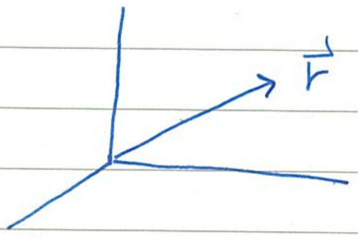
一. 運動學及其相關數學

運動學 (Kinematics)

一只描述物理運動，不考慮運動的起因。

*向量與純量的概念

在古典力學中，第一步的簡化為將質點(或小物件)視為數學上的點
以"向量"(位置向量) $\vec{r}$ 來描述

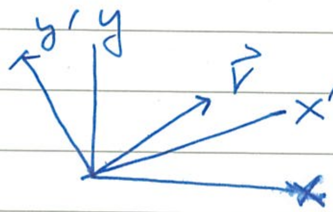


何謂向量？

$\vec{r}$: 有大小，方向之量 (directed line segment)

大小為 r. (距離)

方向的描述與座標有關：

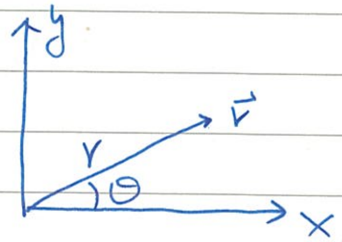


$\vec{r}$ 对 xy 及 x'y' 之相对方向不同

但 $\vec{r}$ 本身與座標的選取無關

如何精確的定義向量呢？(之後，我們將有例子表明大小及方量並不唯一定出向量)

首先，要定量，寫出其分量是一個方法：



$$\vec{r} = x\hat{x} + y\hat{y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

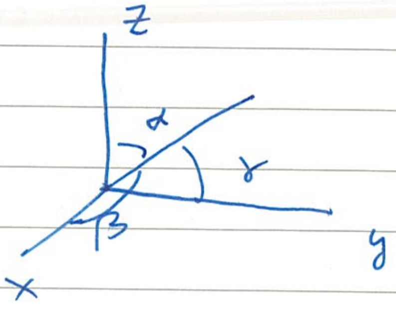
$x = r\cos\theta$

$y = r\sin\theta$

(矩陣的寫法)

或 $(x, y) = \vec{r}^T$ (transpose)

3D



α, β, γ direction cosines

$x = r\cos\beta$

$y = r\cos\gamma$

$z = r\cos\alpha$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

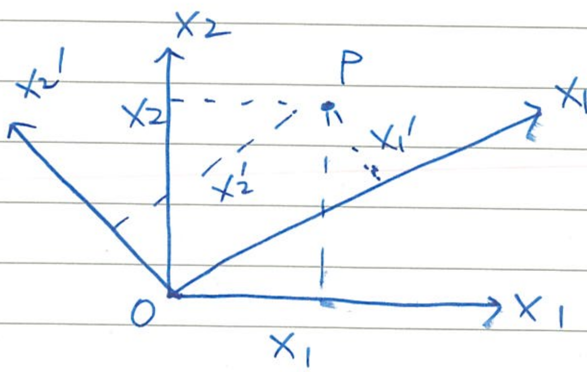
$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{or} \quad (x, y, z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}^T$$

∴向量也可以用一組數（分量）來表示，
但如此產生了以下問題：

是不是任何一組數都可以成為向量？如 $\begin{pmatrix} \text{身高} \\ \text{體重} \end{pmatrix}$ = 向量？
就向量跟座標無關的概念，現在呢？

答案藏在所謂的座標轉換（coordinate transformation）：

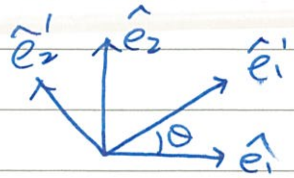
真正的向量，它的分量表示隨座標轉動而變



$$\vec{OP} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{x_1-x_2} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}_{x_1'-x_2'}$$

$$= x_1\hat{e}_1 + x_2\hat{e}_2$$

$$= x_1'\hat{e}_1' + x_2'\hat{e}_2'$$



$$\hat{e}_1' = \cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2$$
$$\hat{e}_2' = -\sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_2$$

或 $$\hat{e}_1 = \cos\theta\,\hat{e}_1' - \sin\theta\,\hat{e}_2'$$
$$\hat{e}_2 = \sin\theta\,\hat{e}_1' + \cos\theta\,\hat{e}_2' \quad \text{— ①}$$

$$\therefore\ x_1\hat{e}_1 + x_2\hat{e}_2 = (x_1\cos\theta + x_2\sin\theta)\,\hat{e}_1' + (-x_1\sin\theta + x_2\cos\theta)\,\hat{e}_2' \quad \text{-- ②}$$
$$= x_1'\hat{e}_1' + x_2'\hat{e}_2'$$

$$\therefore\ x_1' = x_1\cos\theta + x_2\sin\theta$$
$$x_2' = -x_1\sin\theta + x_2\cos\theta$$

Rotational Matrix (transformation)

列 →

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{-- ③}$$

↑ 行

可見分量隨轉動而變！(但不同以大小，$x_1^2 + x_2^2 = x_1'^2 + x_2'^2$ 不變)

⇒ 一般的向量即以此關係推廣

若 $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ 為一向量 ⟺ $v_i' = R_{ij} v_j$

* Rotational Matrix 也可以 direction cosine 來表示，$\lambda_{ij} \equiv \cos(\hat{e}_i', e_j)$ $\therefore\ \cos\theta = \lambda_{11}$, $\lambda_{12} = \sin\theta$, $\lambda_{21} = \cos(\theta + \pi/2) = -\sin\theta$

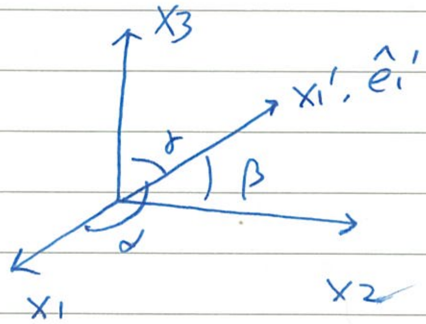
$\lambda_{22} = \cos\theta$, $\therefore R_{ij} = \lambda_{ij}$

以上可推廣到 3維

λ_{ij} 又可稱為 directional cosine

$$R = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}$$

其中



$\lambda_{11} = \cos\alpha$

$\lambda_{12} = \cos\beta$

$\lambda_{13} = \cos\gamma$

$$\hat{e}_1' = \begin{pmatrix} \lambda_{11} \\ \lambda_{12} \\ \lambda_{13} \end{pmatrix}, \text{同理 } \hat{e}_2' = \begin{pmatrix} \lambda_{21} \\ \lambda_{22} \\ \lambda_{23} \end{pmatrix}, \hat{e}_3' = \begin{pmatrix} \lambda_{31} \\ \lambda_{32} \\ \lambda_{33} \end{pmatrix}$$

