Electrodynamics

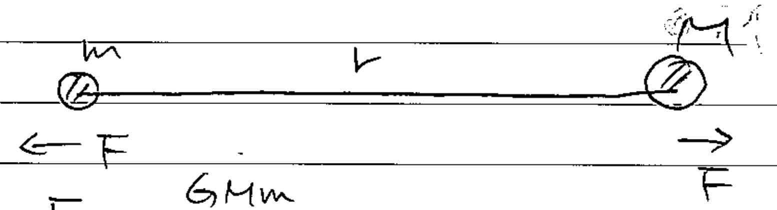
Introduction

Electrodynamics was complete at 1888 when Herz presented his decisive experimental confirmation of Maxwell's theory.

It presents an augment of our understandings about forces (force laws) in Newtonian mechanics:

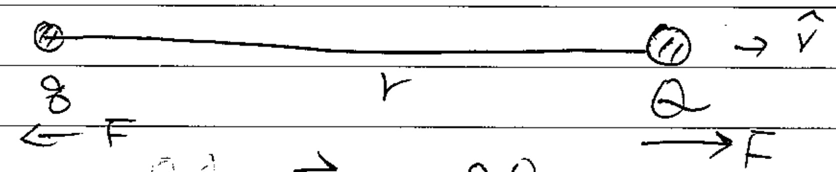
evolution of concept of interactions:

* Action at a distance. (超距作用)



$$F = \frac{GMm}{r^2}$$

or



$$\vec{F} = \frac{qQ}{4\pi\epsilon_0 r^2}\hat{r}$$

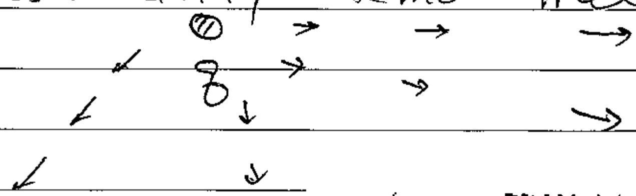
Forces can act ~~over~~ across a distance.

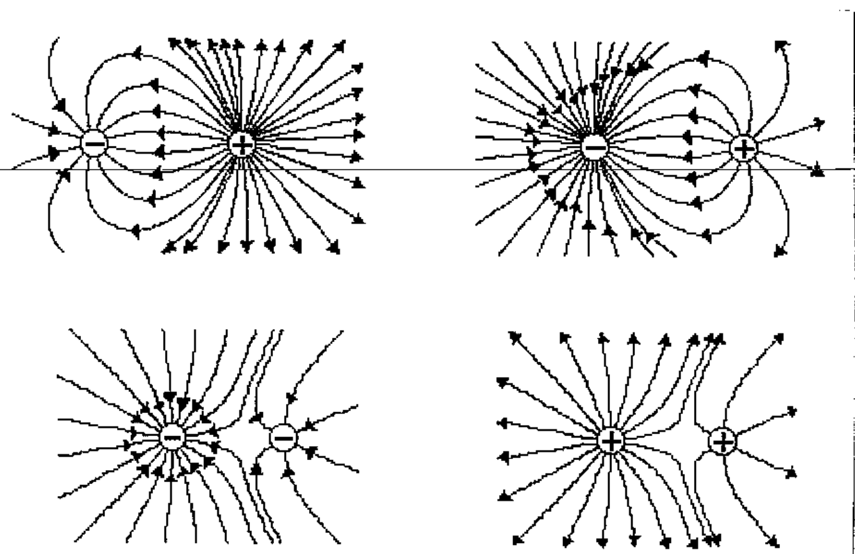
but how & why forces can act across space?

In Newton's theory of gravitation, there is no explanation.

The view about forces in modern era was changed, starting from the development of electrodynamics.

Originated from Faraday, he argued that forces do not simply act across space but space has already been changed when "charges" (or mass) are placed in the space. This was most easily demonstrated using "force lines" as follows





It leads to the concept of fields.

* Fields :

When a charge is placed in the space, the space around the charge is filled by electric (magnetic) fields.

$$\vec{E}(\vec{r},t) = \frac{\vec{F}(\vec{r},t)}{q_0} \quad \text{before}$$

$\uparrow$ test charge

another charge q_0 is introduced.

The ability of filling the strength of fields is called "charge"

example $M \overset{r}{\text{—————}} m_o$ $\quad$ Field $= \dfrac{\frac{GMm_o}{r}}{m_o}$

$$= \frac{GM}{r}\hat{r}$$

$\therefore$ M = charge (gravitational mass)

On the other hand, forces F act on M.

would accelerate M : with a = acceleration

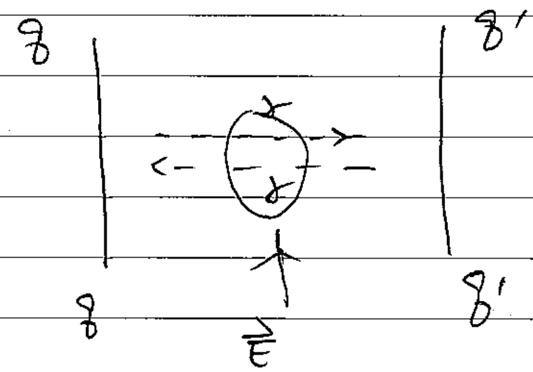
$$F = M_I a$$

M_I = inertia mass.

Expt. $M_I = M$.

* Photons, gluons, ... gauge bosons.

Further development of fields lead (Yukawa, 1948) to the conclusion : actually, charges are emitting particles constantly, and forces/interactions result from exchanging particles between charges



$\Leftrightarrow$ # of photons.

The view of gauge bosons is now the basis for "Standard Model" of elementary particles.

Interactions are characterized by the particles that are being exchanged:

Strong	—	gluons
Electromagnetic	—	photons
Weak	—	$W^{\pm}$, Z
Gravitational	—	graviton (not observed yet)

From the above brief outline of our understanding of interaction, it is clear that electromagnetism is the first example of modern theory of interaction.

In addition, it is clear that one needs mathematical tools to describe fields.

The mathematical tool is the vector analysis, which we shall introduce to you in the followings.

Vector Analysis.

As we discussed in the introductor, the fundamental object in electrodynamics is the field which is a vector that depends on positions $\vec{r}$ and time t. Therefore, we need mathematical tool to describe a vector field.

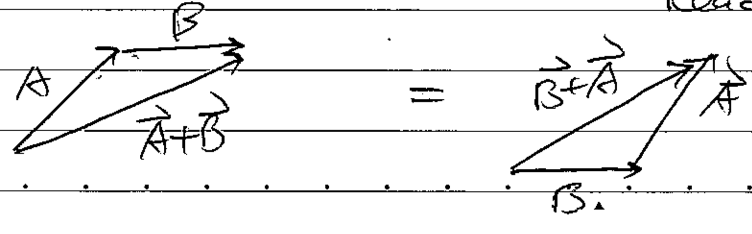
Vector algebra

A vector = an object with direction and magnitude
$\equiv \vec{A}$ (or bold face **A**)
example: displacement $\vec{r}$, $\vec{E}$, $\vec{p}$, ... denoted as an arrow $\nearrow$ with length = magnitude
(Mathematicians have abstract definition of vectors which we shall not explore extensively here) magnitude $\equiv |\vec{A}|$ $(|A|) = A$ $\nearrow$ = direction

The concept of vectors does not rely on cooridinate systems and be operated by addition and multipications

Operations

(i) Addition : $\vec{A}+\vec{B} \equiv$ place tail of $\vec{B}$ at head of $\vec{A}$ (definition)



associative: $(\vec{A}+\vec{B})+\vec{C} = \vec{A}+(\vec{B}+\vec{C})$

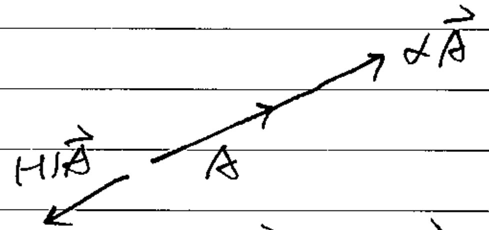
(ii) Multiplication by a scalar

$\alpha \vec{A}$: magnitude $= \alpha |\vec{A}|$

↑

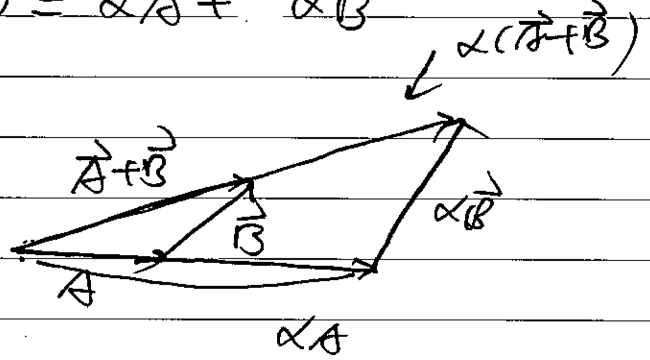
real direction = direction of $\vec{A}$

#



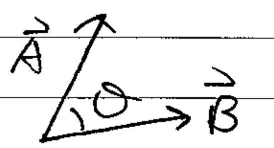
$(-1)\vec{A} \equiv -\vec{A}$ $\quad\therefore\quad \vec{A}+(-\vec{B}) \equiv \vec{A}-\vec{B}$

distributive: $\alpha(\vec{A}+\vec{B}) = \alpha\vec{A}+\alpha\vec{B}$



(iii) Inner product (dot product)

