11/6/96 Revised Harmonic Oscillator

Single harmonic oscillator $\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$, $[\hat{x}, \hat{p}] = i\hbar$

$$= \frac{1}{2m}\hat{p}^\dagger\hat{p} + \frac{1}{2}m\omega^2\hat{x}^\dagger\hat{x}$$

Many systems can be approximated by harmonic oscillators in the lowest order :

(i) $V(x)$ is min at $x = x_0$

$$U(x) = U(x_0) + (x-x_0)\underbrace{U'(x_0)}_{0} + \frac{1}{2!}(x-x_0)^2 U''(x_0) + \underbrace{\ldots}_{\text{anharmonic !}}$$

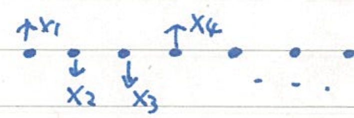
$V''(x_0) > 0$ $\quad \frac{1}{2}V'(x_0) = \frac{1}{2}m\omega^2$

(ii) phonons : quantized elastic waves in solids

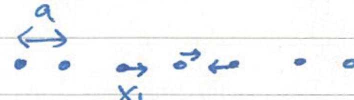
$x_1, \ldots, x_N$ = displacements of atoms from lattice points $(x_0 = 0)$

$$U(x_1, \ldots, x_N) = \frac{1}{2}\sum_{i,j} \underbrace{\frac{\partial^2 U}{\partial x_i \partial x_j}\Big|_0}_{U_{ij} = U_{ji} \text{ (Symmetric)}} x_i x_j + \ldots$$

$$\hat{H} \simeq \frac{1}{2m}\sum_i p_i^2 + \frac{1}{2}\sum_{ij} U_{ij} x_i x_j$$



(transverse)



(longitudinal)

<u>discrete Fourier transf.</u>

$$x_k = \frac{1}{\sqrt{N}}\sum_j x_j e^{+ikja}$$ not conventional !

$$p_k = \frac{1}{\sqrt{N}}\sum_j p_j e^{-ikja}$$

Conventional

$$p_k = \frac{1}{\sqrt{N}}\sum_j p_j e^{ikja}$$

$$\sum_j e^{i(k-k')ja} = N\delta_{k,k'}$$

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$$A(x,t) = \frac{1}{\sqrt{V}} \sum_{k,\alpha} A_{k,\alpha}(t)\, \hat{\epsilon}^{\alpha} e^{ik\cdot x} + A^*_{\alpha,k}(t)\, \hat{\epsilon}^{\alpha} e^{-ik\cdot x}$$

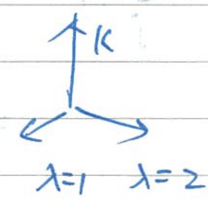
$$\uparrow a_{k\lambda} \qquad\qquad -\frac{1}{c}\frac{\partial}{\partial t}$$

(iii) E.M. radiation field

$$H = \frac{1}{2}\int d^3\vec{r}\,(E^2+B^2) \sim \sum_{k\alpha} 2\left(\frac{\omega}{c}\right)^2 \underset{\uparrow \text{ polarization}}{A^*_{k,\alpha}(t)}\, A_{k,\alpha}(t)$$

$$E = -\frac{1}{c}\frac{\partial A}{\partial t}, \quad B = \nabla\times A \qquad \nabla\cdot A = 0$$

$$\cdots = \frac{1}{2}\sum_{\substack{k\lambda \\ \uparrow \\ \text{polarization}}} \left(P_{k,\lambda}^2 + \underset{\substack{\uparrow \\ k\cdot c}}{\omega_k^2}\, Q_{k\lambda}^2\right)$$



$$P_{k\lambda} = \frac{1}{\sqrt{2\omega_k}}\left(a_{k\lambda} + a_{k\lambda}^*\right)$$

$$Q_{k\lambda} = -i\sqrt{\frac{\omega_k}{2}}\left(a_{k\lambda} - a_{k\lambda}^*\right)$$

$$\vec{A}_{k\lambda} = \frac{\hat{\varepsilon}_{k\lambda}}{\sqrt{2\omega_k}}\, a_{k\lambda}$$

(iv) <u>particle in a magnetic field</u>

$$H = \sum_k \frac{1}{2m}\left[(P_k^+ + im\omega_k X_k^+)(P_k - im\omega_k X_k) - \frac{i\omega_k}{2}[X_k^+, P_k]\right]$$

$$[X_k^+, P_k] = i\hbar \qquad \cancel{X_k^+ P_k - P_k^+ X_k}$$

$$= \sum_k \left(a_k^+ a_k + \frac{1}{2}\right)\hbar\omega_k$$

$$a_k = \sqrt{\frac{m\omega_k}{2\hbar}}\left(X_k + \frac{iP_k}{m\omega_k}\right)$$

$$a_k^+ = \sqrt{\frac{m\omega_k}{2\hbar}}\left(X_k^+ - i\frac{P_k^+}{m\omega_k}\right)$$

$$[a_k, a_{k'}^+] = \sqrt{\frac{m\omega_k}{2\hbar}}\sqrt{\frac{m\omega_{k'}}{2\hbar}} \times \frac{1}{m\omega_k} \times 2\delta_{k,k'}\, i\hbar$$

$$= \delta_{k,k'}$$

$$x_j = \frac{1}{\sqrt{N}} \sum_k e^{-ik\cdot ja} x_k \qquad x_{j+N} = x_j \Rightarrow Nka = 2\pi n$$

$$p_j = \frac{1}{\sqrt{N}} \sum_k e^{ik\cdot ja} p_k \qquad k = \frac{2\pi n}{Na},\ n = 0,1,2 \ldots N-1$$

$$\sum_j p_j^2 = \frac{1}{N} \sum_k \sum_{k'} \sum_j e^{ik\cdot ja} e^{ik'\cdot ja} p_k p_{k'}$$

$$= \frac{1}{N} \sum_k \sum_{k'} \delta_{k,-k'} \cdot N \ p_k p_{k'}$$

$$= \sum_k p_k p_{-k} = \sum_k |p_k|^2$$

$$\left(p_j^* = p_j \Rightarrow p_{-k} = p_k^* \right)$$

$$\sum_{ij} U_{ij} x_i x_j = \frac{1}{N} \sum_{kk'} \sum_{ij} U_{ij} e^{-ik\cdot ia} e^{-ik'\cdot ja} x_k x_{k'}$$

$$U_{ij} = U(i-j) \qquad = \frac{1}{2} \sum_{kk'} \sum_{ij} \frac{U(i-j)}{N} e^{-ik(i-j)a} \times e^{i(k+k')ja} x_k x_{k'}$$

$$= \frac{1}{2} \cdot \sum_k U_k x_k x_{-k}$$

$$\equiv \frac{m}{2} \sum_k \omega_k^2 |x_k|^2 \qquad \text{Assume } (U_k > 0)\ (\because \min)$$

$$H \cong \frac{1}{2m} \sum_k |p_k|^2 + \frac{m}{2} \sum_k \omega_k^2 |x_k|^2$$

$$[x_k^+, p_{k'}] = i\hbar \delta_{k,k'}$$

$$[x_k, p_{k'}^+] = i\hbar \delta_{k,k'}$$

conventional

$$[x_j, p_{j'}] = i\hbar \delta_{jj'}$$

$$[x_k, p_{k'}] = \frac{1}{N} \sum_j i\hbar \, e^{-i(k-k')ja} = i\hbar \delta_{k,k'}$$

$$\hat{H} \cong \frac{1}{2m} \sum_k \hat{p}_k^+ \hat{p}_k + \frac{m}{2} \sum_k \omega_k^2 \hat{x}_k^+ \hat{x}_k$$

→ see over

$\Rightarrow N$ harmonic oscillators! Note the difference : $\hat{p}_k^+ \neq \hat{p}_k$!