└..④

總結：向量是一有大小，方向之量。

其分量表示 v_α 必須滿足 $v_\alpha' = \lambda_{\alpha\beta} v_\beta$.
在旋轉下

反之在旋轉下不變之量 ⇒ scalar (純量)

如：向量之大小即為純量

但這樣的定義還不夠，$\lambda_{\alpha\beta}$ 必須定得更清楚，

為了此目的，我們先了解向量可以有大小，主要是內積之概念來的：

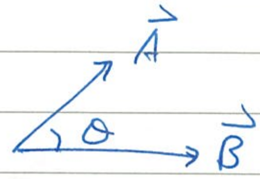
A_α 与 B_α 皆是 向量

(i) $(\vec{A}+\vec{B})_\alpha \equiv A_\alpha + B_\alpha$

(ii) $(C\vec{A})_\alpha = C A_\alpha$

(iii) Scalar product

$$\vec{A}\cdot\vec{B} \equiv \sum_\alpha A_\alpha B_\alpha$$



特例 $A^2 = \vec{A}\cdot\vec{A}$

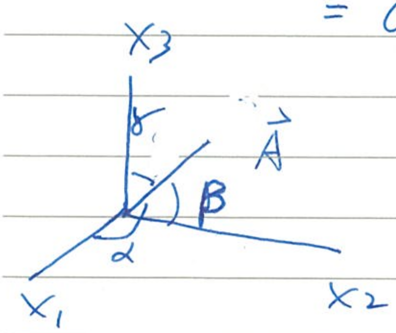
problem 1-2

$$\because \frac{\vec{A}\cdot\vec{B}}{AB} = \sum_\alpha \frac{A_\alpha}{A}\frac{B_\alpha}{B} = \cos\theta$$

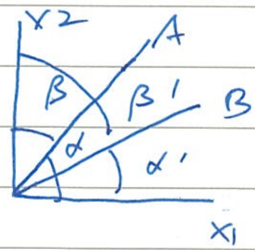
directional cosine of $\vec{A}$

directional cosine of $\vec{B}$

$$= \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'$$



以2維說明



$$\cos\alpha\cos\alpha' + \cos\beta\cos\beta'$$

$$= \cos\alpha\cos\alpha' + \sin\alpha\sin\alpha'$$

$$= \cos(\alpha-\alpha') = \cos\theta$$

$$\therefore \vec{A}\cdot\vec{B} = AB\cos\theta$$

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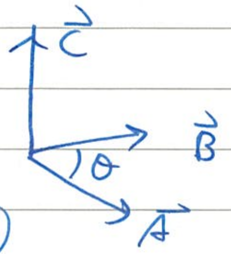
(iv) vector product (cross product)

$$\vec{C} = \vec{A} \times \vec{B}$$

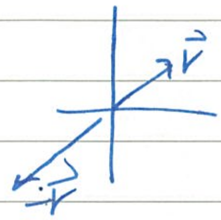


$C = AB\sin\theta$, $\hat{C} \perp \hat{A}$
$\hat{C} \perp \hat{B}$ ... (5)

$\vec{C}$ is an axial vector

(if $\vec{A}$ & $\vec{B}$ are polar vectors)

polar: 对原点反轉變号者，如 $\vec{r}$



若 $\vec{A}$, $\vec{B}$ 皆變号, $\vec{C}$ 不變, ∴ C ⇒ axial vector

行列式表式：

$$\vec{C} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

(可由⑤式証明：$C^2 = A^2B^2(1-\cos^2\theta) = A^2B^2 - (\vec{A}\cdot\vec{B})^2 = \cdots$)

$$= (A_2B_3 - A_3B_2, -(A_1B_3 - A_3B_1), A_1B_2 - A_2B_1)$$

或 $C_i = \sum_{jk} \epsilon_{ijk} A_j B_k$

ϵ_{ijk} = Levi-Civita ~~density~~ symbol (permutation symbol)

$= 0$ 任二对或以上一樣，如 $\epsilon_{112} = 0$

$= 1$ even permutation $\epsilon_{123} = 1$

$= -1$ odd ″

↓交換偶數次之排列

$\epsilon_{132} = -1$

例 $\epsilon_{123} = \epsilon_{231} = \varepsilon_{312} = 1$

$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$

$$\hat{e}_i \times \hat{e}_j = \sum_k \epsilon_{ijk} \hat{e}_k$$

Useful identities

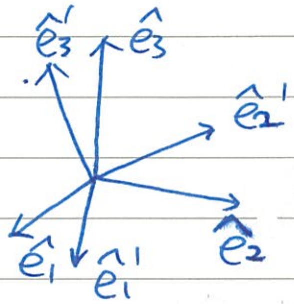
$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

$$= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$$

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

旋轉矩陣之要求：即怎樣的 $R_{\alpha\beta}$
才滿足向量旋轉下轉換所需？
⇒ 正交！



首先 $\hat{e}_1 = x\hat{e}_1' + y\hat{e}_2' + z\hat{e}_3'$

$$x = \hat{e}_1\cdot\hat{e}_1' = \lambda_{11}$$
$$y = \hat{e}_1\cdot\hat{e}_2' = \lambda_{21}$$
$$z = \hat{e}_1\cdot\hat{e}_3' = \lambda_{31}$$

同理 $\hat{e}_2 = \lambda_{12}\hat{e}_1' + \lambda_{22}\hat{e}_2' + \lambda_{23}\hat{e}_3'$
$\hat{e}_3 = \lambda_{13}\hat{e}_1' + \lambda_{23}\hat{e}_2' + \lambda_{33}\hat{e}_3'$ $\Big\}$ $\underline{\hat{e}_i = \hat{e}_j'\lambda_{ji}}$

如此，∵ $x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$

$$= (x_1\lambda_{11} + \lambda_{12}x_2 + \lambda_{13}x_3)\hat{e}_1'$$
$$+ (\lambda_{21}x_1 + \lambda_{22}x_2 + \lambda_{23}x_3)\hat{e}_2'$$
$$+ (\lambda_{31}x_1 + \lambda_{32}x_2 + \lambda_{33}x_3)\hat{e}_3'$$

∴ $x_1' = \lambda_{11}x_1 + \lambda_{12}x_2 + \lambda_{13}x_3$
$x_2' = \lambda_{21}x_1 + \lambda_{22}x_2 + \lambda_{23}x_3$ 即 $x_i' = \underline{\underline{\lambda_{ij}}}\, x_j$
$x_3' = \lambda_{31}x_1 + \lambda_{32}x_2 + \lambda_{33}x_3$