$\vec{A}\cdot\vec{B} \equiv AB\cos\theta$ = product of A times projection of $\vec{B}$ along $\vec{A}$

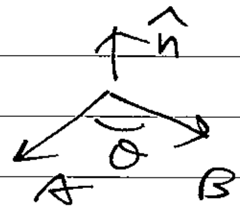


$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ (commutative) $\quad \vec{A}\cdot\vec{A} \equiv A^2$

$\vec{A}\cdot\vec{B} = 0 \quad \vec{A}$ & $\vec{B}$ are perpendicular distributive $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$

(iv) Cross product of two vectors

$$\vec{A}\times\vec{B} \equiv AB\sin\theta\ \hat{n}$$



$\vec{B}\times\vec{A} = -\vec{A}\times\vec{B}$ (non-commutative)

right-hand rule

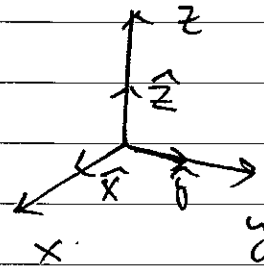
distributive $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$

$$\vec{A}\times\vec{A} = 0$$

Component form

It's more convenient to express vectors in a cooridinate system.

Let $\hat{x}$, $\hat{y}$, & $\hat{z}$ be unit vectors ($|\hat{A}|=1$) parallel to x, y and z axes respectively.



$\hat{x}\cdot\hat{y} = 0$

$\hat{x}\cdot\hat{x} = 1$, $\hat{y}\cdot\hat{y} = 1$, $\hat{z}\cdot\hat{z} = 1$

$\hat{x}\cdot\hat{z} = 0$

$\hat{y}\cdot\hat{z} = 0$ --- ①

$(\hat{x}, \hat{y}, \hat{z})$ are basis vectors

so that any vector $\vec{A}$ can be expressed in terms of $\hat{x}$, $\hat{y}$ & $\hat{z}$. $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$

A_x, A_y, A_z are components of $\vec{A}$.

usually, expressed as $\vec{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$

Furthermore, by using ①, it's easy to see

$$A_x = \vec{A}\cdot\hat{x}$$
$$A_y = \vec{A}\cdot\hat{y}$$
$$A_z = \vec{A}\cdot\hat{z}$$

Using the component form, one can show the following rules

(i) $\alpha\vec{A} = (\alpha A_x)\,\hat{x} + (\alpha A_y)\,\hat{y} + (\alpha A_z)\,\hat{z}$

(ii) $\vec{A}+\vec{B} = (A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) + (B_x\hat{x} + B_y\hat{y} + B_z\hat{z})$

$$= (A_x+B_x)\,\hat{x} + (A_y+B_y)\,\hat{y} + (A_z+B_z)\,\hat{z}$$

(iii) $\vec{A}\cdot\vec{B} = (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\cdot(B_x\hat{x} + B_y\hat{y} + B_z\hat{z})$

$$= A_xB_x + A_yB_y + A_zB_z$$

$\therefore\ A^2 = A_x^2 + A_y^2 + A_z^2$

$\hat{x}\times\hat{x} = 0,\ \hat{y}\times\hat{y} = 0,\ \hat{z}\times\hat{z} = 0,\quad \hat{x}\times\hat{y} = -\hat{y}\times\hat{x} = \hat{z}$

$\hat{y}\times\hat{z} = -\hat{z}\times\hat{y} = \hat{x}\ ;\ \hat{z}\times\hat{x} = -\hat{x}\times\hat{z} = \hat{y}$

$$\vec{A}\times\vec{B} = (A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) \times (B_x\hat{x} + B_y\hat{y} + B_z\hat{z})$$

$$= (A_yB_z - A_zB_y)\hat{x} + (A_zB_x - A_xB_z)\hat{y}$$

$$+ (A_xB_y - A_yB_x)\hat{z}$$

$$\equiv \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad \text{--- (2)}$$

Triple and multiple products

From inner product and cross product, one can form triple or multiple products.

example:

$$\vec{A}\cdot(\vec{B}\times\vec{C}), \quad \vec{A}\times(\vec{B}\times\vec{C}), \quad (\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}), \ldots \text{etc.}$$

These products usually can be reduced/simplified.

For instance, from (2), one sees

$$\vec{C}\cdot(\vec{A}\times\vec{B}) \underset{\uparrow}{=} \begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

replacing
$\hat{x}$ by C_x
$\hat{y}$ by C_y
$\hat{z}$ by C_z

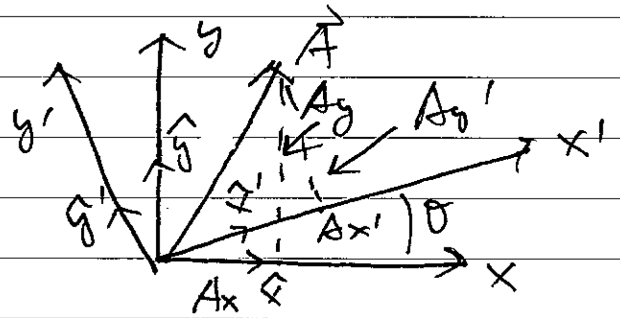
$$\therefore \vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Similarly, one can show

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$$

Transformation of vectors

Vectors are fixed objects and will not be changed if one uses different cooridinates. However, the components change.



For instance, in the left figure, one sees that going from (x,y) to (x',y')

$$\vec{A} = \begin{pmatrix} A_x \\ A_y \end{pmatrix} \rightarrow \vec{A} = \begin{pmatrix} A_{x'} \\ A_{y'} \end{pmatrix}'.$$

In general

$$\because\ \hat{x} = R_{xx}\hat{x}' + R_{yx}\hat{y}'$$

$$\hat{y} = R_{yy}\hat{y}' + R_{xy}\hat{x}'$$

$$\vec{A} = A_x\hat{x} + A_y\hat{y} = A_{x'}\hat{x}' + A_{y'}\hat{y}'$$

$$= A_x(R_{xx}\hat{x}' + R_{yx}\hat{y}') + A_y(R_{yy}\hat{y}' + R_{xy}\hat{x}')$$

$$\therefore\ A_{x'} = R_{xx}A_x + R_{xy}A_y$$

$$A_{y'} = R_{yx} A_x + R_{yy} A_y$$

which are written as

$$\begin{pmatrix} A_{x'} \\ A_{y'} \end{pmatrix} = \underbrace{\begin{pmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{pmatrix}}_{R} \begin{pmatrix} A_x \\ A_y \end{pmatrix} \quad \cdots (3)$$

The matrix R characterizes the transformation.
Note that
For rotations, $R_{xx} = \cos\theta$, $R_{yx} = -\sin\theta$

$R_{xy} = \sin\theta$, $R_{yy} = \cos\theta$

$$\therefore \quad R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

* Conversely, given components that trans-form according to (3), they are components of a vector!

A quantity which is not changed under R → scalar

Differential calculus

Three "derivatives"

As we indicated, to describe electric/magnetic fields, one needs to deal with

$$\vec{E}(x,y,z), \quad \vec{B}(x,y,z).$$

and generally $f(x,y,z)$ (i.e. $\rho(x,y,z)$, charge density)

the derivative for one variable x & scalar
∴ function $f(x)$, $\frac{df}{dx}$
(not change under rotation)

needs to be generalized.