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Dilemma of specific heat :

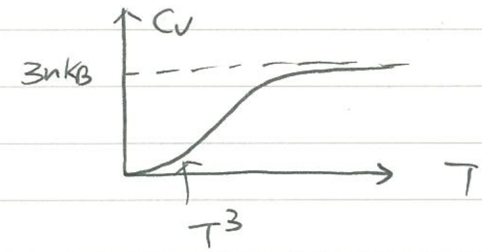
classically : each atom in solids , $H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$

$$\underset{\frac{1}{2}k_BT}{\uparrow} \qquad \underset{\frac{1}{2}k_BT}{\uparrow}$$

$$E = \frac{1}{2}k_BT \times \underset{x,y,z}{3} \times N \times \underset{p^2, x^2}{2} = 3Nk_BT$$

$$\therefore \; C_V = \frac{1}{V}\frac{dE}{dT} = 3nk_B = \text{constant}.$$

but experimentally



resolved by Q.M. : $p(n) = \frac{1}{Z} e^{-\frac{(n+\frac{1}{2})\hbar\omega}{k_BT}}$

$$\Rightarrow \langle n \rangle = \frac{1}{e^{\frac{\hbar\omega}{k_BT}} - 1} , \quad E = \sum_{\omega} \frac{\hbar\omega}{e^{\hbar\omega/k_BT} - 1}$$

$\Rightarrow C_V = \frac{1}{V}\frac{dE}{dT}$, see Ex 7.5.4 for more details.

$$Z = \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\hbar\omega} = e^{-\frac{1}{2}\beta\hbar\omega} \sum_{n=0}^{\infty} e^{-nx} , \; x = \beta\hbar\omega$$

$$= \frac{e^{-\frac{1}{2}\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}}$$

$$\langle n \rangle = \frac{1 - e^{-\beta\hbar\omega}}{e^{-\frac{1}{2}\beta\hbar\omega}} \underbrace{\sum_{n=0}^{\infty} n e^{-(n+\frac{1}{2})x}}$$

$$= e^{-\frac{x}{2}}\left(-\frac{d}{dx}\sum_{n=0}^{\infty} e^{-nx}\right)$$

$$= e^{-x/2}\left(\frac{d}{dx}\frac{-1}{1 - e^{-x}}\right)$$

$$= e^{-x/2}\frac{e^{-x}}{(1 - e^{-x})^2}$$

$$= \frac{1}{e^{\frac{\hbar\omega}{k_BT}} - 1}$$

$$\langle H\rangle \equiv \frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle} \geq 0$$

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~~this equality requires~~

A trivial property:

$$\langle H\rangle \geq 0$$

$$\because \langle H\rangle = \frac{1}{2m}\langle\psi|p^2|\psi\rangle + \frac{1}{2}m\omega^2\langle\psi|x^2|\psi\rangle$$

this step $\Rightarrow \int |D\psi|^2 = \int -\psi^* D^2\psi$ requires

$$\langle\psi|p^2|\psi\rangle = \langle\psi|p^\dagger p|\psi\rangle = \langle\phi|\phi\rangle \geq 0, \quad |\phi\rangle = p|\psi\rangle$$

Similarly, $\langle\psi|x^2|\psi\rangle \geq 0$ $\qquad$ ψ finite at $\pm\infty$

Coordinate Basis (solving differential equation)

1D $\quad \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2\right)\psi = E\psi \; ; \quad \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left(E - \frac{m\omega^2}{2}x^2\right)\psi = 0$

A useful step: make it dimensionless

$\because$ if $\langle\psi|\psi\rangle$ is not finite $\langle H\rangle$ 不見得 $\geq$ ... So for Dirac method, $\varepsilon \neq 2n+1$ $\Rightarrow$ can get $\varepsilon < 0$ $\because \varepsilon \neq 2n+1$ $\langle\psi|\psi\rangle = \infty$

$$x = by, \qquad \psi = \psi(y)$$

$$\frac{d^2\psi}{dy^2} + \Big(\underbrace{\frac{2mb^2}{\hbar^2}E}_{\equiv 2\varepsilon} - \underbrace{\frac{m^2\omega^2}{\hbar^2}b^4}_{\equiv 1}y^2\Big)\psi(y) = 0$$

$\Rightarrow b = \left(\frac{\hbar}{m\omega}\right)^{\frac{1}{2}}$ = unit of length $\Rightarrow y$ = dimensionless

ε has no dimension! $\qquad$ for later convenience!

see over

$$\Rightarrow \psi''(y) + (2\varepsilon - y^2)\psi = 0$$

$\rightarrow$ measuring E in the unit: $\frac{\hbar^2}{2mb^2} = \frac{\hbar^2}{2m}\left(\frac{\hbar}{m\omega}\right)^{-1} = \frac{1}{2}\hbar\omega$

Asymptotic behaviors:

(i) $y \to \infty \qquad \psi'' - y^2\psi = 0 \qquad \psi = Ay^m e^{\pm\frac{y^2}{2}}$ for any m

$$\because \psi'' = Ay^{m+2}e^{\pm y^2/2}\left[1 \pm \frac{2m+1}{y^2} + \frac{m(m-1)}{y^4}\right]$$

$$\rightarrow y^2\psi$$

$$\psi' = A(my^{m-1} \pm y^{m+1})e^{\pm y^2/2}$$

(ii) $y \to 0$, $\psi'' + 2\varepsilon\psi = 0$

$$\psi = A\cos\sqrt{2\varepsilon}\,y + B\sin\sqrt{2\varepsilon}\,y \quad \text{(well-behaved!)}$$

(remove $e^{y^2/2}$ singularity)

Ansatz: $\psi = u e^{-y^2/2}$ (to remove $y^2\psi$ term)

$$\psi' = u' e^{-y^2/2} - y u e^{-y^2/2}$$

$$\psi'' = u'' e^{-y^2/2} - 2yu' e^{-y^2/2} - u e^{-y^2/2} + y^2 u e^{-y^2/2}$$

$$\Rightarrow u'' - 2yu' + (2\varepsilon - 1)u = 0 \quad \text{— ①}$$

$y^2\psi$ is removed!

Originally: if $\psi = \sum C_n y^n$, $\Rightarrow$ relation between C_{n+2}, C_n, C_{n-2} hard to solve

$$u = \sum_{n=0}^{\infty} C_n y^n$$

same order "n"

① implies: $\displaystyle\sum_{n=2}^{\infty} C_n \left[n(n-1) y^{n-2} - 2ny^n + (2\varepsilon - 1) y^n \right] = 0$

$n=1$ $\quad$ $n=0$

$+ C_1(-2+2\varepsilon-1)y + C_0(2\varepsilon-1)$

$n=2$ corresponds here $n=0$

$$\Rightarrow \sum_{n=0}^{\infty} y^n \left[C_{n+2}(n+2)(n+1) + C_n(2\varepsilon - 1 - 2n) \right] = 0$$

$$C_{n+2} = C_n \frac{(2n+1-2\varepsilon)}{(n+1)(n+2)}, \quad C_1, C_0 \text{ undermined!}$$

$$u(y) = C_0\left[1 + \underbrace{\frac{(1-2\varepsilon)}{1\cdot 2}}_{C_2} y^2 + \underbrace{\frac{(1-2\varepsilon)}{1\cdot 2}\frac{(5-2\varepsilon)}{3\cdot 4}}_{C_4} y^4 + \cdots\right]$$
$$+ C_1\left[y + \frac{3-2\varepsilon}{2\cdot 3} y^3 + \frac{3-2\varepsilon}{2\cdot 3}\frac{(7-2\varepsilon)}{4\cdot 5} y^5 + \cdots\right]$$

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For any ε which is not equal to $\frac{2n+1}{2}$ for some n.

Then, the power serieses don't terminate.

Since, $\frac{C_{n+2}}{C_n} \to \frac{2}{n}$ as $n \to \infty$

and this is the same as $y^m e^{y^2} = \sum_{k=0}^{\infty} \frac{y^{2k+m}}{k!}$

$\therefore C_n = \frac{1}{(\frac{n-m}{2})!}$, $\frac{C_{n+2}}{C_n} = \frac{1}{\frac{n+2-m}{2}} \to \frac{2}{n}$,

So $u(y) \sim y^m e^{y^2}$. $\psi(y) \sim y^m e^{\frac{y^2}{2}}$ which blows up at $y \to \pm\infty$, not acceptable!

$\therefore \varepsilon_n = n + 1/2$, $n = 0, 1, 2, \ldots$ ≥ 0, $E_n = (n + 1/2)\hbar\omega$

$(E_n = 2\varepsilon \cdot \frac{1}{2}\hbar\omega)$

n = even, we take $C_1 = 0$,

$2\varepsilon = 2n+1$
1, 5, 9, ...