由此可知 $\lambda_{ij} = \hat{e}_i'\cdot\hat{e}_j$ 可視為單位向量間之變換
係數

由(4)式及內積之特性可知
$$\hat{e}_n'\cdot\hat{e}_n' = 1 \quad (n=1,2,3)$$

$$\therefore \quad \lambda_{11}^2+\lambda_{12}^2+\lambda_{13}^2=1$$
$$\lambda_{21}^2+\lambda_{22}^2+\lambda_{23}^2=1$$
$$\lambda_{31}^2+\lambda_{32}^2+\lambda_{33}^2=1$$

$$\hat{e}_n'\cdot\hat{e}_m'=0 \quad (n\neq m)$$

$$\lambda_{n1}\lambda_{m1}+\lambda_{n2}\lambda_{m2}+\lambda_{n3}\lambda_{m3}=0$$

orthogonal condition: $\sum_i \lambda_{ik}\lambda_{ij}=\delta_{kj}$

δ_{kj} = Kronecker delta symbol

$=1 \quad k=j$

$=0 \quad k\neq j$

$$\therefore R=\begin{pmatrix}\lambda_{11} & \lambda_{12} & \lambda_{13}\\ \lambda_{21} & \lambda_{22} & \lambda_{23}\\ \lambda_{31} & \lambda_{32} & \lambda_{33}\end{pmatrix}$$

→ 單位向量且互相垂直 ($\hat{e}_i$表示成$\hat{e}_i$)

↓ ($\hat{e}_i$表示成$\hat{e}_i'$) 單位向量且互相垂直

正交保証了向量大小不變(如下),因此當我們定向量遵守 $v_\alpha = R_{\alpha\beta}v_\beta$, $R_{\alpha\beta}$ 也須滿足正交,$\vec{v}$才是向量!

簡易矩陣運算

(i) 我們可以

$$\overset{\vec{r}'}{\begin{pmatrix}x_1'\\x_2'\\x_3'\end{pmatrix}}=\overset{R}{\begin{pmatrix}\lambda_{11} & \lambda_{12} & \lambda_{13}\\ \lambda_{21} & \lambda_{22} & \lambda_{23}\\ \lambda_{31} & \lambda_{32} & \lambda_{33}\end{pmatrix}}\overset{\vec{r}}{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}}$$

表示 $x_i'=\sum_j \lambda_{ij}x_j$ $\left[=\lambda_{ij}x_j \text{ (Einstein convention)}\right]$

或 $\vec{r}'=R\vec{r}$,所涉及之乘法⇒為矩陣乘法

如果寫成橫式:用同樣的乘法

$$(x_1',x_2',x_3')=(x_1,x_2,x_3)\begin{pmatrix}\lambda_{11} & \lambda_{21} & \lambda_{31}\\ \lambda_{12} & \lambda_{22} & \lambda_{32}\\ \lambda_{13} & \lambda_{23} & \lambda_{33}\end{pmatrix}$$

記為 $\vec{r}'^T = \vec{r}^T R^T$

T = transpose，即行與列對調

(ii) AB = ?

連續二個旋轉，最後相當某一個旋轉

$\vec{r}_1 = R_1 \vec{r}$

$\vec{r}_2 = R_2 \vec{r}_1$

$\vec{r}_2 = R\vec{r} = R_2 (R_1 \vec{r})$

由一次接一次旋轉可知

$R_2(R_1\vec{r}) \Rightarrow r_{2i} = \sum_j R_{2ij} r_{1j} = \sum_j R^2_{ij} \sum_k R^1_{jk} r_k$

$r_{1j} = \sum_k R^1_{jk} r_k$

$= \sum_k \left(\sum_j R^2_{ij} R^1_{jk} \right) r_k \equiv \sum_k R_{ik} r_k$

$\therefore R_{ik} = \sum_j R^2_{ij} R^1_{jk}$

換句話說，可以不先乘上 r 而直接 R^2 乘以 R^1

以得到 R，$\therefore$ $R = \underset{\text{矩陣乘法}}{R_2 R_1} = \begin{pmatrix} \cancel{R^2_{11}\ R^2_{12}} & R^2_{13} \\ R^2_{21} & R^2_{22} & R^2_{23} \\ R^2_{31} & R^2_{32} & R^2_{33} \end{pmatrix} \begin{pmatrix} R^1_{11} & R^1_{12} & R^1_{13} \\ R^1_{21} & R^1_{22} & R^1_{23} \\ R^1_{31} & R^1_{32} & R^1_{33} \end{pmatrix}$

注意與數不同之處：$AB \neq BA$ (in general)
但 $(AB)C = A(BC)$

(iii)

$$r'^2 = (x', y', z')\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \vec{r}^T R^T R \vec{r}$$

可視為　　　　　　　　　　　　矩陣乘法

$$= \vec{r}^T \vec{r}$$

矩陣乘法 (= 內積)

∴ 可以先算 $R^T R$

$$= \begin{pmatrix} \lambda_{11} & \lambda_{21} & \lambda_{31} \\ \lambda_{12} & \lambda_{22} & \lambda_{32} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{pmatrix}\begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv I$$ (單位矩陣)

∴ $= \vec{r}^T \vec{r} = r^2$　∴ 旋轉不改變長度

(iv) $(A+B)_{ij} = A_{ij} + B_{ij}$

∴ $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1+4 & 2+0 \\ 3+1 & 4+2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 2 \\ 4 & 6 \end{pmatrix}$$

(v) 反矩陣 A^{-1}.　$A^{-1}A = AA^{-1} = I$

Determinants

$$\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{32} \end{pmatrix}$$

$$\det \lambda = |\lambda| = \begin{vmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{vmatrix}$$

$$= \lambda_{11} \begin{vmatrix} \lambda_{22} & \lambda_{23} \\ \lambda_{32} & \lambda_{33} \end{vmatrix} - \lambda_{12} \begin{vmatrix} \lambda_{21} & \lambda_{23} \\ \lambda_{31} & \lambda_{33} \end{vmatrix} + \lambda_{13} \begin{vmatrix} \lambda_{21} & \lambda_{22} \\ \lambda_{31} & \lambda_{32} \end{vmatrix}$$

$$\det(AB) = \det A \det B$$

$\therefore \det \lambda = \Pi$ eigenvalues

$$\det \lambda^T = \begin{vmatrix} \lambda_{11} & \lambda_{21} & \lambda_{31} \\ \lambda_{12} & \lambda_{22} & \lambda_{32} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{vmatrix}$$

$$= \lambda_{11} \begin{vmatrix} \lambda_{22} & \lambda_{32} \\ \lambda_{23} & \lambda_{33} \end{vmatrix} - \lambda_{21} \begin{vmatrix} \lambda_{12} & \lambda_{32} \\ \lambda_{13} & \lambda_{33} \end{vmatrix} + \lambda_{31} \begin{vmatrix} \lambda_{12} & \lambda_{22} \\ \lambda_{13} & \lambda_{23} \end{vmatrix}$$

$$= \det \lambda$$

$\because R^T R = \mathbb{I}$. $\therefore$ $(\det R)^2 = 1$, $\det R = \pm 1$

$\because$ 是3維 $\det R = \Pi$ eigenvalues $\therefore$ $\begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix} \in \{R \mid \det R = -1\}$

$\vec{r}\,' = \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix} \vec{r} = -\vec{r}$ 為所謂的 inversion，不是真正的 rotation，$\therefore \det R = -1$ 稱為 improper rotations.