Gradient

For one variable, function $f(x)$ one has the change of f

$$df = f(x+dx) - f(x) = \frac{df}{dx}dx.$$

For a function of two or three variables (x, y, z), $T(x, y, z)$

Obviously, the change of T depends on change of $\vec{r} \equiv (x, y, z)$, i.e., ∵ dx is replaced by $d\vec{r} = (dx, dy, dz)$

$$\therefore dT \equiv T(\vec{r}+d\vec{r}) - T(\vec{r})$$

$$= T(x+dx, y+dy, z+dz) - T(x, y, z)$$

$$= \frac{\partial T(x,y,z)}{\partial x}dx + \frac{\partial T(x,y,z)}{\partial y}dy + \frac{\partial T(x,y,z)}{\partial z}dz$$

$$= \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right)\cdot(dx, dy, dz)$$

∴ the object $\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right) \equiv \nabla T$

determines the change of T

∇T is called gradient of T.

With $\vec{\nabla} \equiv$ gradient which can be viewed as an

operator as $\frac{d}{dx}$ but $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$= \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$

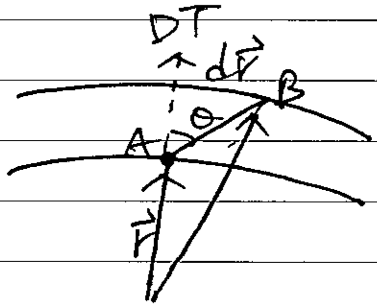
$=$ del operator

it acts on object that follows

$\vec{\nabla} f \neq f\vec{\nabla}$

Geometrical meaning: ∵ $dT = \vec{\nabla} T \cdot d\vec{r}$

$d\vec{r} = (dx, dy, dz)$ is the dispacement vector from $\vec{r}$ to $\vec{r} + d\vec{r}$



∴ $dT = |\vec{\nabla} T| \cdot |d\vec{r}| \cdot \cos\theta$

θ is the angle between $\vec{\nabla} T$ & $d\vec{r}$.

Obviously, when B moves around with fixed $|d\vec{r}|$,

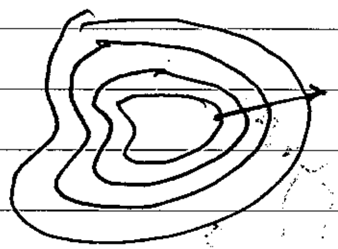
(i) $\theta = 0$, dT is the greatest! i.e. when $d\vec{r} \parallel \vec{\nabla} T$

$dT = T(\vec{r}+d\vec{r}) - T(\vec{r})$ is the largest !

$\therefore$ $\vec{\nabla}T$ is the direction with largest change of T (maximum increase!)

In the contour plot ($T(x,y,z) = \text{Const}$ plotting curves with)

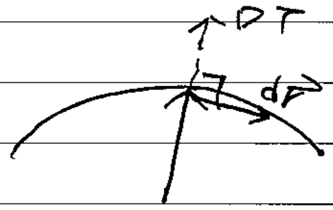
$\vec{\nabla}T$ is the direction of steep ascent



(ii) when $\vec{\nabla}T \perp d\vec{r}$, $dT = d\vec{r} \cdot \nabla T = 0$

$\therefore$ on the contour of T (i.e. $T = \text{const}$),

$\nabla T \perp$ contour



(iii) $\vec{\nabla}T = 0$ (x,y,z) is a stationary point ($\because dT = 0$)

This is similar to one variable case:

$\left.\frac{df(x)}{dx}\right|_{x=x_0} = 0$ — x_0 is a stationary point.

Differentiations of vector fields

The differentiation becomes more complicated when the function (T) becomes a vector:

$$\vec{v}(x, y, z) \quad (=\vec{v}(\vec{r}))$$
$$=(v_x(\vec{r}), v_y(\vec{r}), v_z(\vec{r}))$$

Clearly, one can discuss changes of each component $v_i(\vec{r})$ along different directions: $\frac{\partial}{\partial x} v_i(\vec{r})$, $\frac{\partial}{\partial y} v_i(\vec{r})$, $\frac{\partial}{\partial z} v_i(\vec{r})$

$\therefore$ The most general "derivative" would be $\vec{\nabla} v_i(\vec{r})$, $i = x, y, z$,

hence there are nine derivatives:

$$\partial_j v_i \equiv \frac{\partial}{\partial x_j} v_i \qquad x_1 = x,\ x_2 = y,\ x_3 = z$$

which can be viewed as a matrix denoted by

$$\vec{\nabla}\vec{v} \equiv \overleftrightarrow{T} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

$$T_{ij} = \partial_j v_i$$

which is known as <u>2nd rank tensor</u>.

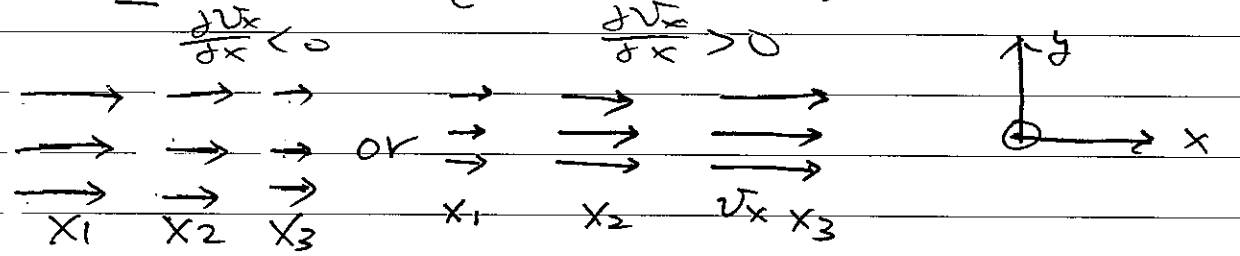
It turns out that there are two special combinations that have clear physical meaning for characterizing $\vec{v}(\vec{r})$.

(i) The divergence. $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \equiv \vec{\nabla} \cdot \vec{v}$

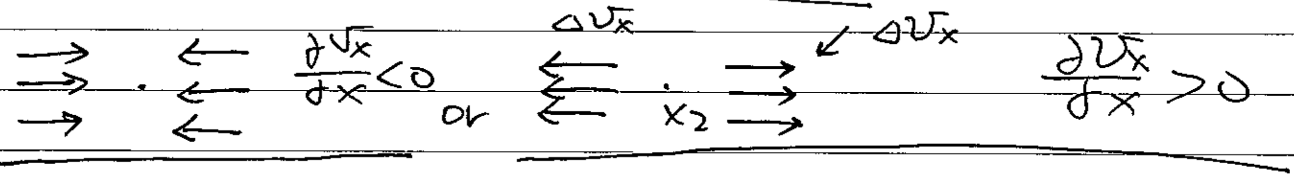
For each component, just as $T(x,y,z)$,

$\frac{\partial}{\partial x} v_x$ describes changes of $\underline{v_x}$ along

$\underline{x}$ direction (same direction):



relative to x_2, one has



$\therefore \left.\frac{\partial v_x}{\partial x}\right|_{\text{at } \vec{r}_0}$ describes "net outflow / inflow" in x direction at $\vec{r}_0$

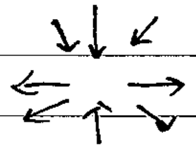
Adding all directions, one defines

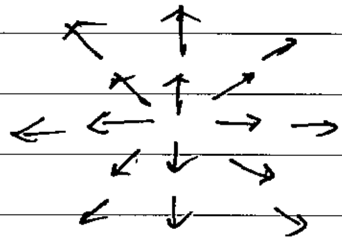
the divergence $\underset{\text{of } \vec{v}}{\equiv} \vec{\nabla} \cdot \vec{v} = \frac{\partial}{\partial x} v_x + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

In the most general situation, it's possible to have net outflow/inflow at a given point $\vec{r}_0$

In that case, the situation is like a sprinkler sprinkling water.