$u_n(y) = C_0 + C_2 y^2 + C_4 y^4 + \cdots + C_n y^n$ even

n = odd, take $C_0 = 0$

$2\varepsilon = 2n+1$
$n = 1, 3, 5, \ldots$
$2\varepsilon = 3, 7, \ldots$

$u_n(y) = C_1 y + C_3 y^3 + \cdots + C_n y^n$ odd.

$$C_{n+2} = C_n \frac{(2n+1-2\varepsilon)}{(n+2)(n+1)}$$

Example: $n = 0$, $u_0 = C_0$. $\psi(x) = C_0 e^{-y^2/2}$

$= C_0 e^{-\frac{m\omega}{2\hbar}x^2}$

normalization

$$C_0^2 \int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2} dx = 1$$

$$C_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}$$

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$n=1$ $\quad \psi(x) = C_1\, y\, e^{-y^2/2} \equiv \hat{C}_1\, \underbrace{2}\, y\, e^{-y^2/2}$

Convention : largest power's coefficient $= 2^n$

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = \sqrt{\frac{\hbar}{m\omega}}\, |\hat{C}_1|^2 \cdot 4 \int_{-\infty}^{\infty} dy\, y^2 e^{-y^2}$$

$$= 4 |\hat{C}_1|^2 \sqrt{\frac{\hbar}{m\omega}}\, \frac{\sqrt{\pi}}{2}$$

$$|\hat{C}_1| = \left(\frac{m\omega}{\pi\hbar\, 2^2}\right)^{\frac{1}{4}}$$

Such a convention leads to

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar\, 2^{2}(n!)^2}\right)^{\frac{1}{4}} e^{-y^2/2} H_n(y) \quad , \quad y = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$G(x,t) = e^{-t^2+2tx}$ is the generating function

$$= \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)\, t^n$$

Remarks :

(i) momentum space

$$\left[\frac{p^2}{2m} + \frac{1}{2} m\omega^2 \left(i\hbar \frac{d}{dp}\right)^2\right] \Phi(p) = E\, \Phi(p)$$

$$\left(\frac{-\hbar^2 m}{2}\omega^2 \frac{d^2}{dp^2} + \frac{p^2}{2m}\right) \Phi(p) = E\, \Phi(p)$$

Let $m^* = \frac{1}{m\omega^2}$, $\quad \omega^{*2} = \frac{1}{m}\frac{1}{m^*} = \omega^2$, $\quad$ m^*: dual mass

(i.e. $\omega^* = \omega$)

$$\Rightarrow \left(\frac{-\hbar^2}{2m^*}\frac{d^2}{dp^2} + \frac{1}{2} m^* \omega^{*2} p^2\right) \Phi(p) = E\, \Phi(p)$$

$$\therefore E = E_n \quad , \quad \Phi_n(p) = C_n\, e^{-\frac{m^*\omega^*}{2\hbar} p^2} H_n\left(\sqrt{\frac{m^*\omega^*}{\hbar}}\, p\right)$$

~~which has a~~

$$\langle p|n\rangle = (-i)^n \left[\frac{1}{\pi\hbar\, m\omega\, 2^{2n}(n!)^2}\right]^{\frac{1}{4}} e^{-\frac{p^2}{2m\hbar\omega}} H_n\left(\frac{p}{\sqrt{m\hbar\omega}}\right)$$

$$= C_n\, e^{-\frac{p^2}{2\hbar m\omega}}\, H_n\left(\sqrt{\frac{1}{\hbar m\omega}}\, p\right)$$

$$\Phi_n(p) = \int_{-\infty}^{\infty} e^{-\frac{ipx}{\hbar}} \varphi_n(x) \frac{dx}{\sqrt{2\pi\hbar}} \quad \text{——} \quad (1)$$

$$\therefore \int_{-\infty}^{\infty} dp\, |\Phi_n(p)|^2 = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} \int dx\, dx'\, e^{i\frac{p(x-x')}{\hbar}} \varphi_n(x)\varphi_n^*(x')$$

$$= \frac{2\pi\hbar}{2\pi\hbar} \int_{-\infty}^{\infty} dx\, |\varphi_n(x)|^2 = \frac{2\pi\hbar}{2\pi\hbar} = 1$$

$$\therefore |C_n|^2 \int_{-\infty}^{\infty} dp\, e^{-y_p^2} H_n(y_p) H_n(y_p) = 1$$

$$y_p = \sqrt{\frac{m^*\omega^*}{\hbar}}\, p$$

$$|C_n|^2 \sqrt{\frac{\hbar}{m^*\omega^*}} \underbrace{\int_{-\infty}^{\infty} dx\, e^{-x^2} [H_n(x)]^2}_{\pi^{\frac{1}{2}} 2^n n!} = 1$$

$$|C_n|^2 = \sqrt{\frac{m^*\omega^*}{\hbar}} \frac{1}{\sqrt{\pi}\, 2^n n!}, \quad |C_n| = \left[\frac{1}{(m\omega)\pi\hbar 2^{2n} (n!)^2}\right]^{\frac{1}{4}}$$

Direct calculation (from table)

$$\int_{-\infty}^{\infty} e^{-\frac{ipx}{\hbar}} \varphi_n(x) \frac{dx}{\sqrt{2\pi\hbar}}$$

$$= \int_{-\infty}^{\infty} e^{-\frac{ip}{\hbar}\sqrt{\frac{\hbar}{m\omega}} y}\, e^{-y^2/2} H_n(y) \frac{dy}{\sqrt{2\pi\hbar}} \sqrt{\frac{\hbar}{m\omega}} \left[\frac{m\omega}{\pi\hbar\, 2^{2n}(n!)^2}\right]^{\frac{1}{4}}$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi\hbar}}\, e^{-\frac{1}{2}\left(p\sqrt{\frac{1}{m\omega\hbar}}\right)^2} H_n\left(p\sqrt{\frac{1}{m\omega\hbar}}\right)(-i)^n \quad (\cdots)$$

$$= (-i)^n\, e^{-\frac{p^2}{2\hbar m\omega}} H_n\left(\frac{p}{\sqrt{m\omega\hbar}}\right) \left[\frac{1}{(m\omega)\, \pi\hbar\, 2^{2n} (n!)^2}\right]^{\frac{1}{4}}$$

$$= \left[\frac{1}{\hbar^2} \frac{m\omega}{\pi\hbar\, 2^{2n}(n!)^2} \frac{\hbar^2}{m^2\omega^2}\right]^{\frac{1}{4}}$$

$$C_n = |C_n|\, e^{i\frac{3n}{2}\pi} !$$

Φ_n

(ii) The levels are spaced uniformly. $\Delta E = \hbar\omega$

$\therefore$ When the system is at $E_n = (n+\frac{1}{2})\hbar\omega$,
we may think this as n quanta, each with $\hbar\omega$.

When the system moves from E_n to E_{n+1},
we say we add one more quanta to the system.

When $n=0$, there is no quanta, $E_0 = \frac{1}{2}\hbar\omega \neq 0$

$\Rightarrow$ zero-point energy!

$\Leftrightarrow$ Classical $E_{min} = 0$ $\because$ x and p can be zero at the same time!

Minimum case: Consider $\langle p \rangle = 0$, $\langle x \rangle = 0$ for any ψ
(can be always achieved by shifting coordinates.)

$\hookrightarrow$ $\langle x \rangle = 0$, $\langle p \rangle$ must $= 0$ otherwise $\langle x \rangle \neq 0$

$$\langle \psi | H | \psi \rangle = \frac{1}{2m}\langle p^2 \rangle + \frac{1}{2}m\omega^2 \langle x^2 \rangle$$

$$= \frac{1}{2m}(\Delta p)^2 + \frac{1}{2}m\omega^2(\Delta x)^2 \underbrace{+ \frac{1}{2m}\langle p \rangle^2 + \frac{1}{2}m\omega^2\langle x \rangle^2}_{\text{min by } \langle x \rangle = 0,\ \langle p \rangle = 0}$$

1D: $E \Rightarrow \psi_E$ real
$\langle p \rangle = 0$!
$\downarrow$
suffice to consider ψ = real
$\because \psi = \psi_R + i\psi_I$
$\langle \psi | H | \psi \rangle$
$= \langle \psi_R | H | \psi_R \rangle$
$+ \langle \psi_I | H | \psi_I \rangle$

$\because \Delta x \Delta p \geq \hbar/2$

$$\therefore \langle H \rangle \geq \frac{\hbar^2}{8m\Delta x^2} + \frac{1}{2}m\omega^2(\Delta x)^2$$

min case "=" holds

$$\therefore \langle H \rangle = \frac{\hbar^2}{8m\Delta x^2} + \frac{1}{2}m\omega^2(\Delta x)^2$$

$$\psi(x) = \psi(0)\, e^{-\frac{1}{2\Delta^2}x^2}$$ (see previous discussions on Heisenberg principle.) $\frac{\Delta^2}{2} = (\Delta x)^2$

next, we need to minimize $\langle H\rangle$ by choosing appropriate Δx!

$$\therefore \frac{d\langle H\rangle}{d(\Delta X)^2} = 0 = \frac{1}{2}m\omega^2 + \frac{-\hbar^2}{8m(\Delta X)^4}$$

$$(\Delta x)^2 = \frac{\hbar}{2m\omega}$$

$$\langle H\rangle_{min} = \frac{1}{2}\hbar\omega$$

$$\psi(x) = \psi(0)\, e^{-\frac{m\omega}{2\hbar}x^2} \xrightarrow[\text{normalize}]{} \psi_{min}(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$$

$\because$ For any ψ, $\langle\psi|H|\psi\rangle \geq \langle H\rangle_{min}$

$\therefore$ The ground state $= \psi_{min}(x)$

(iii) Classical limit $(n\to\infty)$:

(a) $m = 2g$, $\omega = 1\ \text{rad/sec}$ $\left(\frac{2\pi}{T}\right)$, $(T\sim 2\pi)$

$x_0 \sim 1\text{cm}$ (max displacement)

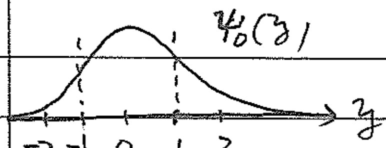
$E = \frac{1}{2}m\omega^2 x_0^2 \sim 1\ \text{erg}$

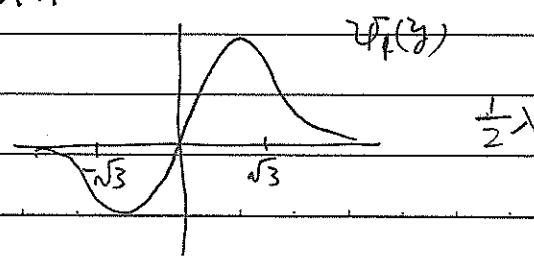
$\Delta E = \hbar\omega \simeq 10^{-27}\ \text{erg}$ $\quad n\sim 10^{27} \gg 1$

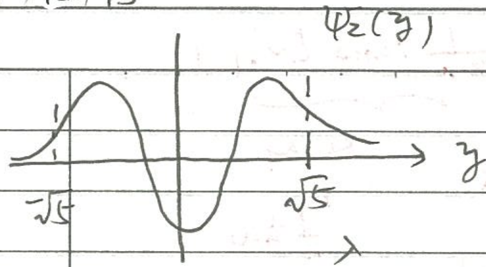
$\Delta E \ll E$ $\therefore$ $n+1$, n are almost indistinguishable!