而 $\det R = 1$ 則是 proper rotation

因此向量的定義：$\{v_\alpha\}$

$$v_\alpha' = \sum_\beta R_{\alpha\beta} v_\beta$$

因此对 $\det R = -1$ 之轉換不一定 $\begin{cases} \text{polar} \\ \text{axial} \end{cases}$

其中 $R^T R = I \quad \det R = 1$

$R_1 R_2 = R_3$ (2次旋轉 = 一次淨旋轉)

例：$\vec{A}$ 及 $\vec{B}$ 為向量

則 $\vec{C} = \vec{A} \times \vec{B}$ 亦為向量.

設旋轉矩陣 $R = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}$

則旋轉後 $A_i' = \sum_j \lambda_{ij} A_j$

$B_i' = \sum_j \lambda_{ij} B_j$

此時 $\vec{C}' = \vec{A}' \times \vec{B}'$，我們需要証明

$$C_i' = \sum_j \lambda_{ij} C_j$$

对任意三個向量 $\vec{D}, \vec{E}, \vec{F}$

$\vec{D} \cdot \vec{E} \times \vec{F}$ 為 $\vec{D}, \vec{E}$ 及 $\vec{F}$ 所張之平行六面体之体積，此体積不隨旋轉而變 ($\det R = 1$)

$\therefore \vec{D}' \cdot \vec{E}' \times \vec{F}' = \vec{D} \cdot \vec{E} \times \vec{F}$

$\therefore \sum_{i'j'k'} \epsilon_{i'j'k'} D_{i'}' E_{j'}' F_{k'}' = \sum_{ijk} \epsilon_{ijk} D_i E_j F_k$

$$\therefore \sum_{ijk}\left(\sum_{i'j'k'} \epsilon_{i'j'k'} \lambda_{i'i} \lambda_{j'j} \lambda_{k'k}\right) D_i E_j F_k$$

$$= \sum_{ijk} \epsilon_{ijk} D_i E_j F_k$$

$$\therefore \sum_{i'j'k'} \epsilon_{i'j'k'} \lambda_{i'i} \lambda_{j'j} \lambda_{k'k} = \epsilon_{ijk}$$

$$\because \sum_{i} \lambda_{i'i} \lambda_{i''i} = \delta_{i'i''}$$

∴上式 $\times \lambda_{i''i}$ 並取 $\sum_i$ 可得

$$\sum_i \lambda_{i''i} \epsilon_{ijk} = \sum_{j'k'} \epsilon_{i''j'k'} \lambda_{j'j} \lambda_{k'k} \quad \leftarrow ①$$

因為 $C_{i'} = \sum_{j'k'} \epsilon_{i'j'k'} A_{j'} B_{k'}$

$$= \sum_{jk}\sum_{j'k'} \epsilon_{i'j'k'} \lambda_{j'j} \lambda_{k'k} A_j B_k$$

①式 $\rightarrow$ $$= \sum_{jk}\sum_{i} \lambda_{i'i} \epsilon_{ijk} A_j B_k$$

$$= \sum_i \lambda_{i'i} C_i$$

故得証 $\vec{C}$ 為一向量

<u>Polar v.s. axial</u> 雖然 $\vec{C}$ 為一向量，但与一般向量不同，一般向量在 inversion 下，$\vec{a} \rightarrow -\vec{a}$
$\vec{C}$ 則是 $\vec{C} \rightarrow \vec{C}$，故 $\vec{C}$ 稱 axial vector，而一般向量則是 polar vector 以為区别

並不只是有大小，方向而已，有 handness

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如：向量之定義不限制 $\det R = 1$

則 $\vec{L} = \vec{r} \times \vec{p}$

$\vec{L}' = \vec{r}' \times \vec{p}'$

$\vec{r}' = R\vec{r}$
$\vec{p}' = R\vec{p}$ 考慮 $R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$，$\vec{r}' = -\vec{r}$，$\vec{p}' = -\vec{p}$

則 $\vec{L}' = \vec{r} \times \vec{p} = \vec{L}$ 並不 $\neq R\vec{L}\ (-\vec{L})$！

則 $\vec{L}'$ 不是向量，與我們認知不同，

∴ $\vec{L}' = \vec{r}' \times \vec{p}' = \vec{L}$ 與 $\vec{L}' = R\vec{L} = -\vec{L}$ 產生矛盾，故要求 $\det R = 1$

向量空間

若 $\vec{a}' = R\vec{a}$　　$\vec{b}' = R\vec{b}$

則 $(\vec{a}' + \vec{b}') = R(\vec{a} + \vec{b})$

且 $c\vec{a}' = R(c\vec{a})$　　c = 常數

因此，若 $\vec{a}$ 及 $\vec{b}$ 是向量，則 $\vec{a} + \vec{b}$ 及 $c\vec{a}$ 也是向量，所有向量所成之集合 ⇒ 向量空間

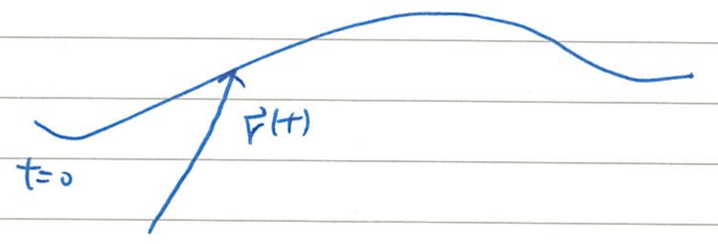
因此，$\vec{r}$ 是向量　$\vec{p}$，$\vec{a}$，$\vec{L} = \vec{r} \times \vec{p}$ 皆是向量！

$\int \vec{a}\, dt$

* 向量的微(積)分運算

古典力學中，質點的狀態是以 $\vec{r}(t)$ 來描述.

$\vec{r}(t) = (x(t), y(t), z(t))$, t = 時間，描述出一曲線



這種追蹤粒子走的描述又稱為 Lagranian description

古典力學是假設 $\vec{r}(t)$ 是可以精確量出的！
也就是測量對質點之影響(原則上)是可以降到幾乎是零，所以所測得之 $\vec{r}(t)$ 即質點沒被擾動時之 $\vec{r}(t)$！因此，可以單獨的抽出 $\vec{r}(t)$ 討論而不管造成 $\vec{r}(t)$ 之力與原因，此稱為運動學(Kinetics)
(量子物理的發現推翻了這種說法，測量會造成運動的改變，且改變之大小無法縮為零，因此，並沒有所謂的未擾動的 $\vec{r}(t)$！
所以 Kinetics 並不能單獨存在，必須與測量結合！)

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一旦知道 $\vec{r}(t)$，可以得到 $\vec{v}(t)=\frac{d\vec{r}}{dt}$ 及 $\vec{a}=\frac{d\vec{v}}{dt}=\frac{d^2\vec{r}}{dt^2}$.

