferromagnetic. It only occurs for temperature below some critical temperature T_C, known as Curie temperature. For instance, Fe, $T_C = 770°C$

The average magnetization depends on temperatures and show similar T dependence



In real materials, interactions between spins are more complicated. There are also materials in which neighbouring spins are locked in opposite directions, known as Antiferromagnetism.

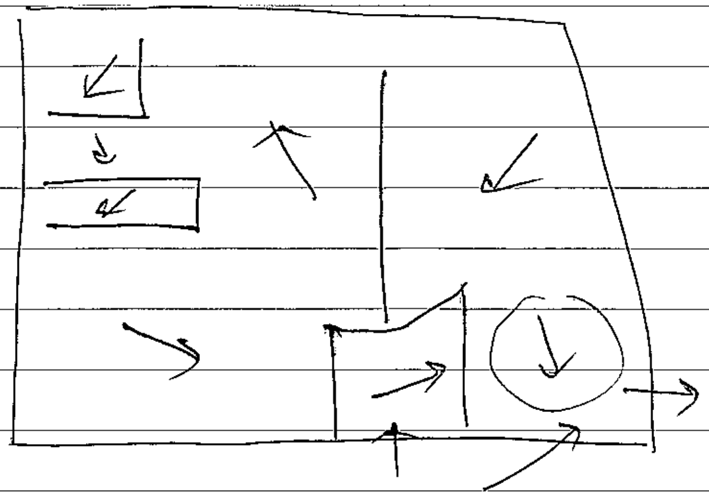
Antiferromagnetism ↑ ↓ ↑ ↓
↓ ↑ ↓ ↑
↑ ↓ ↑ ↓

Domains & hysteresis

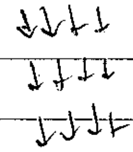
Magnets are made by ferromagnetic materials. If one looks into how magnetic moments are distributed inside a magnet, one finds that all microscopic dipoles point to the same directions.

Instead, alignment of dipoles occurs in patches.

These patches are known domains.



Inside each domains, there are many microscopic dipoles that all line up.



Domains

When there is no external field, $\vec{M}_i$ in each domain points in directions that are random (due to their history). $\therefore \sum_i \vec{M}_i = 0$

There is no net magnetization.

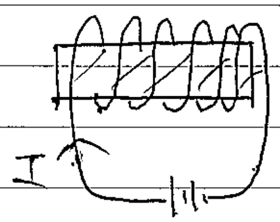
However, when one puts, say, a piece of iron, into a strong magnetic field $\vec{B}$, each domain experiences a torque

$$\vec{\tau}_i = \vec{M}_i \times \vec{B}.$$

Dipoles in each domain tends to align together. Therefore, domain most nearly parallel to $\vec{B}$ will stay in the same direction, other domains will be converted to the same direction.

As a result, boundaries of domains will move. Domains parallel to $\vec{B}$ grow and others shrink. If B is strong enough, there will be one domain that takes over entirely. The iron is said to be saturated.

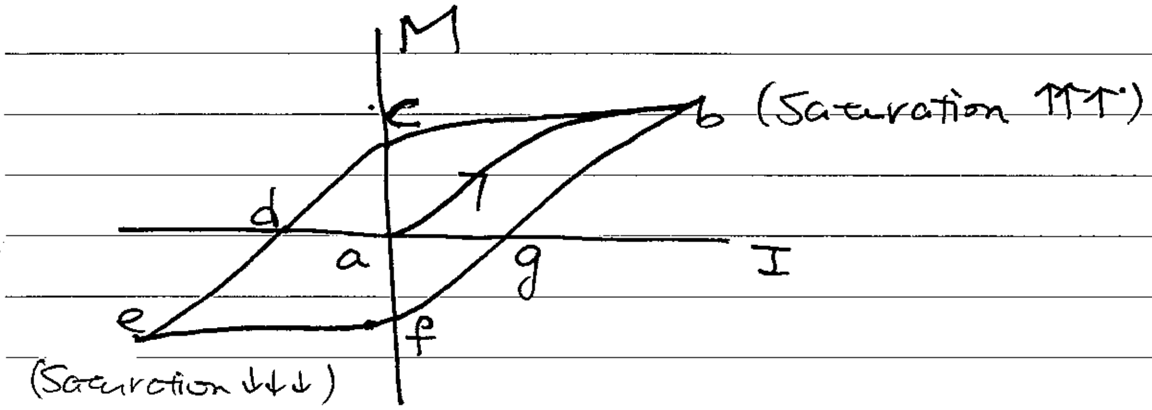
To magnetize ferromagnetic materials, say, iron, one usually uses a solenoid to wrap around the material as follows.



It is found that the above growing process of domains is not reversible. When B is turned off, there will be some domains return to random orientations but there are also a lot of domains remaining in parallel to $\vec{B}$. As a result, one gets a magnetized magnet even if $B=0$.