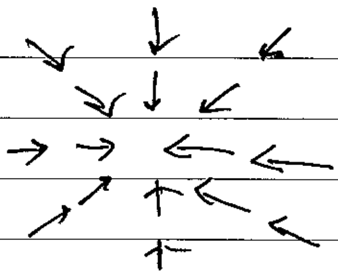
or mixed





"Source"
"Sprinkling"

or



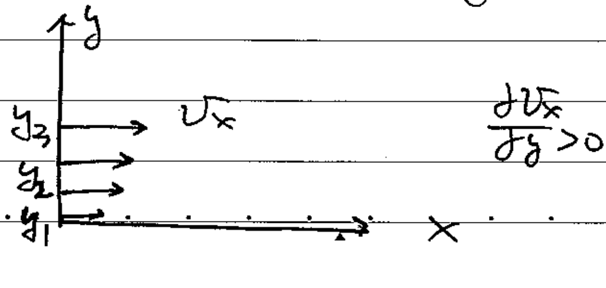
"Sink"
"draining"

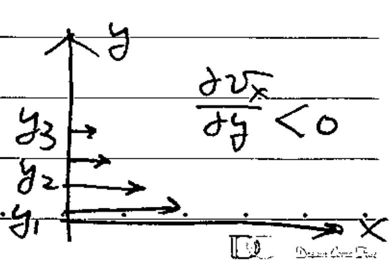
example: $\vec{v} = \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$\vec{\nabla}\cdot\vec{v} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

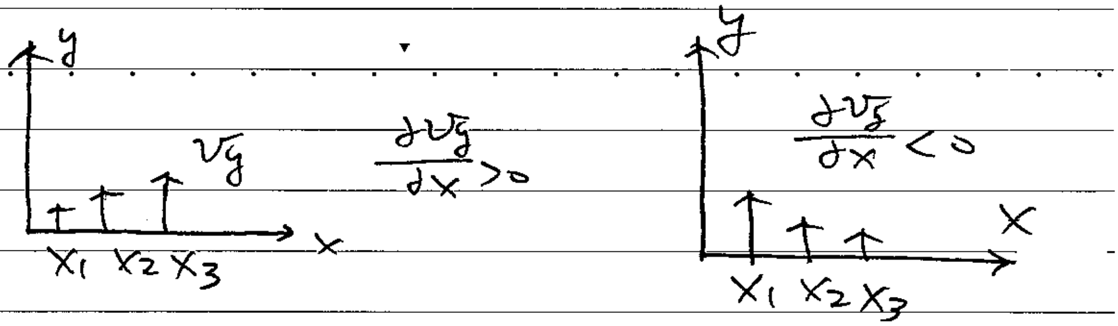
(ii) The curl $\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$

For each component, say v_x, in addition to change along x direction, one can also have changes along y or z directions. In that case, one has "shearing" (like scissors):

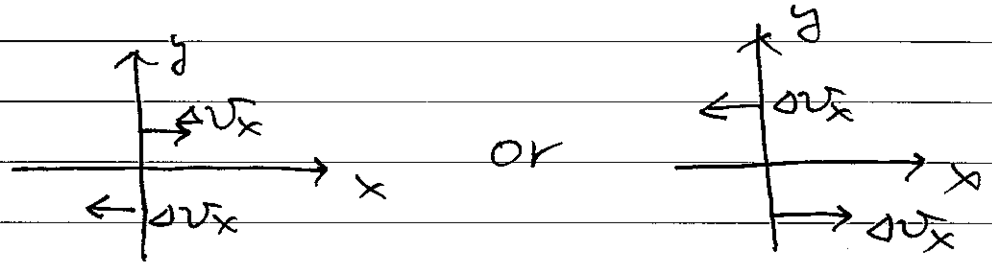




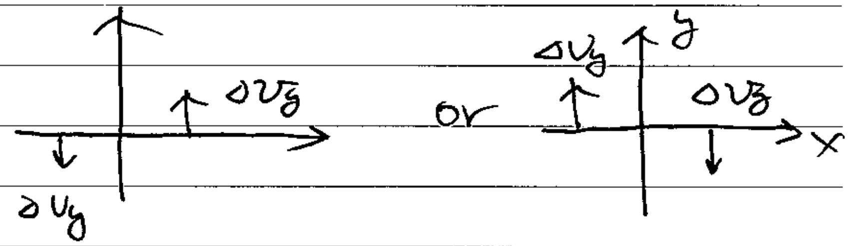
Or



relative to y_2, one has



relative to x_2



One sees that $\left.\frac{\partial v_x}{\partial y}\right|_{\vec{r}_0}$ & $\left.\frac{\partial v_y}{\partial x}\right|_{\vec{r}_0}$ measures swirls about $\vec{r}_0$ and furthermore, when $\frac{\partial v_x}{\partial y}$ & $\frac{\partial v_y}{\partial x} > 0$, the directions of swirls are opposite: $\frac{\partial v_x}{\partial y} > 0$ $\frac{\partial v_y}{\partial x} > 0$

$\therefore$ one defines $\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}$ as positive along z direction: $(\vec{\nabla} \times \vec{v})_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}$

Extension to other directions leads to the definition of curl

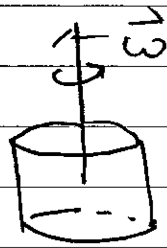
$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

to measure "vortex" like object in $\vec{v}$

Example : $\vec{v} = \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$\nabla\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0 \qquad \text{(irrotational)}$$

Example : Rotation of water



$$\vec{v} = \vec{\omega}\times\vec{r} \qquad \vec{\omega} = (0, 0, \omega)$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = (-y\omega, x\omega, 0)$$

$v_x = -y\omega, \quad v_y = x\omega, \quad v_z = 0$

$$\nabla\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial w} \\ -\omega y & \omega x & 0 \end{vmatrix} = \left(\frac{\partial x\omega}{\partial x} + \frac{\partial y\omega}{\partial y}\right)\hat{z}$$

$$= 2\omega\hat{z}$$

Products of differentiations

Distribution

$$\vec{\nabla}(f+g) = \vec{\nabla}f + \nabla g$$

$$\vec{\nabla}\cdot(\vec{A}+\vec{B}) = \vec{\nabla}\cdot\vec{A} + \vec{B}\cdot\vec{B}$$

$$\vec{\nabla}\times(\vec{A}+\vec{B}) = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{B}$$

Products

(i) $\vec{\nabla}(fg) = (\vec{\nabla}f)g + f\vec{\nabla}g \quad \left(\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}\right)$

(ii) $\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}$

(iii) $\vec{\nabla}\cdot(f\vec{A}) = f\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}f$

(iv) $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot\vec{\nabla}\times\vec{A} - \vec{A}\cdot\vec{\nabla}\times\vec{B}$

(v) $\vec{\nabla}\times(f\vec{A}) = f\vec{\nabla}\times\vec{A} - \vec{A}\times\nabla f$

(vi) $\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - B(\vec{\nabla}\cdot\vec{A})$

These identities can be checked by writing out components.

For instance,

$$\vec{\nabla}\cdot(f\vec{A}) = \underbrace{\frac{\partial}{\partial x}(fA_x)}_{\frac{\partial f}{\partial x}A_x + f\frac{\partial A_x}{\partial x}} + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z)$$

$$= \frac{\partial f}{\partial x}A_x + \frac{\partial f}{\partial y}A_y + \frac{\partial f}{\partial z}A_z + f\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

$$= \vec{\nabla}f\cdot\vec{A} + f\vec{\nabla}\cdot\vec{A}$$

Note that $\vec{\nabla}$ always acts on object that follows $\vec{\nabla}$, $\therefore$ $\vec{A}\cdot\vec{\nabla} \neq \vec{\nabla}\cdot\vec{A}$!

Second derivatives

When one forms products that involve with $\vec{\nabla}$, one can also form products of $\vec{\nabla}$ and $\vec{\nabla}$.

There are 5 different ways of products that involve two $\vec{\nabla}$'s:

(i) Acting on scalar function T

$$① \ \vec{\nabla}\cdot(\vec{\nabla}T) \equiv \nabla^2 T$$

$$② \ \vec{\nabla}\times(\vec{\nabla}T)$$

① $\underline{\vec{\nabla}\cdot(\vec{\nabla}T)} = \vec{\nabla}\cdot\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right)$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) T$$

$$\therefore \ \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \text{Laplacian}$$

$$\nabla^2 T = \text{Laplacian of } T$$

Note that ∇^2 can also act on a vector $\vec{v}$

$$\nabla^2\vec{v} = (\nabla^2 v_x, \nabla^2 v_y, \nabla^2 v_z)$$

② $\underline{\vec{\nabla}\times(\vec{\nabla}T)} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} \end{vmatrix}$ $\quad \left(\vec{\nabla}T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right)\right)$

$$= \left(\frac{\partial^2 T}{\partial y\partial z} - \frac{\partial^2 T}{\partial z\partial y}, \frac{\partial^2 T}{\partial z\partial x} - \frac{\partial^2 T}{\partial x\partial z}, \frac{\partial^2 T}{\partial x\partial y} - \frac{\partial^2 T}{\partial y\partial x}\right)$$

$\therefore \ \vec{\nabla}\times(\vec{\nabla}T) = 0$ if the second derivatives of T

Commute : $\frac{\partial^2 T}{\partial x_i\partial x_j} = \frac{\partial^2 T}{\partial x_j\partial x_i}$

$\vec{\nabla}\times(\vec{\nabla}T)=0$ is consistent with the naive expectation: $\vec{\nabla}\times\vec{\nabla}=0$

(A vector cross-product with itself $=0$)

(gradient field has no vortex!)