一般而言，一個 $\vec{A}(s)$ 給定後

$$\frac{d\vec{A}(s)}{ds} = \lim_{\Delta s\to 0}\frac{\vec{A}(s+\Delta s)-\vec{A}(s)}{\Delta s}$$

$$= \left(\frac{dA_x(s)}{ds}, \frac{dA_y(s)}{ds}, \frac{dA_z(s)}{ds}\right)$$

在直角座標中　　成為对三個分量分別微分

由此可知 $$\frac{d(\vec{A}+\vec{B})}{ds} = \frac{d\vec{A}}{ds}+\frac{d\vec{B}}{ds}$$

$$\frac{d\vec{A}\cdot\vec{B}}{ds} = \frac{d\,(A_xB_x+A_yB_y+A_zB_z)}{ds}$$

$$= \frac{dA_xB_x}{ds}+\frac{dA_yB_y}{ds}+\frac{dA_zB_z}{ds} = \cdots$$

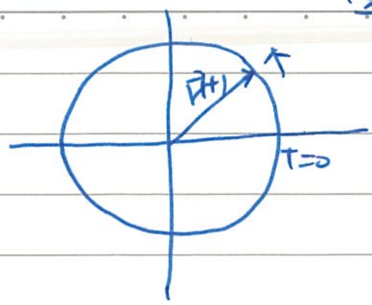
$$= \vec{A}\cdot\frac{d\vec{B}}{ds} + \frac{d\vec{A}}{ds}\cdot\vec{B},$$ $\left(\frac{d\vec{v}^2}{dt} = 2\vec{v}\cdot\frac{d\vec{v}}{dt} = \frac{dv^2}{dt} = 2v\frac{dv}{dt}\right)$

$$\frac{d\vec{A}\times\vec{B}}{ds} = \vec{A}\times\frac{d\vec{B}}{ds} + \frac{d\vec{A}}{ds}\times\vec{B}$$

$$\frac{d\,\phi(s)\vec{A}(s)}{ds} = \frac{d\phi}{ds}\vec{A}(s) + \phi(s)\frac{d\vec{A}}{ds}$$

(以上可以从分量之微分了解)

例：圓周運動



$$\vec{r}(t) = (R\cos\omega t,\ R\sin\omega t)$$

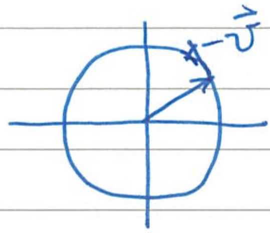
$$\vec{v} = \frac{d\vec{r}}{dt} = (-R\omega\sin\omega t,\ R\omega\cos\omega t)$$

<0 >0 for small t

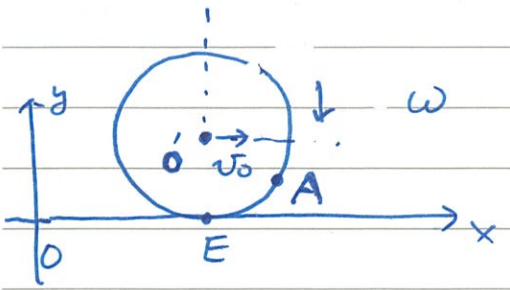
$$\vec{r}\cdot\frac{d\vec{r}}{dt} = 0$$

↳ 指向 ↖ 切線

∴



$$\vec{a} = \frac{d\vec{v}}{dt} = -R\omega^2(\cos\omega t,\ \sin\omega t) = -\omega^2\vec{r}$$



$$\vec{OA} = \vec{OO'} + \vec{O'A}$$

$$= (v_0 t,\ R) + (R\cos\omega t,\ -R\sin\omega t)$$

$$\vec{v_A} = \frac{d\vec{OA}}{dt} = (v_0,\ 0) + R\omega(-\sin\omega t,\ -\cos\omega t)$$

$\omega t = \frac{\pi}{2}$, $\Rightarrow A = E$, $\vec{v_E} = (v_0,\ 0) + (-R\omega,\ 0)$

$\therefore$ if $v_0 = R\omega$, $\vec{v_E} = 0$

此為純滾動之情形

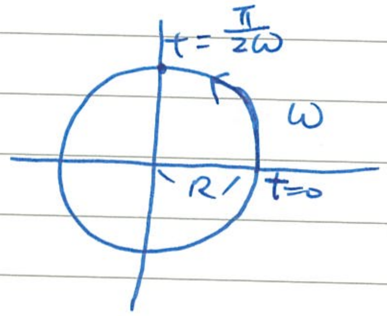
相对微分，向量的積分也是以分量定義的：

$$\int \vec{A}(s)\,ds = \int [A_x(s)\hat{e}_1 + A_y(s)\hat{e}_2 + A_z(s)\hat{e}_3]\,ds$$

$$= \left(\int A_x(s)\,ds\right)\hat{e}_1 + \left(\int A_y(s)\,ds\right)\hat{e}_2 + \left(\int A_z(s)\,ds\right)\hat{e}_3$$

$$= \left(\int A_x(s)\,ds,\ \int A_y(s)\,ds,\ \int A_z(s)\,ds\right)$$

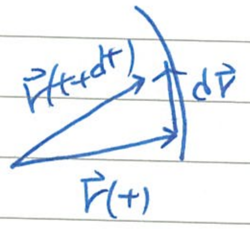
例：



求質点由 $t=0$ 到 $t=\frac{\pi}{2\omega}$ 所走之弧長

$$\vec{r}(t) = (R\cos\omega t,\ R\sin\omega t)$$

$$d\vec{r} = (-R\omega\sin\omega t\,dt,\ R\omega\cos\omega t\,dt)$$



$$= \vec{r}(t+dt) - \vec{r}(t)$$

$$|d\vec{r}| = ds = (R\omega)\,dt$$

$$\therefore\ s = \int_0^{\frac{\pi}{2\omega}} |d\vec{r}| = \int_0^{\frac{\pi}{2\omega}} \left|\frac{d\vec{r}}{dt}\right| dt$$

$$= \int_0^{\frac{\pi}{2\omega}} R\omega\,dt = \frac{\pi}{2}R$$

一般曲線的弧長（以 $t=0$ 定為起点）

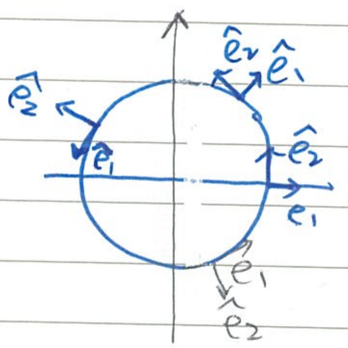
$$s(t) = \int_0^t ds = \int_0^t |d\vec{r}| = \int_0^t \left|\frac{d\vec{r}}{dt}\right| dt$$

$$= \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

非直角座標(曲線座標)之向量微積分

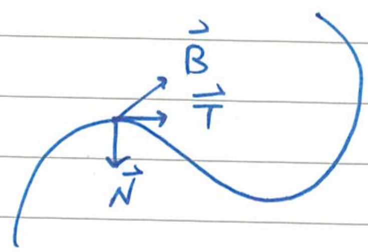
以上的微分及積分之所以可以一個分量一個分量做是因為 $\hat{e}_1, \hat{e}_2$ 及 $\hat{e}_3$ 與 s 無關之故。實際上,由於常需要使用曲線座標,此時,局部上可以找到互相垂直的單位向量 ($\hat{e}_i(s)$, $|\hat{e}_i|=1$),但它們不再與 s 無關。這樣一組 $\{\hat{e}_i\}$ 又稱為 moving trihedral.