The irreversibility yields the so-call hysteresis (磁滞) phenomenon.

Starting from $M=0$, increasing I will take M to saturation. This is curve shown, starting from a to b.



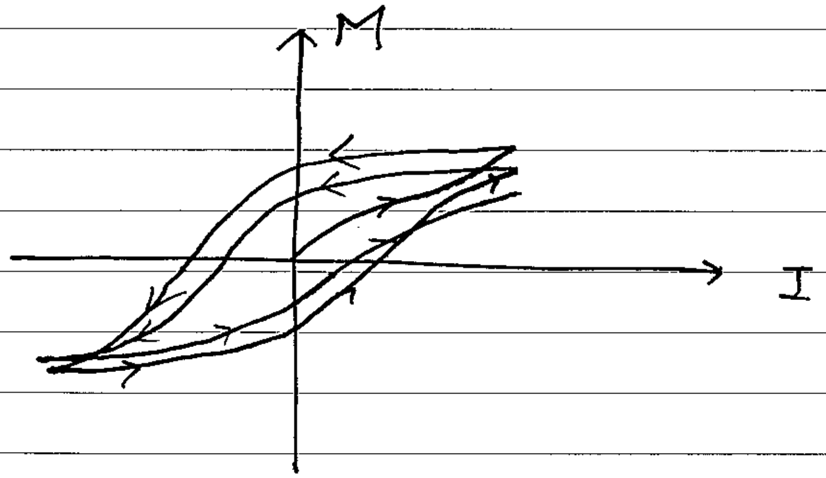
However, as we indicated, the growing process of M is not reversible, if we decrease I to 0, M will not go to zero.

Instead, it may go to c by following $b \to c$. In order to remove M, one now has to apply inverse current, i.e., $-I$, resulting in $c \to d$ ($M=0$). One can further apply $-I$ so that it is saturated in opposite direction at e point.

Now, if one reverse the direction of current by increasing current, it follows symmetric path $e \to f \to g \to b$

The resulting curve $b \to c \to d \to e \to f \to g \to b$ forms a loop. Called hysteresis loop.

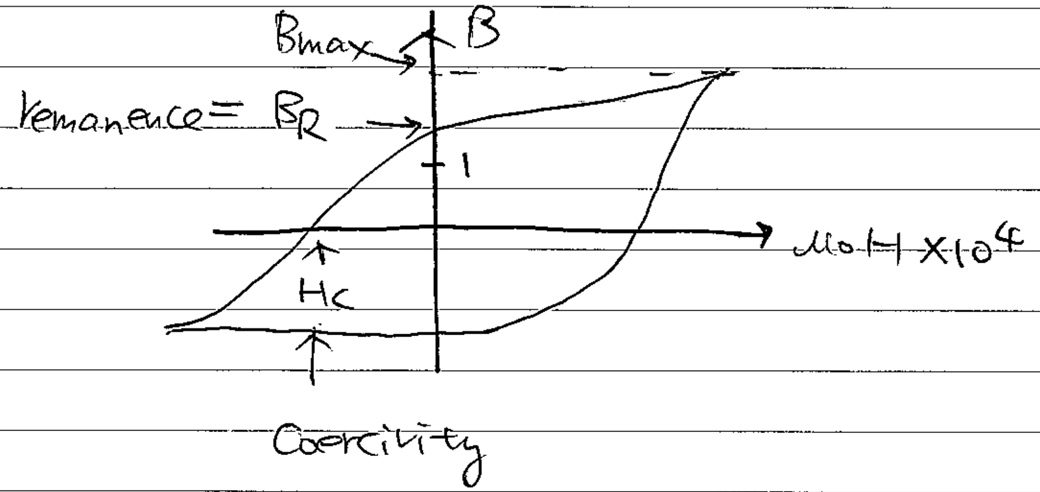
Note that the actual path depends on history of $\vec{M}$ and also on rate of change of current. Hence the real M-I curves may look like



There exists a largest area of loop when the rate of change of current is very slow.

Since $H = nI$, $\vec{B} = \mu_0(\vec{H} + \vec{M})$ & $\vec{M}$ is very large in comparing to H, so that $B \simeq \mu_0 M$.

One often plots B v.s. $\mu_0 H$



B is usually much larger than $\mu_0 H$

Hence in the above plot, $\mu_0 H$ is enlarged to $\mu_0 H \times 10^4$

It is clear that usage of ferromagnetic material enhances B a lot:

without iron ; $B_0 = \mu_0 H = \mu_0 n I$

with iron : $B \sim \mu_0 M \sim 10^4 B_0$!

The extra magnetic field comes from the contribution of the aligned dipoles in ferromagnetic materials.

That is why one tries to wrap the coil around an iron core to make a powerful electromagnet.

Finally, note that the same phenomenon also occurs ferroelectric materials where electric dipoles play similar roles.