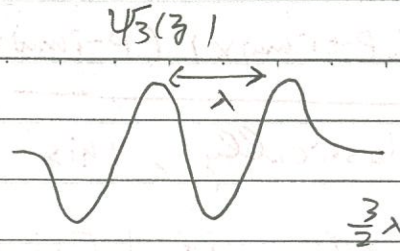
(b) $\psi_n''(y) + (2n+1-y^2)\psi(y) = 0$

turning points : $y = \pm\sqrt{2n+1}$







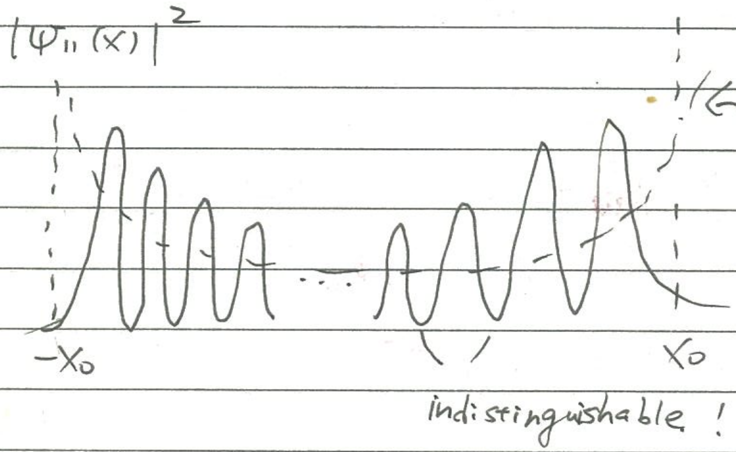


$$\begin{cases} \psi_n(-x) = \psi_n(x), & n \in \text{even} \\ \psi_n(-x) = -\psi_n(x), & n \in \text{odd} \end{cases}$$

n large, $\lambda \downarrow$

roughly: $2\sqrt{2n+1} = \frac{n}{2}\lambda_n$

$$\lambda_n \sim \frac{1}{\sqrt{n}} \to 0 \text{ as } n \to \infty$$



classical $P(x) \sim \frac{1}{v(x)} \sim \frac{1}{\sqrt{x_0^2 - x^2}}$

turning points

However, $\langle x \rangle_{n\to\infty}$, $\langle p \rangle_{n\to\infty}$ do not act like classical particles! (static!)

Dirac's operator method:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

Classically, $\hat{H} = \left(\frac{p}{\sqrt{2m}} + i\sqrt{\frac{m\omega^2}{2}}x\right)\left(\frac{p}{\sqrt{2m}} - i\sqrt{\frac{m\omega^2}{2}}x\right)$

$$a \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right) \quad \xleftarrow{\text{try}}$$

$$a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right) \neq a$$

method of factorization! ↳ see over

$$[a, a^+] = \frac{m\omega}{2\hbar}\left\{\left[x, \frac{-i}{m\omega}p\right] + \left[\frac{ip}{m\omega}, x\right]\right\}$$

$$= \frac{1}{2\hbar}\left\{[p, x]i - i[x, p]\right\} = 1$$

NO.
DATE

因式分解 (factorization)

$$H = \frac{1}{2m}\underbrace{(p+im\omega x)(p-im\omega x)}_{\text{classically, this is enough}} - \underbrace{\frac{1}{2m}\, im\omega\, [x,p]}_{\substack{Q.M \\ = \frac{1}{2}\hbar\omega}}$$

$$= \underbrace{\frac{1}{2m}(m\omega)^2\left[x-\frac{ip}{m\omega}\right]\left[x+\frac{ip}{m\omega}\right]}_{\Downarrow} + \frac{1}{2}\hbar\omega$$

make it dimensionless

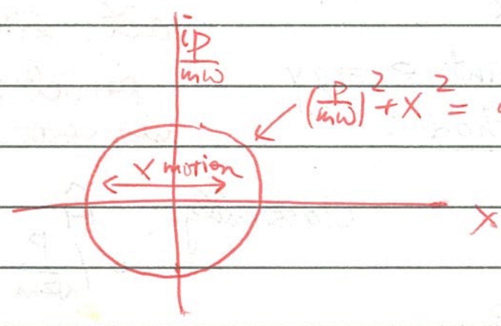
$$= \hbar\omega\left[\underbrace{\frac{m\omega}{2\hbar}}_{=\frac{1}{2b^2}}\underbrace{\left(x-\frac{ip}{m\omega}\right)\left(x+\frac{ip}{m\omega}\right)}_{=a^+a}\right] + \frac{1}{2}\hbar\omega$$

因式分解法

Classical: $p_0 = 0$. $x = x_0\cos\omega t$

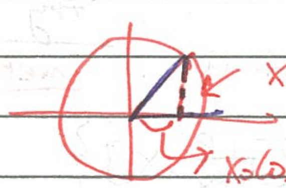
$p = -m\omega x_0 \sin\omega t$

$$\begin{cases} x+\dfrac{ip}{m\omega} = e^{-i\omega t}x_0 \quad \left(+\dfrac{p_0}{m\omega}\, i e^{-i\omega t}\right) \\ \text{"normal modes"} \\ x-\dfrac{ip}{m\omega} = x_0 e^{i\omega t} \end{cases}$$



$\left(\frac{p}{m\omega}\right)^2 + x^2 = \text{const}.$

⟺ 物理意義：SHO 為 圓周運動之投影的 表示



$x_0\sin\omega t = \frac{p}{m\omega} = \frac{v}{\omega}$ (速度)

① 為了一般較不提！

Energy is a const

∴只好分 長度角度

Angel

Normal modes in classical SHO

$$H = \frac{1}{2} m\omega^2 \underbrace{\left[x - \frac{ip}{m\omega}\right]}_{A} \underbrace{\left[x + \frac{ip}{m\omega}\right]}_{A^*}$$

$$\dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p}$$

$$\dot{A} = \dot{x} - \frac{i}{m\omega}\dot{p} = \frac{\partial H}{\partial p} + \frac{i}{m\omega}\frac{\partial H}{\partial x}$$

$$\frac{\partial H}{\partial p} = \frac{\partial H}{\partial A}\left(-\frac{i}{m\omega}\right) + \frac{\partial H}{\partial A^*}\frac{i}{m\omega}$$

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial A} + \frac{\partial H}{\partial A^*}$$

$$\therefore \dot{A} = \frac{2i}{m\omega}\frac{\partial H}{\partial A^*} \quad \text{——— ①}$$

Similarly

$$\dot{A}^* = \dot{x} + \frac{i}{m\omega}\dot{p} = \frac{-2i}{m\omega}\frac{\partial H}{\partial A} \quad \text{— ②}$$

$$\because \quad H = \frac{1}{2} m\omega^2 |A|^2$$

$$\therefore ① \Rightarrow \quad \dot{A} = i\omega A \qquad A = A_0 e^{i\omega t}$$

$$② \Rightarrow \quad \dot{A}^* = -i\omega A^* \qquad A^* = A_0 e^{-i\omega t}$$

$\Longleftrightarrow$ just another way to say

$$\dot{x} = \frac{p}{m}$$

$$\dot{p} = -Kx = -m\omega^2 x$$

linear combinations

$$\dot{A} = \dot{x} - \frac{i}{m\omega}\dot{p} = i\omega x + \frac{p}{m} = i\omega A$$

$$a^\dagger a = \frac{m\omega}{2\hbar}\left(x - \frac{ip}{m\omega}\right)\left(x + \frac{ip}{m\omega}\right)$$

$$= \frac{m\omega}{2\hbar}\left[x^2 - \frac{i}{m\omega}[p,x] + \frac{p^2}{m^2\omega^2}\right]$$

$$= \frac{1}{\hbar\omega}\left[\frac{1}{2}mp^2 + \frac{1}{2}m\omega^2 x^2\right] - \frac{1}{2}$$

$$= \frac{H}{\hbar\omega} - \frac{1}{2}, \quad \therefore \quad H = \hbar\omega(a^\dagger a + \tfrac{1}{2})$$