例:

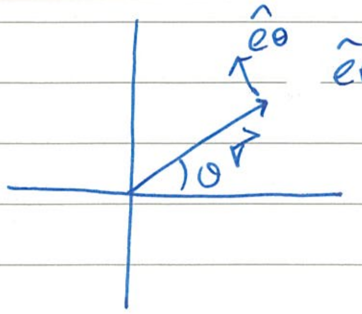


$\underbrace{\hat{e}_1=\hat{r},\ \hat{e}_2=\hat{\theta}}_{\text{極座標}}$ 與弧長有關

另一例見習題

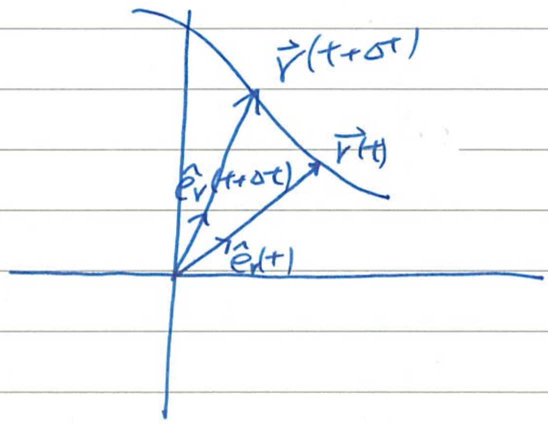


(2D) 2維極座標 (r, θ)



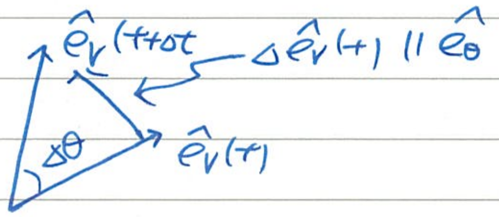
$$\vec{r} = r\hat{e}_r$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{e}_r + r\underbrace{\frac{d\hat{e}_r}{dt}}_{\neq 0\ !}$$



很明顯的

$$\lim_{\Delta t \to 0} \frac{\Delta \hat{e}_r(t)}{\Delta t} \perp \hat{e}_r$$

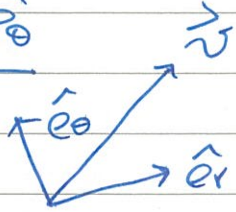


數學上，∵ $\hat{e}_r \cdot \hat{e}_r = 1$ ∴ $\frac{d\,\hat{e}_r \cdot \hat{e}_r}{dt} = 0$

∴ $\hat{e}_r \cdot \frac{d\hat{e}_r}{dt} = 0$ ∴ $\frac{d\hat{e}_r}{dt} \perp \hat{e}_r$

∵ $|\Delta\hat{e}_r| = |\hat{e}_r|\Delta\theta$ ∴ $\left|\frac{d\hat{e}_r}{dt}\right| = \frac{d\theta}{dt} = \omega$

∴ $d\hat{e}_r = d\theta\,\hat{e}_\theta$, $\underline{\frac{d\hat{e}_r}{dt} = \dot{\theta}\,\hat{e}_\theta}$ ---- (6)

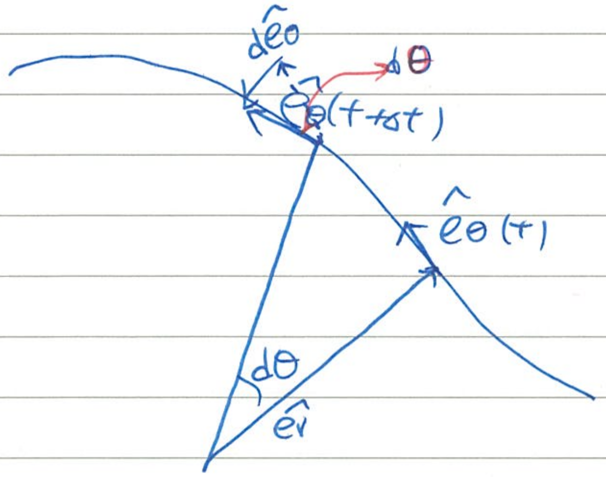


$$\vec{v} = \underbrace{\dot{r}\hat{e}_r}_{\text{徑向速度 } v_r} + \underbrace{r\dot{\theta}\hat{e}_\theta}_{\text{角速度, } v_\theta}$$

徑向速度 v_r　~~切線~~角速度, v_θ

同理，為了求 $\vec{a}=\frac{d\vec{v}}{dt}$，也須求 $\frac{d\hat{e_\theta}}{dt}$

$$\because \vec{a} = \ddot{r}\,\hat{e_r} + \dot{r}\,\frac{d\hat{e_r}}{dt} + \frac{d\,r\dot{\theta}}{dt}\,\hat{e_\theta} + r\dot{\theta}\,\frac{d\hat{e_\theta}}{dt}$$



由左圖可知

$d\hat{e_\theta} \parallel -\hat{e_r}$

$\because |d\hat{e_\theta}| = |e_\theta|\cdot d\theta$

$\therefore d\hat{e_\theta} = -\hat{e_r}\,d\theta$

$$\therefore \frac{d\hat{e_\theta}}{dt} = -\dot{\theta}\,\hat{e_r} \quad \text{---(7)}$$

$$\therefore \vec{a} = \ddot{r}\,\hat{e_r} + \dot{r}\dot{\theta}\,\hat{e_\theta} + \dot{r}\dot{\theta}\,\hat{e_\theta} + r\ddot{\theta}\,\hat{e_\theta} - r\dot{\theta}^2\,\hat{e_r}$$

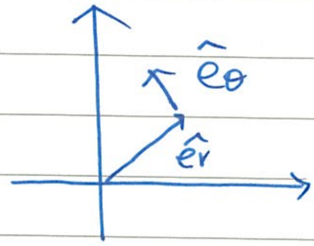
$$= \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{\text{向心加速度 } a_r}\hat{e_r} + \underbrace{(r\ddot{\theta} + 2\dot{r}\dot{\theta})}_{\text{角加速度 } a_\theta}\hat{e_\theta}$$

特例： $\dot{\theta} = \omega$ = 常數，r = 常數

等速圓周運動

$$\vec{a} = -r\omega^2\,\hat{e_r}$$

注意 ⑥ & ⑦ 式 也可以 直接 在 直角座標系 中推導：



$$\hat{e}_r = (\cos\theta,\ \sin\theta)$$

$$\hat{e}_\theta = (\cos(\theta + \pi/2),\ \sin(\theta+\pi/2))$$

$$= (-\sin\theta,\ \cos\theta)$$

$$\frac{d\hat{e}_r}{dt} = \left(\frac{d\cos\theta}{d\theta}\dot{\theta},\ \frac{d\sin\theta}{d\theta}\dot{\theta} \right)$$