(ii) Acting on a vector field

③ $\vec{\nabla}(\vec{\nabla}\cdot\vec{v})$ $(\neq\nabla^2\vec{v})$ (gradient-of-divergence)

④ $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v})$

⑤ $\vec{\nabla}\times(\vec{\nabla}\times\vec{v})$

④ $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v})=0$ (curl "vortex" has no divergence!)

this is similar to the vector identity

$\vec{A}\cdot(\vec{A}\times\vec{B})=0$

However, $\vec{v}\cdot(\vec{\nabla}\times\vec{v})\neq 0$ while $\vec{B}\cdot(\vec{A}\times\vec{B})=0$

$$\vec{v}\cdot\vec{\nabla}\times\vec{v} = v_x(\vec{\nabla}\times\vec{v})_x + v_y(\vec{\nabla}\times\vec{v})_y + v_z(\vec{\nabla}\times\vec{v})_z$$

$$= v_x\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)+v_y\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)+v_z\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)$$

$$\neq 0$$

⑤ $\vec{\nabla}\times(\vec{\nabla}\times\vec{v})=\vec{\nabla}(\vec{\nabla}\cdot\vec{v})-\nabla^2\vec{v}$

$\Leftrightarrow \vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})$

$\therefore$ curl-of-curl = ③ − Laplacian

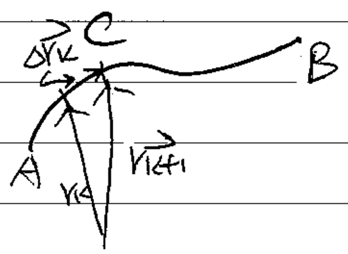
$\therefore$ Only two kinds of 2nd derivatives: Laplacian + gradient-of-divergence

Integrations of vector fields

There are many situations in electrodynamics in which one needs to perform integrations over vector fields. The most important are:

Line integral

Given a curve C,



$$\int_{A\,(C)}^{B} \vec{v}\cdot d\vec{r}$$

Closed curve $\Rightarrow \oint_C \vec{v}\cdot d\vec{r}$

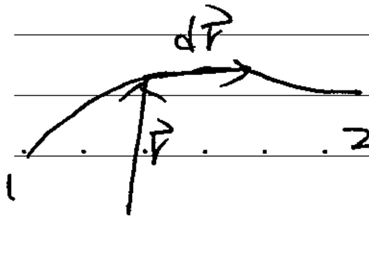
Which is defined by cutting the curve into $\Delta\vec{r}_1, \Delta\vec{r}_2, \dots$ and perform the sum:

$$\int_{A\,(C)}^{B} \vec{v}\cdot d\vec{r} = \lim_{N\to\infty} \sum_{k=0}^{N-1} \vec{v}(\vec{r}_k)\cdot\Delta\vec{r}_k$$

The most familiar example is the work done by a force $\vec{F}(\vec{r})$:

during a small displacement $d\vec{r}$

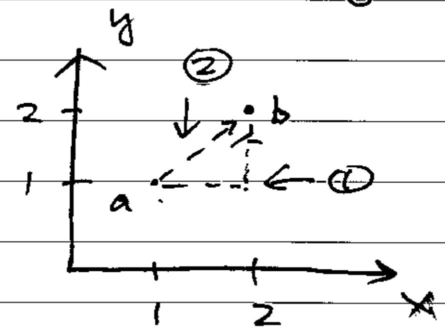
$$dw = \vec{F}(\vec{r})\cdot d\vec{r}$$



$$W_{1\to 2} = \int_1^2 \vec{F}\cdot d\vec{r}$$

example :

$$\vec{v} = y^2 \hat{x} + 2x(y+1)\hat{y}$$



path ①
$$\int_{①} \vec{v} \cdot d\vec{r} = \int_{(1,1)}^{(2,1)} \vec{v} \cdot \underbrace{d\vec{r}}_{dx\,\hat{x}} + \int_{(2,1)}^{(2,2)} \vec{v} \cdot \underbrace{d\vec{r}}_{dy\,\hat{y}}$$

$$= \int_1^2 dx \; y^2 \Big|_{y=1} + \int_1^2 dy \; 2x(y+1) \Big|_{x=2}$$

$$= 1 + 4\int_1^2 dy\,(y+1) = 1 + 4\left[\tfrac{1}{2}(4-1) + 1\right]$$

$$= 11$$

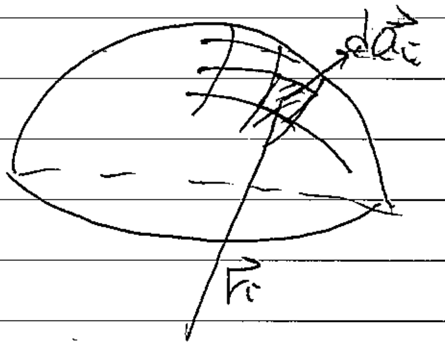
path ② On path ②, $x = y = t$, $d\vec{r} = (dt, dt)$

$$\vec{v} = (t^2, 2t(t+1))$$

$$\therefore \int_{①} \vec{v}\, d\vec{r} = \int_1^2 \left[t^2 dt + 2t(t+1)dt\right]$$

$$= \int_1^2 (3t^2 + 2t)\,dt = t^3 + t^2 \Big|_1^2 = 10$$

Surface integral



Given a surface S, one can divide it into infinitely many infinitesimal patches of area $d\vec{a}_i$

$d\vec{a}_i = da_i \, \hat{n}_i$ $\hat{n}_i$ is a unit normal to the surface,

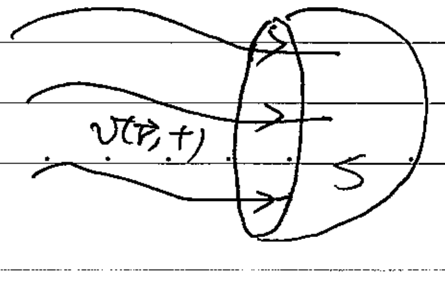
then $\displaystyle\int_S \vec{v} \cdot d\vec{a} = \lim_{N\to\infty} \sum_i \vec{v}(\vec{r}_i) \cdot \hat{n}_i \, da_i$

To be able to define $\hat{n}$, the surface has to be orientable. There are surfaces that one can't define $\hat{n}$ (i.e. in & out) such as Möbius strip. in that case, one can't define surface integrals on these surfaces

For close surfaces, one denote $\displaystyle\oint_S \vec{v} \cdot d\vec{a}$

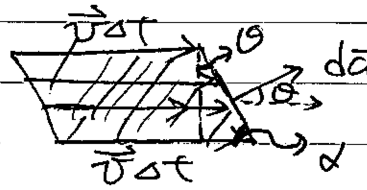
Example : flux $\vec{A} = \rho \vec{v}$ $\vec{v}$ = flow velocity of water

ρ = density of water



$\displaystyle\int_S \rho \vec{v} \cdot d\vec{a}$ = mass of water that flows across S per unit time

Consider a small patch $d\vec{a}$, during Δt,



as shown in the left figure,

all volume marked by dashed line will pass through the patch.