~~Define~~ $\cancel{\hat{H} = \frac{H}{\hbar\omega} = a^\dagger a + \frac{1}{2}}$, we now have to solve $\hat{H}|\varepsilon\rangle = \varepsilon|\varepsilon\rangle$

This is equivalently to solve

$a^\dagger a|n\rangle = n|n\rangle$ $\quad \varepsilon \to n$ to remind us $a^\dagger a|n\rangle = n|n\rangle$

Let $N \equiv a^\dagger a$. $N^\dagger = a^\dagger a = N$

$[N, a] = [a^\dagger a, a] = a^\dagger[a,a] + [a^\dagger, a]a = -a$

take "$\dagger$"

$[N, a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger[a, a^\dagger] + [a^\dagger, a^\dagger]a = a^\dagger$

Hence, if $N|n\rangle = n|n\rangle$, n needs not to be an integer up to now!

$$N a^\dagger|n\rangle = a^\dagger N|n\rangle + a^\dagger|n\rangle$$

$$= (n+1)(a^\dagger|n\rangle)$$

$a^\dagger|n\rangle$ is also an eigenket of $\hat{N}$ with eigenvalue $n+1$.

$\therefore a^\dagger$ increases the eigenvalue by 1 : creation operator

Similarly, a decreases " " : annihilation "

$\because \quad N(a|n\rangle) = (n-1)(a|n\rangle)$

For 1D, there is no degeneracy. $\therefore a^\dagger|n\rangle \propto |n+1\rangle$

$a|n\rangle \propto |n-1\rangle$

$\therefore (a^+)^m |n\rangle \propto |n+m\rangle$

$(a)^m |n\rangle \propto |n-m\rangle$

$\not\parallel$

eigenvalue $= n-m$

Now, since $\langle H\rangle \geq 0$, so, one can not indefinitely decrease the eigenvalue of a^+a by applying $(a)^m$ to $|n\rangle$ $(m \to \infty,\ n-m \to -\infty)$.

The only possibility is n itself must

(i) be a non-negative integer !!

$\therefore \hat{H}|n\rangle = \hbar\omega(n+1/2)|n\rangle$

$E_n = \hbar\omega(n+\frac{1}{2})$ $n = 0, 1, 2, 3, \ldots \Leftrightarrow N =$ number operator

(ii) Also, there exists a state $|\psi\rangle$ such that $a|\psi\rangle = 0$ ← null vector

$\because a^+a|\psi\rangle = 0$ ← null vector $\therefore$ we denote $|\psi\rangle = |0\rangle$ $\hat{H}|0\rangle = \frac{1}{2}\hbar\omega|0\rangle$.

From $|0\rangle$, one can build all other states $|n\rangle$ by $(a^+)^n|0\rangle$

Since $|0\rangle$ is unique (no degeneracy in one dimensional), $|n\rangle$ is unique.

<u>normalization</u> $a|n\rangle \propto |n-1\rangle$

$\therefore a|n\rangle = c|n-1\rangle$, $|n\rangle, |n-1\rangle$ normalized

i.e. $\langle n|n\rangle = \langle n-1|n-1\rangle = 1$

$\underbrace{\langle n|a^+a|n\rangle}_{=n} = |c|^2$

$\therefore c = \sqrt{n}\, e^{i\phi_n}$, take $\phi_n = 0$ (convention)

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

$$a^\dagger (a|n\rangle) = \sqrt{n}\, a^\dagger |n-1\rangle$$

$$\therefore \quad n|n\rangle = \sqrt{n}\, a^\dagger |n-1\rangle$$

$$a^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle$$

Just remember, the larger number appears under the square root.

<u>Matrix form of a & $a^\dagger$</u> :

$$\langle n'|a|n\rangle = \sqrt{n}\, \delta_{n',n-1}$$

$$\langle n'|a^\dagger|n\rangle = \sqrt{n+1}\, \delta_{n',n+1}$$

$$\begin{array}{c|c}
 & n=0 \quad 1 \quad 2 \quad \cdots \\
\begin{matrix} n'=0 \\ 1 \\ 2 \\ \vdots \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & & \\ 0 & 0 & \sqrt{2} & 0 & \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ \vdots & & \ddots & & \end{pmatrix}
\end{array} = a$$

not Hermitian

$$\begin{array}{c|c}
 & n=0 \quad 1 \quad 2 \quad 3 \quad 4 \\
\begin{matrix} n'=0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & \cdots & \\ 0 & \sqrt{2} & 0 & \cdots & \\ 0 & 0 & \sqrt{3} & & \\ \vdots & & & \ddots & \end{pmatrix}
\end{array} = a^\dagger$$

$$H = \begin{pmatrix} \frac{1}{2}\hbar\omega & & & \\ & \frac{3}{2}\hbar\omega & & 0 \\ 0 & & \frac{5}{2}\hbar\omega & \\ & & & \ddots \end{pmatrix}$$

$$X = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger) = \sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} 0 & 1 & 0 & 0 & \\ 1 & 0 & \sqrt{2} & 0 & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \cdot \\ 0 & 0 & \sqrt{3} & 0 & \cdot \\ \vdots & \vdots & & & \ddots \end{pmatrix}$$

$$p = i\sqrt{\frac{m\hbar\omega}{2}}\,(a^\dagger - a)$$

$$= i\sqrt{\frac{m\hbar\omega}{2}}\begin{pmatrix} 0 & -1 & 0 & 0 & \cdots \\ 1 & 0 & -\sqrt{2} & 0 & \cdots \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \cdot \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}$$

<u>Eigenkets</u> :

$$|1\rangle = a^\dagger |0\rangle$$

$$|2\rangle = \frac{1}{\sqrt{2}} a^\dagger |1\rangle = \frac{1}{\sqrt{2}} (a^\dagger)^2 |0\rangle$$

$$|3\rangle = \frac{1}{\sqrt{3}} a^\dagger |2\rangle = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{2}} (a^\dagger)^2 |1\rangle = \frac{1}{\sqrt{3!}} (a^\dagger)^3 |0\rangle$$

$$\vdots$$

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle$$

<u>in position space</u> :

$$0 = \langle x | a | 0\rangle = \sqrt{\frac{m\omega}{\hbar}} \langle x | \hat{x} + \frac{i\hat{p}}{m\omega} | 0\rangle$$

$$\therefore\ x\,\psi_0(x) + \underbrace{\frac{\hbar}{m\omega}}_{b^2} \frac{d}{dx}\psi_0(x) = 0$$

$$, \quad x = by$$

$$\left(y + \frac{d}{dy}\right)\psi_0(y) = 0$$

$$\psi_0 = C\, e^{-\frac{y^2}{2}} = C\, e^{-\frac{m\omega}{2\hbar}x^2}$$

normalization : $C = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}$

$$\therefore\ \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$$

12/12/95

In general, $\because$ $a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$

$$= \frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right)$$

$\therefore$ $\psi_n(x) = \langle x | \frac{1}{\sqrt{n!}} (a^+)^n | 0\rangle$

$$= \frac{1}{\sqrt{n!}}\left[\frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right)\right]^n \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}y^2}$$

not on coefficient of $e^{-\frac{y^2}{2}}$

$\frac{d}{dy}$ only acts on $e^{-\frac{1}{2}y^2}$

Compared with

$$\psi_n(x) = \left[\frac{m\omega}{\pi\hbar\, 2^{2n}(n!)^2}\right]^{\frac{1}{4}} e^{-y^2/2} H_n(y).$$

for largest power has coefficient $= 2^n$

We have $H_n(y) = e^{\frac{y^2}{2}}\left(y - \frac{d}{dy}\right)^n e^{-\frac{1}{2}y^2}$ $\Rightarrow$ recursion relation.. see 8-1

~~Equation of Motion and coherent states:~~

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

$$\frac{dp}{dt} = \frac{-1}{i\hbar}[H, p] = \frac{-1}{i\hbar}\frac{1}{2}m\omega^2[x^2, p]$$

$$= -m\omega^2 x$$

$$\frac{dx}{dt} = \frac{p}{m}$$

$$\frac{da}{dt} = \frac{-1}{i\hbar}[H, a] = \frac{-1}{i\hbar}\hbar\omega[a^+a, a] = -i\omega a$$

classically also true

$$\frac{da^+}{dt} = i\omega a^+$$

$$a(t) = a(0)\, e^{-i\omega t} \qquad a^+(t) = a^+(0)\, e^{i\omega t}$$

$$x(t) = \sqrt{\frac{\hbar}{2m\omega}}\,(a(t) + a^+(t)) = \sqrt{\frac{\hbar}{2m\omega}}\left(a(0)\, e^{-i\omega t} + a^+(0)\, e^{i\omega t}\right)$$