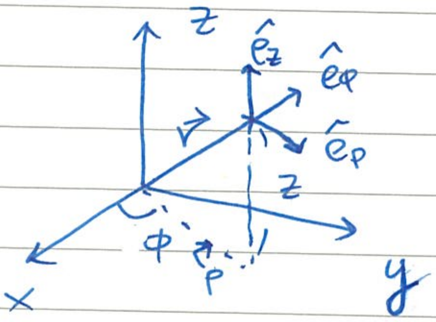
$$= (-\sin\theta\,\dot{\theta},\ \cos\theta\,\dot{\theta}) = \dot{\theta}\,\hat{e}_\theta$$

$$\frac{d\hat{e}_\theta}{dt} = \left(-\frac{d\sin\theta}{d\theta}\dot{\theta},\ \frac{d\cos\theta}{d\theta}\dot{\theta} \right)$$

$$= -\dot{\theta}(\cos\theta,\ \sin\theta) = -\hat{e}_r\,\dot{\theta}$$

其他 常用之座標系統

3維圓柱型座標 (3D, Cylinder coordinates)



$$\vec{r} = (x, y, z) = \rho\hat{e}_\rho + z\hat{e}_z$$

$$x = \rho\cos\phi$$

$$y = \rho\sin\phi$$

$$\rho = \sqrt{x^2+y^2}, \quad \phi = \tan^{-1}\frac{y}{x}$$

$$\hat{e}_z = (0, 0, 1) = \frac{d\vec{r}}{dz} \Big/ \left|\frac{d\vec{r}}{dz}\right|$$

$$\hat{e}_\rho = (\cos\phi, \sin\phi, 0) = \frac{d\vec{r}}{d\rho} \Big/ \left|\frac{d\vec{r}}{d\rho}\right|$$

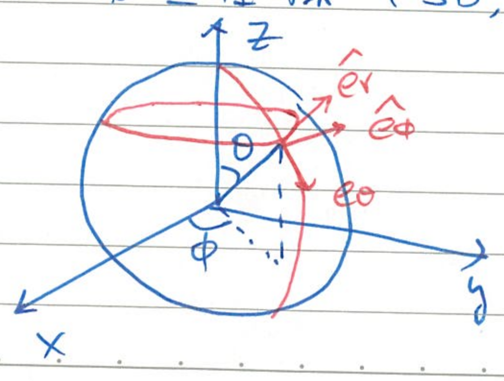
$$\hat{e}_\phi = (-\sin\phi, \cos\phi, 0) = \frac{d\vec{r}}{d\phi} \Big/ \left|\frac{d\vec{r}}{d\phi}\right|$$

$$\frac{d\hat{e}_z}{dt} = 0, \quad \frac{d\hat{e}_\rho}{dt} = \dot{\phi}\hat{e}_\phi, \quad \frac{d\hat{e}_\phi}{dt} = -\dot{\phi}\hat{e}_\rho$$

$$\therefore \quad \vec{v} = \dot{\rho}\hat{e}_\rho + \rho\dot{\phi}\hat{e}_\phi + \dot{z}\hat{e}_z$$

$$\vec{a} = (\ddot{\rho} - \rho\dot{\phi}^2)\hat{e}_\rho + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{e}_\phi + \ddot{z}\hat{e}_z$$

3維球型座標 (3D, spherical coordinates)



$$(r, \theta, \phi)$$

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$\hat{e}_r = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)\ \left(= \frac{d\vec{r}}{dr}\Big/\left|\frac{d\vec{r}}{dr}\right|\right)$$

$$\hat{e}_\theta = \frac{d\vec{r}}{d\theta}\Big/\left|\frac{d\vec{r}}{d\theta}\right| \quad (見圖中之紅線)$$

$$= (\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ -\sin\theta)$$

$$\hat{e}_\phi = \frac{d\vec{r}}{d\phi}\Big/\left|\frac{d\vec{r}}{d\phi}\right| = (-\sin\phi,\ \cos\phi,\ 0)$$

因此

$$\frac{d\hat{e}_r}{dt} = \frac{d\hat{e}_r}{d\theta}\dot{\theta} + \frac{d\hat{e}_r}{d\phi}\dot{\phi}$$

$$= \dot{\theta}\,\hat{e}_\theta + \sin\theta\,\dot{\phi}\,\hat{e}_\phi$$

$$\frac{d\hat{e}_\theta}{dt} = -\dot{\theta}\,\hat{e}_r + \cos\theta\,\dot{\phi}\,\hat{e}_\phi$$

$$\frac{d\hat{e}_\phi}{dt} = -\dot{\phi}(\cos\phi,\ \sin\phi,\ 0)$$

$$\because\ \hat{e}_r = \sin\theta(\cos\phi,\sin\phi,0) + \cos\theta(0,0,1)$$

$$\hat{e}_\theta = \cos\theta(\cos\phi,\sin\phi,0) - \sin\theta(0,0,1)$$

$$\therefore\ \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta = (\cos\phi,\sin\phi,0)$$

$$\therefore\ \frac{d\hat{e}_\phi}{dt} = -\dot{\phi}(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta)$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin\theta\,\dot{\phi}\,\hat{e}_\phi$$

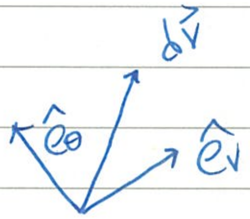
$$\vec{a} = (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2)\,\hat{e}_r$$

$$+ \left[\frac{1}{r}\frac{d(r^2\dot{\theta})}{dt} - r\sin\theta\cos\theta\,\dot{\phi}^2\right]\hat{e}_\theta$$

$$+ \left[\frac{1}{r\sin\theta}\,\frac{d}{dt}\left(r^2\sin^2\theta\,\dot{\phi}\right)\right]\hat{e}_\phi \qquad (\text{exercise})$$

相關之概念：面元及体元

亦即另一個寫法：



$$d\vec{r} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta$$

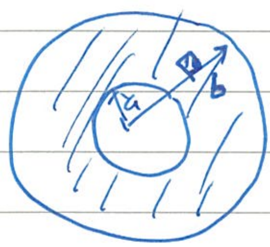
$\because \hat{e}_r \perp \hat{e}_\theta$ 且 $|\hat{e}_\theta| = 1$, $|\hat{e}_r| = 1$