$\because$ volume $= da \cdot \underbrace{(v\Delta t \sin\alpha)}_{\text{height}} = da\, v\Delta t \cos\theta$

$= \vec{v} \cdot d\vec{a}\, \Delta t$

$\therefore$ during Δt, mass of water that passes through $d\vec{a}$ $= \rho \vec{v} \cdot d\vec{a}\, \Delta t$

Hence $\int_S \rho \vec{v} \cdot d\vec{a} =$ mass of water that flows across S

$\equiv$ flux

<u>Volume integral</u>

$\int_V T(x,y,z)\, d\tau$

$d\tau = dx\,dy\,dz$ example. $T =$ mass density ρ

$\rho \cdot d\tau = dm$ $\int_V \rho\, d\tau =$ total mass

Fundamental theorems in vector calculus

There are relations between line integrals, surface integrals and volume integrals. These relations are generalizations of the fundamental theorem in calculus :

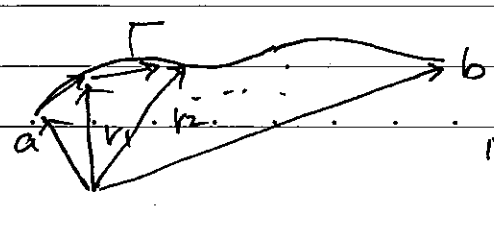
$$\underbrace{\int_a^b \frac{df(x)}{dx}\,dx}_{\text{line integrals}} = \underbrace{f(b)-f(a)}_{\text{boundary of line}} \quad \text{--- (4)} \qquad a \text{———} b$$

which can be viewed as a relation between line integral and related function at boundaries (a & b - points) of the line

Fundamental theorem for gradient

$$\vec{\nabla} T \cdot d\vec{r} = dT = T(\vec{r}+d\vec{r}) - T(\vec{r})$$

$$\therefore \int_{a\,\Gamma}^{b} \vec{\nabla} T \cdot d\vec{r} = \int_{a\,\Gamma}^{b} dT = T(\cancel{\vec{r}_1}) - T(a) + \cancel{T(\vec{r}_2)} - \cancel{T(\vec{r}_1)} + T(\vec{r}_3) - T(\cancel{\vec{r}_2}) + \cdots + T(b) - T(\vec{r}_N)$$



$$= T(b) - T(a)$$

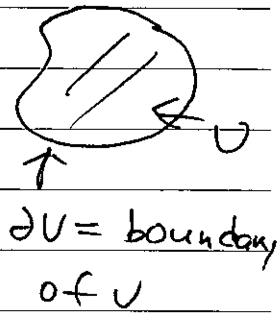
independent of the path Γ.

This directly generalizes eq. (4).

Corollary: $a=b$ $\quad \oint \vec{\nabla}T \cdot d\vec{l} = T(a) - T(a) = 0$

Divergence theorem (Green's theorem (2D), Gauss's theorem)

$$\int_V \vec{\nabla}\cdot\vec{v}\, d\tau = \oint_{\partial V} \vec{v}\cdot d\vec{a} \quad \text{--- (5)}$$



physical picture: $\vec{v}$ = flow velocity of water

$\oint_{\partial V} \rho\vec{v}\cdot d\vec{a}$ = mass of water that flows across ∂V.

i.e. flow out of region V.

(per unit time)

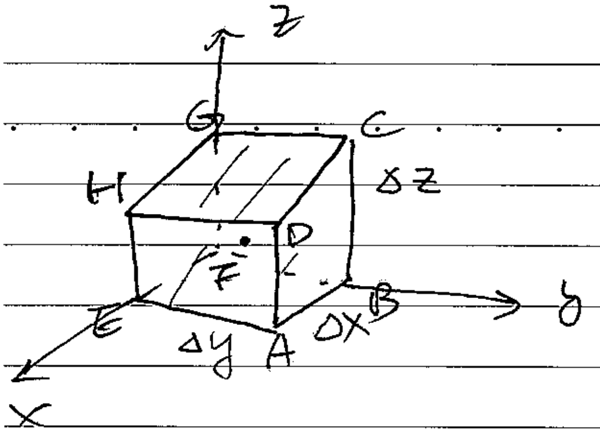
$\vec{\nabla}\cdot\vec{v}$ = Spreading source/sink inside V

Water is incompressible, ∴ source flow in/out rate in V

= net flow out rate

∴ This is the picture of (5).

Eq. (5) is also a generalization of eq. (4), which can be best illustrated by considering a small cube located at (x, y, z)



$\vec{F} = (x, y, z)$

$$\iiint_{\text{cube}} \frac{\partial v_x}{\partial x}\, dx\, dy\, dz$$

$$= \iint dy\, dz \int_{x}^{x+\Delta x} dx\, \frac{\partial v_x(x,y,z)}{\partial x}$$

$$= \iint dy\, dz\; v_x(x+\Delta x, y, z) - v_x(x, y, z)$$

For $\square EADH$, $d\vec{a} = \Delta y\, \Delta z\, \hat{x}$ at $x + \Delta x$

$$\therefore \int_{\square EADH} d\vec{a} \cdot \vec{v} = \iint dy\, dz\, v_x(x+\Delta x, y, z)$$

For $\square FBCG$, $d\vec{a} = \Delta y\, \Delta z\, (-\hat{x})$ at x

$$\int_{\square FBCG} d\vec{a} \cdot \vec{v} = -\iint dy\, dz\, v_x(x, y, z)$$

$$\therefore \iiint_{\text{cube}} \frac{\partial v_x}{\partial x}\, dx\, dy\, dz = \int_{\square EADH} d\vec{a} \cdot \vec{v} - \int_{\square FBCG} d\vec{a} \cdot \vec{v}$$

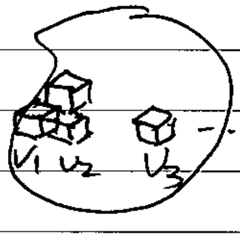
Similarly $\displaystyle\iiint_{\text{cube}} \frac{\partial v_y}{\partial y}\, dx\, dy\, dz = \int_{\square ABCD} d\vec{a} \cdot \vec{v} - \int_{\square EFGH} d\vec{a} \cdot \vec{v}$

$$\iiint_{\text{cube}} \frac{\partial v_z}{\partial z}\, dx\,dy\,dz = \int_{\square HDCG} d\vec{a}\cdot\vec{v} - \int_{\square EABD} d\vec{a}\cdot\vec{v}$$

$\therefore$ Altogether we get

$$\iiint_{\text{cube}=V} \vec{\nabla}\cdot\vec{v}\, d\tau = \oint_{\partial V} \vec{v}\cdot d\vec{a}$$

For general shapes, we can approximate V by small cubes V_i, $i=1,2,\dots N$

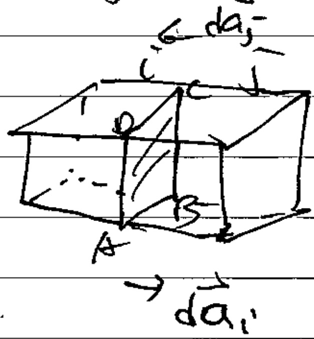


For each V_i,

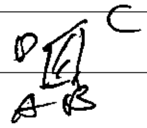
$$\int_{V_i} \vec{\nabla}\cdot\vec{v}\, d\tau = \oint_{\partial V_i} \vec{v}\cdot d\vec{a}$$

$$\therefore \sum_i \int_{V_i} \vec{\nabla}\cdot\vec{v}\, d\tau = \sum_i \oint_{\partial V_i} \vec{v}\cdot d\vec{a}$$

$\because$ For adjacent cubes, $\int \vec{v}\cdot d\vec{s}$ gets cancelled at overlapped surface



$d\vec{a}_j = -d\vec{a}_i$ $\quad \vec{v}_i = \vec{v}_j$ at



$$\therefore \int_{\square ABCD} \vec{v}\cdot d\vec{a}_i + \int_{\square ABCD} \vec{v}\cdot d\vec{a}_j = 0$$

$\therefore$ Only integrations on ∂V_i on surface of V don't get

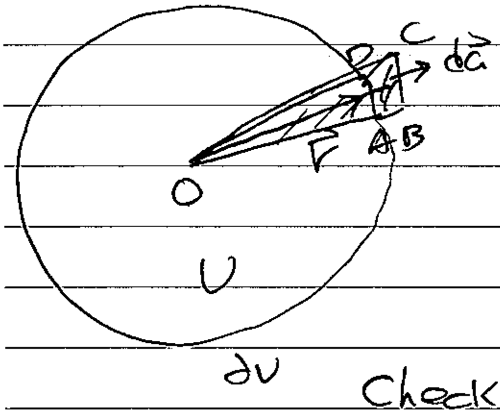
Cancelled

$$\therefore \sum_i \int_{\partial V_i} \vec{v}\cdot d\vec{a} = \int_{\partial V} \vec{v}\cdot d\vec{a}$$
$(N \to \infty)$

$$\therefore \int_V \vec{\nabla}\cdot\vec{v}\, d\tau = \int_{\partial V} \vec{v}\cdot d\vec{a}$$

Example

volume $= \frac{1}{3}\vec{r}\cdot d\vec{a}$



$$\therefore \oint_{\partial V} \frac{1}{3}\vec{r}\cdot d\vec{a} = V$$

Check by the divergence theorem:

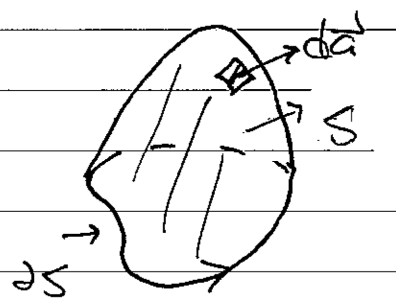
$$\oint_{\partial V} \vec{r}\cdot d\vec{a} = \int_V \vec{\nabla}\cdot\vec{r}\, d\tau$$

$$\vec{\nabla}\cdot\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\therefore \oint_{\partial V} \vec{r}\cdot d\vec{a} = 3\int_V d\tau = 3V$$

$\frac{1}{3}\oint_{\partial V} \vec{r}\cdot d\vec{a} = V$ is reproduced.