12/12/95

(i) By construction, $\langle n|m\rangle = \delta_{nm}$

$$\therefore \left[\frac{m\omega}{\pi\hbar\, 2^{2n}(n!)^2}\right]^{\frac{1}{2}} \int dx\; e^{-y^2} H_n(y) H_m(y) = \delta_{nm}$$

$$y = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$\therefore \int_{-\infty}^{\infty} dy\; e^{-y^2} H_n(y) H_m(y) = \delta_{nm} \sqrt{\pi}\, 2^n (n!)$$

(ii) $\langle x|a^{+}|n\rangle = \sqrt{n+1}\; \psi_{n+1}(x)$

$$\frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right) \psi_n(x) = \sqrt{n+1}\; \psi_{n+1}(x)$$

$$\because \psi_n(x) = \left[\frac{m\omega}{\pi\hbar\, 2^{2n}(n!)^2}\right]^{\frac{1}{4}} e^{-y^2/2} H_n(y)$$

$$\psi_{n+1}(x) = \left[\quad '' \quad\right]^{\frac{1}{4}} \cdot \frac{1}{\sqrt{2}} \frac{1}{\sqrt{n+1}}\, e^{-y^2/2} H_{n+1}(y)$$

$$\Rightarrow \left(y - \frac{d}{dy}\right) e^{-y^2/2} H_n(y) = e^{-y^2/2} H_{n+1}(y)$$

$$2y H_n(y) - \frac{dH_n(y)}{dy} = H_{n+1}(y) \qquad \text{— ①}$$

Similarly, from $\langle x|a|n\rangle = \sqrt{n}\; \psi_{n-1}(x)$

$$\Rightarrow \frac{1}{2}\left(y + \frac{d}{dy}\right) e^{-y^2/2} H_n(y) = n\, e^{-y^2/2} H_{n-1}(y)$$

$$\frac{dH_n(y)}{dy} = 2n\, H_{n-1}(y) \qquad \text{— ②}$$

$$① + ② \Rightarrow \quad 2y H_n = H_{n+1} + 2n H_{n-1}$$

(iii) From ②, $H_0 = 1$, $H_0(-x) = H_0(x)$

$H_1(-x) = (-1) H_1(x)$, $H_2(-x) = H_2(x)$, $\ldots$ $H_n(-x) = (-1)^n H_n(x)$

$\psi_n(-x) = (-1)^n \psi_n(x)$

(iv) generating function : will be an excercise!
(related to coherent states!)

綠意林

∴ We call them coherent states.

There are more reasons that will come up later.
Before we derive $|\lambda\rangle$, we first explicitly state what we want to construct:

(i) We want to construct $|\lambda\rangle$ so that it reflects classical harmonic oscillator's motion. Since the motion of the center of the wave packet obeys the law of classical mechanics,

$$\langle\lambda|X|\lambda\rangle \equiv X_c(t)$$
$$\langle\lambda|\hat{P}|\lambda\rangle \equiv P_c(t),$$

we wish $H = \frac{1}{2m}P_c^2 + \frac{1}{2}m\omega^2 X_c^2$ as in classical case.

Indeed, if $a|\lambda\rangle = \lambda|\lambda\rangle$, $H = \frac{1}{2}\hbar\omega + \hbar\omega\langle\lambda|a^+|\lambda\rangle\langle\lambda|a|\lambda\rangle$ $= \frac{1}{2}\hbar\omega + \hbar\omega\lambda^*\lambda$

$= \frac{1}{2}\hbar\omega + \hbar\omega\cdot\frac{m\omega}{2\hbar}\langle\lambda|X - \frac{iP}{m\omega}|\lambda\rangle\langle\lambda|X + \frac{iP}{m\omega}|\lambda\rangle$

$= \frac{1}{2}\hbar\omega + \frac{1}{2}m\omega^2 X_c^2 + \frac{1}{2m}P_c^2$. (match the classical result up to $\frac{1}{2}\hbar\omega$!) ⇐ i.e. replace all operators by class values!

(ii) Since $|\lambda\rangle$ represents a state in which the particle oscillates back and forth between two turning point, $|\lambda\rangle$ must be non-stationary state!
($|n\rangle$ is stationary state. ∴ $\langle n|\hat{X}|n\rangle$ is indept. of t!)

Stationary state: if $|\varphi\rangle$ is stationary, for any observable $\hat{A}$ is time-independent

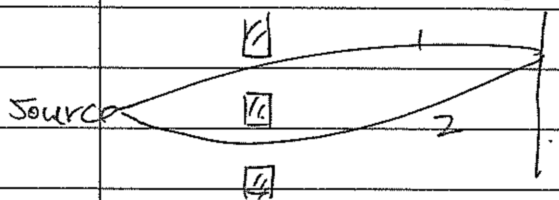
$\langle\varphi|\hat{A}|\varphi\rangle$ is time-independent!

($\hat{A}$ has no explicit time-dependence)

Firstly, why "Coherent"? "協調"

"Coherence" is actually a vague concept.

It was originally aimed to situations such as:



$\Psi(r,t) = \psi_1(r,t) + \psi_2(r,t)$

$$P(r,t) = \langle |\Psi(r,t)|^2 \rangle$$

average over noise in the source.

If the source's noise is purely randomized, we have

$$P = \langle |\psi_1(r,t)|^2 \rangle + \langle |\psi_2(r,t)|^2 \rangle$$

the cross term $\langle \varphi_1^* \varphi_2 \rangle$ or $\langle \varphi_1 \varphi_2^* \rangle$ vanishes!

This is an extreme case: no "coherence" at all.

If, on the other hand, no noise is there,

$$\langle \varphi_1^* \varphi_2 \rangle = \varphi_1^* \psi_2 \Rightarrow \text{max coherence.}$$

In general, if some function is factorized this way $\langle G(r t r' t') \rangle = \psi^*(r t)\,\psi(r' t')$, $\Rightarrow$ we call it first-order coherence.

It turns out that for coherent states $|\lambda\rangle$

$$\langle \lambda | H | \lambda \rangle = \frac{1}{2}\hbar\omega + \hbar\omega \langle \lambda | a^+ a | \lambda \rangle$$

$$= \frac{1}{2}\hbar\omega + \hbar\omega\, \lambda^* \lambda = \frac{1}{2}\hbar\omega + \hbar\omega \underbrace{\langle \lambda | a^+ | \lambda \rangle \langle \lambda | a | \lambda \rangle}_{\text{factorized!}}$$

$$a(0) = \sqrt{\frac{m\omega}{2\hbar}}\left(x(0) + \frac{i\,p(0)}{m\omega}\right)$$

$$a^+(0) = \sqrt{\frac{m\omega}{2\hbar}}\left(x(0) - \frac{i\,p(0)}{m\omega}\right)$$

$$\therefore\ x(t) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t$$

$$p(t) = -m\omega\, x(0)\sin\omega t + p(0)\cos\omega t$$

$\Longleftrightarrow$ same as classical results!

Note: these are expressions for operators!

They don't mean that $\langle x\rangle_n$ & $\langle p\rangle_n$ oscillate in the same way as classical particles

In fact, $\langle x\rangle_n = 0$, $\langle p\rangle_n = 0$! even though as we saw $n\to\infty$ $P(x) \sim P_{classical}(x)$

To observe oscillations reminiscent of the classical motion, we have to look at a superposition of different $|n\rangle$. e.g. $|\alpha\rangle = C_0|0\rangle + C_1|1\rangle$.

How can we construct such a superposition to mimic classical oscillators?

In wave-function language, we need to construct a wave packet that bounce back and forth without spreading in space.

Note that as we have shown, the energy eigenstate $\langle x|n\rangle$ behaves more like a standing wave inside two turning points! $\therefore$ $\langle x\rangle_n$, $\langle p\rangle_n = 0$ are not surprising! No net moving!

It turns out the coherent state defined by $a|\lambda\rangle = \lambda|\lambda\rangle$ will do the job.

$\because$ a is not Hermitian, $\therefore$ in general, λ is complex

λ can be any complex number!

12/18 /95

From Ehrenfest theorem,

$$\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}\langle [H,A]\rangle = 0$$

$\therefore \langle [H,A]\rangle = 0$ for any $\hat{A}$!

i.e. $\langle \psi | HA | \psi\rangle = \langle \psi | AH | \psi\rangle$

(note, $[H,A]$ not necessarily $=0$!)

Therefore, if $|\psi\rangle$ is an eigenket of energy,

it must be a stationary state !

$\langle \varphi | HA | \varphi\rangle = E\langle \psi | A | \varphi\rangle = \langle \varphi | AH | \varphi\rangle$

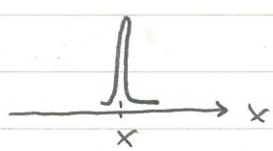
(iii) Preferably, we ask Δx, Δp = t-independent constants

(also has the meaning of "coherence"), so that

the wave packet doesn't spread !