$\therefore dr$ 及 $r\,d\theta$ 為 $d\vec{r} = (dx, dy)$ 在局部座標上之投影

$\therefore$ 面元 $= dx\,dy \Rightarrow dr \cdot r\,d\theta = r\,dr\,d\theta = da$

$$\therefore \iint dx\,dy \Rightarrow \iint r\,dr\,d\theta$$

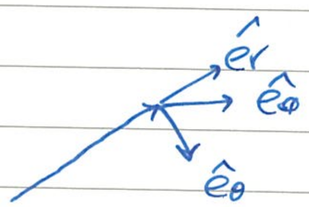
例



$$面積 = \int_a^b r\,dr \int_0^{2\pi} d\theta$$

$$= 2\pi \times \frac{1}{2}(b^2 - a^2) = \pi(b^2 - a^2)$$

体元



$$d\vec{r} = dr\,\hat{e_r} + r\,d\theta\,\hat{e_\theta} + r\sin\theta\,d\phi\,\hat{e_\phi}$$

$$\therefore\ dv = dx\,dy\,dz \Rightarrow dv = dr\ r\,d\theta\ r\sin\theta\,d\phi$$
$$= r^2\sin\theta\,d\theta\,d\phi\,dr.$$

面元, $da = r\,d\theta \cdot r\sin\theta\,d\phi$
$$= r^2\sin\theta\,d\theta\,d\phi$$

$$\therefore\ dv = da \cdot dr.$$

同理, 对 cylindrical coordinates

$$dv = \rho\,d\rho\,d\phi\,dz$$

$$da = \rho\,d\rho\,d\phi$$

一般曲線座標

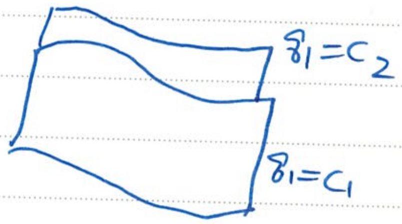
$q_1 = q_1(x, y, z)$

$q_2 = q_2(x, y, z)$

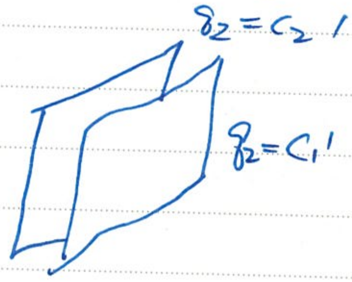
$q_3 = q_3(x, y, z)$

$\vec{r} = \{x(q_1, q_2, q_3),\ y(q_1, q_2, q_3),\ z(q_1, q_2, q_3)\}$

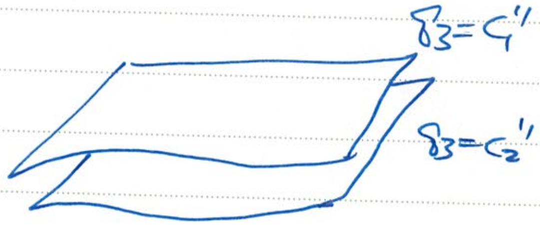
$q_1 =$ 常數 $\Rightarrow$ 曲面



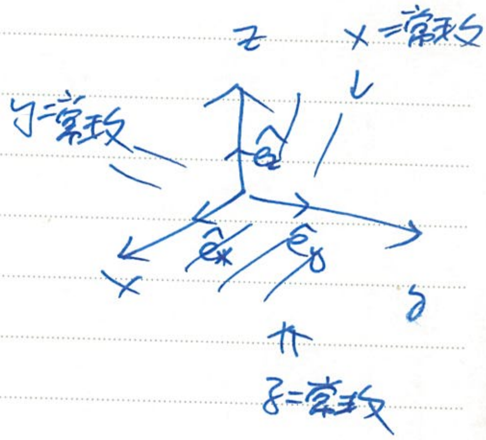
同理 $q_2 =$ 常數 $\Rightarrow$ 曲面



$q_3 =$ 常數 $\Rightarrow$ 曲面



這種情況和 xyz 座標相似

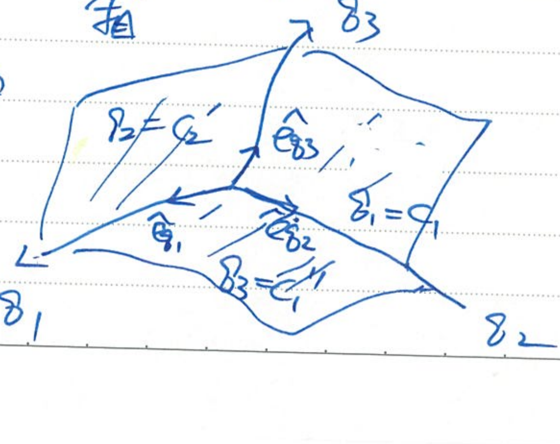


$x = C$, $y = C \Rightarrow$ xz plane

yz plane $\qquad z = C \Rightarrow$ xy plane

$\therefore q_1 = C_1,\ q_2 = C_1',\ q_3 = C_2''$ 之相交曲線

定出 q_1, q_2, q_3 之 3軸



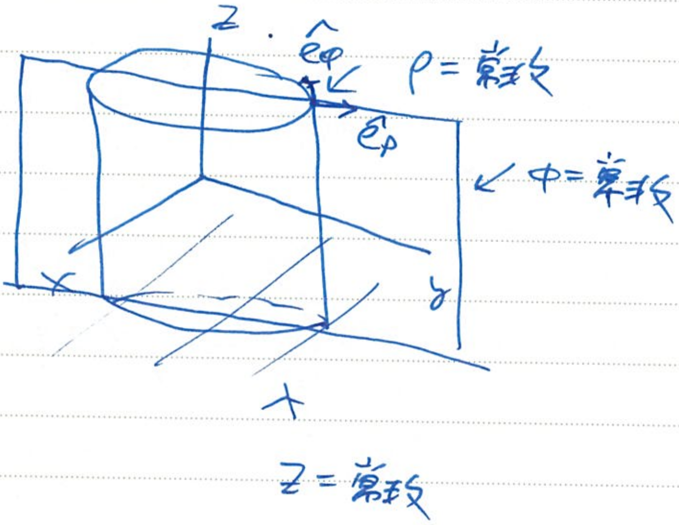
∴局部之單位向量可定為

$$\hat{e}_{q_1} = \frac{\partial\vec{r}/\partial q_1}{\left|\frac{\partial\vec{r}}{\partial q_1}\right|}$$

$$\hat{e}_{q_2} = \frac{\frac{\partial\vec{r}}{\partial q_2}}{\left|\frac{\partial\vec{r}}{\partial q_2}\right|}$$

$$\hat{e}_{q_3} = \frac{\frac{\partial\vec{r}}{\partial q_3}}{\left|\frac{\partial\vec{r}}{\partial q_3}\right|}$$

例：圓柱座標 (ρ, z, ϕ)



$$\hat{e}_\phi = \frac{\frac{\partial\vec{r}}{\partial\phi}}{\left|\frac{\partial\vec{r}}{\partial\phi}\right|}$$

$$\hat{e}_\rho = \frac{\frac{\partial\vec{r}}{\partial\rho}}{\left|\frac{\partial\vec{r}}{\partial\rho}\right|}$$

$$\hat{e}_z = \frac{\partial\vec{r}/\partial z}{\left|\frac{\partial\vec{r}}{\partial z}\right|}$$