The Stokes' theorem



$$\int_S \vec{\nabla}\times\vec{v}\cdot d\vec{a} = \oint_{\partial S} \vec{v}\cdot d\vec{\ell}$$

$\oint_{\partial S} \vec{v}\cdot d\vec{\ell}$ is often called the circulation of $\vec{v}$ along ∂S

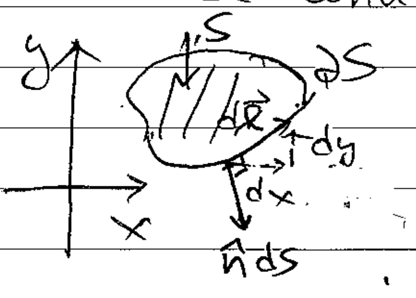
Corollary : $\oint_S \nabla\times\vec{v}\cdot d\vec{a} = 0$

S ↑ closed surface

Physical picture : $\vec{\nabla}\times\vec{v}$ = source of vortices

Sum of vorticities = circulation

The Stokes' theorem can be understood by using the divergence theorem. For this purpose, one considers the circulation of $\vec{v}$ along a curve on a plane which can be conveniently set as xy plane



$$\vec{v}\cdot d\vec{\ell} = v_x\,dx + v_y\,dy$$
$$= (v_y, -v_x)\cdot(dy, -dx)$$

$\because (dy, -dx)\cdot(dx, dy) = 0$

$\therefore (dy, -dx) \perp (dx, dy) = d\vec{l}$

With the same length $\sqrt{(dx)^2+(dy)^2} = ds$

$\therefore (dy, -dx) = \hat{n}\, ds$ $\hat{n}$ is normal to $d\vec{l}$.

$$\therefore \oint_{\partial S} \vec{v} \cdot d\vec{l} = \oint_{\partial S} (v_y, -v_x) \cdot ds\,\hat{n}$$

Using the divergence theorem,

$$\oint_{\partial S} (v_y, -v_x) \cdot ds\,\hat{n}$$

$$= \int_S \nabla \cdot (v_y, -v_x)\, da$$

$$= \int_S \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) da$$

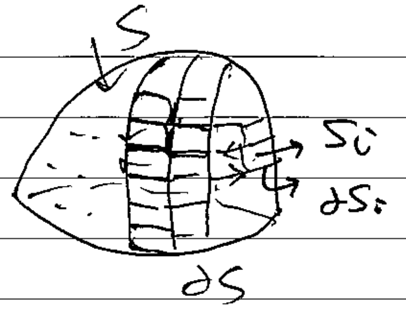
$$= \int_S (\vec{\nabla} \times \vec{v}) \cdot (da\,\hat{z}) \qquad (\vec{\nabla} \times \vec{v})_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}$$

$$\therefore \int_S \vec{\nabla} \times \vec{v} \cdot d\vec{a} = \oint_{\partial S} \vec{v} \cdot d\vec{l} \quad \text{is correct}$$

for a small curve on a plane.

Similar to the proof of the divergence theorem, for general surfaces, one

Can approximate the surface by many small squares S_i as shown on the following.



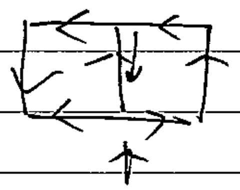
For each S_i, one has

$$\int_{S_i} \vec{\nabla}\times\vec{v}\cdot d\vec{a} = \oint_{\partial S_i} \vec{v}\cdot d\vec{l}$$

$i=1,2,\ldots N$

$$\int_S \vec{\nabla}\times\vec{v}\cdot d\vec{a} = \sum_i \int_{S_i} \vec{\nabla}\times\vec{v}\cdot d\vec{a} = \sum_i \oint_{\partial S_i} \vec{v}\cdot d\vec{l}$$

$\int \vec{v}\cdot d\vec{l}$ gets cancelled for adjacent squares.



$\int \vec{v}\cdot d\vec{l}$ gets cancelled

$$\therefore \sum_i \oint_{\partial S_i} \vec{v}\cdot d\vec{l} = \int_{\partial S} \vec{v}\cdot d\vec{l}$$

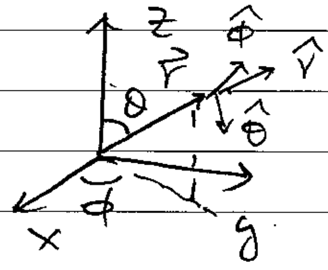
Taking $N\to\infty$, one gets

$$\int_S \vec{\nabla}\times\vec{v}\cdot d\vec{a} = \int_{\partial S} \vec{v}\cdot d\vec{l}$$

Curvilinear coordinates

For many problems in electrodynamics, it will be more convenient to use different coordinate systems. Those are curvilinear coordinates. The most general curvilinear coordinates are discussed in Appendix A. We shall not go into details but just mention two most common used coordinates.

Spherical coordinates



θ = polar angle

ϕ = azimuthal angle

$$z = r\cos\theta$$
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$

Just as $\hat{x}, \hat{y}, \hat{z}$ are defined as unit vectors along the increasing direction of x, y & z $\left(\hat{x}_i = \dfrac{\frac{d\vec{r}}{dx_i}}{\left|\frac{d\vec{r}}{dx_i}\right|}\right)$, $\hat{r}, \hat{\theta}, \hat{\phi}$ are similarly defined, shown in the above figure. In terms of Cartesian coordinates, one has $(\hat{r} \perp \hat{\theta} \perp \hat{\phi})$

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z} = \frac{\frac{\partial\vec{r}}{\partial r}}{\left|\frac{\partial\vec{r}}{\partial r}\right|}$$

$$\hat{\theta} = \frac{\frac{\partial\vec{r}}{\partial\theta}}{\underbrace{\left|\frac{\partial\vec{r}}{\partial\theta}\right|}_{=r}} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\phi} = \frac{\frac{\partial\vec{r}}{\partial\phi}}{\left|\frac{\partial\vec{r}}{\partial\phi}\right| \leftarrow r\sin\theta} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

Any vector $\vec{A}$ can be expressed as

$$\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$$

For general infinitesimal displacement,

$$d\vec{r} = \frac{\partial\vec{r}}{\partial r}dr + \frac{\partial\vec{r}}{\partial\theta}d\theta + \frac{\partial\vec{r}}{\partial\phi}d\phi$$

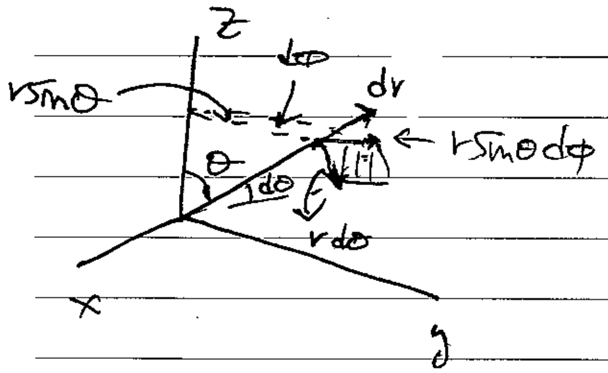
$$= \underset{h_r}{\uparrow} dr\,\hat{r} + \underbrace{r}_{h_\theta} d\theta\,\hat{\theta} + \underbrace{r\sin\theta}_{h_\phi} d\phi\,\hat{\phi}$$

∴ The infinitesimal length

along $\hat{r}$ $= dr$

" $\hat{\theta}$ $= r\,d\theta$

" $\hat{\phi}$ $= r\sin\theta\,d\phi$



∴ area element on a surface with fixed r

$$da = r\,d\theta \cdot r\sin\theta\,d\phi = r^2\sin\theta\,d\phi\,d\theta$$

Volume element $d\tau = r\,d\theta \cdot r\sin\theta\,d\phi \cdot dr = r^2\sin\theta\,dr\,d\theta\,d\phi$

Just as any vector can be expressed in terms $\hat{r}, \hat{\theta}, \hat{\phi}$, $\vec{\nabla}$ operators can be also expressed in terms of $\hat{r}, \hat{\theta}, \hat{\phi}$.