Coherent state: "協調"

* Translational operator : $\hat{T}(a)$ (revised 12/20/95)

$|x\rangle =$ 

$\hat{T}(a)|x\rangle = |x+a\rangle$ 

(active)

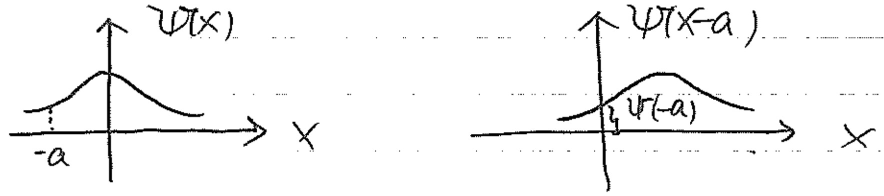
$$\hat{T}(a)|\psi\rangle = \hat{T}(a)\int dx\, |x\rangle\langle x|\psi\rangle$$

$$= \int dx\, |x+a\rangle\, \psi(x)$$

$$= \int dx\, |x\rangle\, \psi(x-a)$$

$$\therefore \langle x|\hat{T}(a)|\psi\rangle = \psi(x-a) = \psi(x) - a\frac{d}{dx}\psi(x) + \frac{a^2}{2!}\frac{d^2}{dx^2}\psi(x) + \cdots = e^{-a\frac{d}{dx}}\psi(x)$$

$$= e^{-\frac{i}{\hbar}a\hat{p}}\,\psi(x) \Rightarrow \hat{T}(a) = e^{-\frac{i}{\hbar}a\hat{p}}$$



$\therefore$ particle $\underline{\rightarrow a}$!

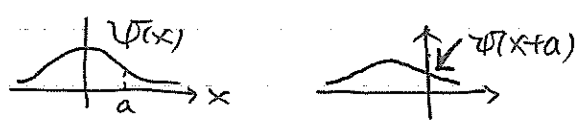
$|\phi\rangle \equiv \hat{T}(a)\,|\psi\rangle$.

$$\langle\phi|\,\hat{x}\,|\phi\rangle = \int dx\ \phi^*(x)\, x\, \phi(x)$$
$$= \int dx\ \psi^*(x-a)\,\psi(x-a)\,x$$
$$= \int dy\ \psi^*(y)\,\psi(y)\,(y+a)$$
$$= \langle x\rangle + a$$

$$\hat{x} \rightarrow \hat{T}^\dagger(a)\,\hat{x}\,\hat{T}(a) = e^{\frac{i}{\hbar}a\hat{p}}\,\hat{x}\,e^{-\frac{i}{\hbar}a\hat{p}} = \underline{\hat{x} + a}$$

i.e. $\hat{x}\,\hat{T}(a) = \hat{T}(a)(\hat{x}+a)$, $[\hat{x}, \hat{T}(a)] = a\,\hat{T}(a)$

Passive : Coordinate system shift $+a$. Wave function (particle) is fixed. Let the operator be $\hat{T}'(a)$ (shift origin a). $\langle x|\hat{T}'(a)|\psi\rangle = \psi(x+a)$



For time-translation, since there is no state $|t\rangle$ (t is not a dynamical variable), $\therefore$ one always adopts passive point of view. $\therefore$ We consider $\psi(t+\Delta t) = \psi(t) + \Delta t \frac{d}{dt}\psi(t) + \frac{\Delta t^2}{2!}\psi''(t) + \cdots$

$$= e^{\Delta t \frac{d}{dt}}\,\psi(t) = e^{-\frac{i}{\hbar}\Delta t\,\hat{H}}\,\psi(t)$$

$T(\Delta t) = e^{-\frac{i}{\hbar}\Delta t\,\hat{H}}$, $\hat{p}$ = generator of translation

$\hat{H}$ = " of time-translation

Similarly,

$$\because \Phi(p-p_0) = e^{-p_0\frac{d}{dp}}\,\Phi(p), \quad i\hbar\frac{d}{dp} = \hat{x}$$
$$= e^{\frac{i}{\hbar}p_0\hat{x}}\,\Phi(p), \quad \Rightarrow T_p(p_0) = e^{\frac{i}{\hbar}p_0\hat{x}}$$

In summary,

$$\hat{T}(a) = e^{-\frac{i}{\hbar}a\hat{p}}$$

$$\therefore \quad \hat{T}_p(P_0) = e^{+\frac{i}{\hbar}P_0\hat{X}}$$

* <u>Construction</u>:

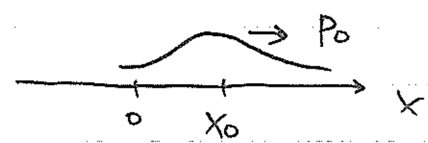
Now, in classical mechanics, $X_{cl} = X_0\cos\omega t + \frac{P_0}{m\omega}\sin\omega t$.

$\therefore$ We want $\langle x \rangle = X_0\cos\omega t + \frac{P_0}{m\omega}\sin\omega t$.

First, we realized that even in the basis $|n\rangle$, only $|0\rangle$ is a wave packet. So, to make the best

use of $|n\rangle$, we can try to shift the initial position and momentum of $|0\rangle$:

$$e^{-\frac{i}{\hbar}Ht}\underbrace{e^{\frac{i}{\hbar}P_0\hat{X}}\,e^{-\frac{i}{\hbar}X_0\hat{P}}\,|0\rangle}_{t=0} \quad \Rightarrow$$



By using $e^A e^B = e^{A+B}\,e^{\frac{1}{2}[A,B]}$,

we have $e^{\frac{i}{\hbar}P_0\hat{X}}\,e^{-\frac{i}{\hbar}X_0\hat{P}} = e^{\frac{i}{\hbar}(P_0\hat{X}-X_0\hat{P})}\,e^{\frac{i}{\hbar}\frac{P_0X_0}{2}}$

Since $e^{\frac{i}{\hbar}P_0X_0}$ is a pure phase factor, $\therefore$ we may drop it.

$$\therefore \quad |\lambda\rangle \equiv e^{\frac{i}{\hbar}(P_0\hat{X}-X_0\hat{P})}\,|0\rangle$$

$$|\lambda t\rangle = e^{-\frac{i}{\hbar}Ht}\,|\lambda\rangle$$

$$\frac{i}{\hbar}(P_0\hat{X}-X_0\hat{P}) = \frac{i}{\hbar}\left[P_0\sqrt{\frac{\hbar}{2m\omega}}\,(a+a^{+}) - iX_0\sqrt{\frac{m\hbar\omega}{2}}\,(a^{+}-a)\right]$$

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$$= \left[\frac{ip_0}{\sqrt{2\hbar m\omega}} - x_0\sqrt{\frac{m\omega}{2\hbar}} \right] a + \left[\frac{ip_0}{\sqrt{2\hbar m\omega}} + x_0\sqrt{\frac{m\omega}{2\hbar}} \right] a^+$$

$$\equiv \lambda a^+ - \lambda^* a$$ displacement operator, $\lambda = x_0\sqrt{\frac{m\omega}{2\hbar}} + \frac{ip_0}{\sqrt{2\hbar m\omega}}$

$= \sqrt{\frac{m\omega}{2\hbar}}\left(x_0 + \frac{ip_0}{m\omega}\right)$

$= a_0 \Leftarrow$ not $\Rightarrow$ operators!

$$\therefore \underline{|\lambda\rangle = e^{\lambda a^+ - \lambda^* a}|0\rangle}$$

$$\because e^{\lambda a^+ - \lambda^* a} = e^{\lambda a^+} e^{-\lambda^* a} e^{-\frac{1}{2}[\lambda a^+, -\lambda^* a]}$$

$$= e^{\lambda a^+} e^{-\lambda^* a} e^{-\frac{1}{2}|\lambda|^2}$$

$$\therefore \underline{|\lambda\rangle = e^{-\frac{1}{2}|\lambda|^2} e^{\lambda a^+}|0\rangle} \quad (\because a|0\rangle = 0)$$

$$= e^{-\frac{1}{2}|\lambda|^2} \sum_n \frac{\lambda^n}{n!}(a^+)^n|0\rangle \qquad \left(a^+|\lambda\rangle = \frac{\partial}{\partial\lambda}|\lambda\rangle\right)$$

$$= e^{-\frac{1}{2}|\lambda|^2} \sum_n \frac{\lambda^n}{\sqrt{n!}}|n\rangle$$

$\therefore \langle\lambda|a|\lambda\rangle_{\text{classical}}$
$= a_0 =$ value at initial