<u>For gradient</u>, ∵ along $\hat{r}, \hat{\theta}, \hat{\phi}$, the lengths are dr, $r d\theta$, $r \sin\theta \, d\phi$,

$$\therefore \quad \vec{\nabla} T = \frac{\partial T}{\partial r}\hat{r} + \frac{\partial T}{r\,\partial\theta}\hat{r} + \frac{\partial T}{r\sin\theta\,\partial\phi}\hat{\phi}$$

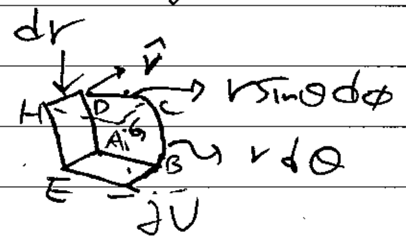
The divergence is more complicated as if $\vec{v} = v_r \hat{r} + v_\theta \hat{\theta} + v_\phi \hat{\phi}$

$$\vec{\nabla}\cdot\vec{v} = \vec{\nabla}\cdot(v_r\hat{r}) + \vec{\nabla}\cdot(v_\theta\hat{\theta}) + \vec{\nabla}\cdot(v_\phi\hat{\phi})$$

$\vec{\nabla}$ will not only act on v_r, v_θ & v_ϕ but also act on $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$.

The best way to write down $\vec{\nabla}\cdot\vec{v}$ is to use the divergence theorem as follows.

For divergence, along constant r direction,



One considers a slab shown in the left figure $dr \to 0$

$$\therefore \oint_{\partial V} \vec{v}\cdot d\vec{a} \cong \int_{DABCD} + \int_{DZFGH} = \int_{DABCD} \vec{v}\cdot r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$$

$$+\int_{DEFGH} \vec{v} \cdot r^2 \sin\theta\, d\theta\, d\phi\, (-\hat{r})$$

$$= \int_{DABCO} v_r r^2 \sin\theta\, d\theta\, d\phi - \int_{DEFGH} v_r r^2 \sin\theta\, d\theta\, d\phi$$

$$= \int_V \frac{\partial}{\partial r}(v_r \underbrace{r^2 \sin\theta}_{h_\theta h_\phi})\, dr \quad d\theta\, d\phi$$

$$\equiv \int_V \vec{\nabla} \cdot \vec{v}\, d\tau = \int_V \vec{\nabla} \cdot \vec{v}\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

divergence theorem

$$\therefore (\vec{\nabla} \cdot \vec{v})_r = \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial r}(v_r r^2 \sin\theta) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)$$

Similarly, by considering a slab along θ direction,

one gets $\frac{1}{h_r h_\theta h_\phi} \frac{\partial}{\partial \theta}(v_\theta h_r h_\phi) = (\vec{\nabla} \cdot \vec{v})_\theta$

$$(\vec{\nabla} \cdot \vec{v})_\theta = \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}(v_\theta r \sin\theta)$$

$$= \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(v_\theta \sin\theta)$$

Along ϕ direction, $(\vec{\nabla} \cdot \vec{v})_\phi = \frac{1}{h_r h_\theta h_\phi} \frac{\partial}{\partial \phi}(v_\phi h_r h_\theta)$

$$= \frac{1}{r \sin\theta} \frac{\partial v_\phi}{\partial \phi}$$

$$\therefore \vec{\nabla} \cdot \vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r \sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

For the curl, the same derivation (as the one done for the divergence) leads to

$$\vec{\nabla}\times\vec{v} = \frac{1}{h_r h_\theta h_\phi}\begin{vmatrix} h_r\hat{r} & h_\theta\hat{\theta} & h_\phi\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ h_r v_r & h_\theta v_\theta & h_\phi v_\phi \end{vmatrix}$$

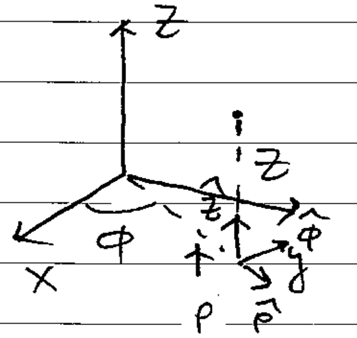
$$= \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ v_r & r v_\theta & r\sin\theta\cdot v_\phi \end{vmatrix}$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r}$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

Cylindrical coordinate



$x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$

(ρ, ϕ, z) is the cylindrical coordinate.

$$\hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\hat{z} = \hat{z}$$

Similar to the spherical coordinates, we have

$$d\vec{r} = \frac{\partial \vec{r}}{\partial \rho} d\rho + \frac{\partial \vec{r}}{\partial \phi} d\phi + \frac{\partial \vec{r}}{\partial z} dz$$

$$= \underset{\substack{\uparrow \\ h_\rho}}{d\rho}\, \hat{\rho} + \underset{\substack{\uparrow \\ h_\phi}}{\rho\, d\phi}\, \hat{\phi} + \underset{\substack{\uparrow \\ h_z}}{dz}\, \hat{z}$$

$$\therefore \quad \vec{\nabla} T = \frac{1}{h_\rho} \frac{\partial T}{\partial \rho} \hat{\rho} + \frac{1}{h_\phi} \frac{\partial T}{\partial \phi} \hat{\phi} + \frac{1}{h_z} \frac{\partial T}{\partial z} \hat{z}$$

$$= \frac{\partial T}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial T}{\partial \phi} + \frac{\partial T}{\partial z} \hat{z}$$

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{h_\rho h_\phi h_z} \left[\frac{\partial}{\partial \rho} (v_\rho h_\phi h_z) + \frac{\partial}{\partial \phi} (v_\phi h_\rho h_z) + \frac{\partial}{\partial z} (v_z h_\rho h_\phi) \right]$$

$$(\vec{v} = v_\rho \hat{\rho} + v_\phi \hat{\phi} + v_z \hat{z})$$

$$= \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho v_\rho) + \frac{\partial}{\partial \phi} (v_\phi) + \frac{\partial}{\partial z} (\rho v_z) \right]$$

$$= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho v_\rho) + \frac{1}{\rho} \frac{\partial}{\partial \phi} v_\phi + \frac{\partial v_z}{\partial z}$$

$$\vec{\nabla} \times \vec{v} = \frac{1}{h_\rho h_\phi h_z} \begin{vmatrix} h_\rho \hat{\rho} & h_\phi \hat{\phi} & h_z \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ h_\rho v_\rho & h_\phi v_\phi & h_z v_z \end{vmatrix}$$

$$= \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ v_\rho & \rho v_\phi & v_z \end{vmatrix}$$

$$= \left(\frac{1}{\rho}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial v_\rho}{\partial z} - \frac{\partial v_z}{\partial\rho}\right)\hat{\phi} + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho v_\phi) - \frac{\partial v_\rho}{\partial\phi}\right]\hat{z}$$

The Helmholtz theorem

As we have seen, for a given vector field $\vec{F}(\vec{r})$, $\vec{\nabla}\cdot\vec{F}(\vec{r})$ has the meaning as sources/sinks while $\vec{\nabla}\times\vec{F}(\vec{r})$ has the meaning as swirls (vortices).

When $\vec{\nabla}\times\vec{F}=0$ in a region R, $\vec{F}$ is called irrotational.

On the other hand, if $\vec{\nabla}\cdot\vec{F}=0$, $\vec{F}$ is called solenoidal.

$\vec{\nabla}\times\vec{F}$ & $\vec{\nabla}\cdot\vec{F}$ appears to two most important features to characterize a vector field, which is summary as the Helmholtz theorem:

$\vec{F}$ is continuous and differentiable in a simply-connected region V and $\vec{F}$ is single-valued.
(without holes)

Then $\vec{F}$ can be completely decomposed into the sum of an irrotational field $\vec{\Lambda}(\vec{r})$ and a solenoidal field $\vec{\Omega}(\vec{r})$

i.e. $\vec{F} = \vec{\Lambda} + \vec{\Omega}$

$$\nabla \cdot \vec{F} = \nabla \cdot \vec{\Lambda}$$

$$\nabla \times \vec{F} = \nabla \times \vec{\Omega}$$