* $$\therefore \underline{a|\lambda\rangle} = e^{-\frac{1}{2}|\lambda|^2} \sum_n \frac{\lambda^n}{\sqrt{n!}}\sqrt{n}\,|n-1\rangle$$

$$= e^{-\frac{1}{2}|\lambda|^2} \sum_n \frac{\lambda^n}{\sqrt{(n-1)!}}|n-1\rangle \ \underline{= \lambda|\lambda\rangle}$$

* normalized: $$\langle\lambda|\lambda\rangle = e^{-|\lambda|^2}\langle 0|e^{\lambda^* a} e^{+\lambda a^+}|0\rangle$$

$$= e^{-|\lambda|^2}\langle 0|e^{\lambda^* a + \lambda a^+} e^{\frac{|\lambda|^2}{2}}|0\rangle$$

$$= e^{-\frac{1}{2}|\lambda|^2}\langle 0|e^{\lambda a^+} e^{\lambda^* a} e^{\frac{1}{2}|\lambda|^2}|0\rangle$$

$$= \langle 0|e^{\lambda a^+} e^{\lambda^* a}|0\rangle$$

$$= \langle 0|0\rangle = 1 \qquad (a|0\rangle = 0)$$

* $|\lambda t\rangle = e^{-\frac{i}{\hbar}Ht}|\lambda\rangle$, $\quad a|\lambda t\rangle = e^{-\frac{i}{\hbar}Ht}\underbrace{\left(e^{\frac{i}{\hbar}Ht} a\, e^{-\frac{i}{\hbar}Ht}\right)}_{\equiv\, a_H}|\lambda\rangle$

緑意林

$$\frac{da_H}{dt} = \frac{i}{\hbar}[H, a_H] = -i\omega a_H, \quad a_H(t) = e^{-i\omega t} a_H(0) = e^{-i\omega t} a$$

$$\therefore a|\lambda t\rangle = e^{\frac{-iH}{\hbar}t} e^{-i\omega t} a|\lambda\rangle = \lambda e^{-i\omega t}|\lambda t\rangle$$

$$\lambda(t) = \lambda e^{-i\omega t}$$

= classical value of $a(t)$

$$\therefore \langle\lambda t|a|\lambda t\rangle = \lambda e^{-i\omega t} \quad \text{--- ①}$$

Similarly, $\because \langle\lambda t|a^+ = \langle\lambda t|\lambda^* e^{i\omega t}$

$$\therefore \langle\lambda t|a^+|\lambda t\rangle = \lambda^* e^{i\omega t} \quad \text{--- ②}$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right), \quad a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$$

$\therefore$ ① & ② imply
$$\langle x\rangle + \frac{i}{m\omega}\langle p\rangle = \sqrt{\frac{2\hbar}{m\omega}}\lambda e^{-i\omega t}$$
$$\langle x\rangle - \frac{i}{m\omega}\langle p\rangle = \sqrt{\frac{2\hbar}{m\omega}}\lambda^* e^{i\omega t}$$

$$\therefore \langle x\rangle = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\left[\lambda e^{-i\omega t} + \lambda^* e^{i\omega t}\right], \quad \lambda = \sqrt{\frac{m\omega}{2\hbar}}\left(x_0 + \frac{ip_0}{m\omega}\right)$$

$$= \frac{1}{2}\left\{\left(x_0 + \frac{ip_0}{m\omega}\right)e^{-i\omega t} + \left(x_0 - \frac{ip_0}{m\omega}\right)e^{i\omega t}\right\}$$

$$= x_0\cos\omega t + \frac{p_0}{m\omega}\sin\omega t$$ as what we wanted!

* What about uncertainties? 同理 $\langle p\rangle = p_0\cos\omega t - m\omega x_0\sin\omega t$

$$\because x = \sqrt{\frac{\hbar}{2m\omega}}(a^+ + a)$$

$$x^2 = \frac{\hbar}{2m\omega}\left[(a^+)^2 + a^2 + \underbrace{a^+a + aa^+}_{=2a^+a+1}\right]$$

$$\langle\lambda t|x^2|\lambda t\rangle = \frac{\hbar}{2m\omega}\left[\langle\lambda t|a^{+2}|\lambda t\rangle + \cdots\right]$$

$$= \frac{\hbar}{2m\omega}\left[(\lambda^* e^{i\omega t})^2 + (\lambda e^{-i\omega t})^2 + 2|\lambda|^2 + 1\right]$$

$$\therefore (\Delta x)^2 = \langle x^2\rangle - (\langle x\rangle)^2 = \frac{\hbar}{2m\omega}\left[(\lambda^* e^{i\omega t})^2 + (\lambda e^{-i\omega t})^2 + 2|\lambda|^2 + 1\right.$$

$$-(\lambda e^{-i\omega t} + \lambda^* e^{i\omega t})^2] = \frac{\hbar}{2m\omega}$$

$$\therefore \ \Delta x = \sqrt{\frac{\hbar}{2m\omega}}$$

Similarly, $\because \ p = i\sqrt{\frac{m\hbar\omega}{2}}\,(a^+ - a)$

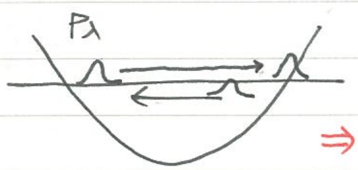
$$\langle p^2 \rangle = -\frac{m\hbar\omega}{2}\left[(\lambda^* e^{i\omega t})^2 + (\lambda e^{-i\omega t})^2 - 2|\lambda|^2 - 1 \right]$$

$$\langle p \rangle = -i\sqrt{\frac{\hbar m\omega}{2}}\left[\lambda e^{-i\omega t} - \lambda^* e^{i\omega t} \right]$$

$$(\Delta p)^2 = \frac{m\hbar\omega}{2}, \quad \Delta p = \sqrt{\frac{\hbar\omega m}{2}}. \quad \Delta x \Delta p = \frac{\hbar}{2}$$

X-representation $\quad \langle x | \lambda t \rangle \equiv \psi_\lambda(x,t)$

Exercise $\quad \psi_\lambda(x,t) = \left(\frac{1}{2\pi\Delta x^2}\right)^{\frac{1}{4}} e^{-\frac{(x-x_{cl}(t))^2}{4\Delta x^2} + \frac{i}{\hbar}p_{cl}(t)(x-x_{cl}(t))}$



$$P_\lambda(x,t) = \sqrt{\frac{1}{2\pi\Delta x^2}}\, e^{-\frac{(x-x_{cl}(t))^2}{2\Delta x^2}}$$

$\Rightarrow$ in particular $\psi_\lambda(x,0)$ gives the generating function for $H_n(x)$!!

Overcompleteness & completeness relation:

Completeness relation: $\sum |n\rangle\langle n| = \mathbb{1}$ or $\int d\xi\, |\xi\rangle\langle\xi| = \mathbb{1}$

But $|n\rangle$ or $|\xi\rangle$ may be over complete!

This is because # of $|n\rangle$ is not necessarily equal to # of linearly-independent states (i.e., dimension) of the Hilbert space of the considered system!

Example: 2D vector space $\quad \hat{x}\hat{x} + \hat{y}\hat{y} = \mathbb{1}$

$$|x_1\rangle = \frac{1}{\sqrt{2}}\hat{x} \qquad |x_3\rangle = \frac{1}{2}(\hat{x} + \hat{y})$$

$$|x_2\rangle = \frac{1}{\sqrt{2}}\hat{y} \qquad |x_4\rangle = \frac{1}{2}(\hat{x} - \hat{y})$$

$$|x_1\rangle\langle x_1| + |x_2\rangle\langle x_2| + |x_3\rangle\langle x_3| + |x_4\rangle\langle x_4| = \hat{x}\hat{x} + \hat{y}\hat{y} = \mathbb{1}$$

but $|x_1\rangle, |x_2\rangle, |x_3\rangle, |x_4\rangle$ are overcomplete!

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It's convenient to use $|\phi\rangle = \sum_n \frac{1}{\sqrt{n!}} \phi^n |n\rangle = e^{\phi a^+}|0\rangle$

instead of $|\lambda\rangle = \underline{\underline{e^{-\frac{1}{2}|\lambda|^2}}} \sum_n \frac{1}{\sqrt{n!}} \lambda^n |n\rangle$.

* non-orthogonal : $\langle \phi' | \phi \rangle = \langle 0 | e^{\phi'^* a} e^{\phi a^+} |0\rangle$

$$= \langle 0 | e^{\phi a^+} e^{\phi'^* a} e^{\phi'^* \phi} |0\rangle$$

$$\uparrow$$

$$e^A e^B = e^B e^A e^{[A,B]}$$

$$= e^{\phi'^* \phi} \qquad (\because \ e^{\phi'^* a}|0\rangle = |0\rangle)$$

*

e.g. $\boxed{\langle \phi | n \rangle = \frac{1}{\sqrt{n!}} (\phi^*)^n}$

ϕ - representation $\quad \langle x | \psi \rangle \Leftrightarrow \langle \phi | \psi \rangle$

$$\Uparrow \qquad\qquad \Uparrow$$

$$\psi(x) \qquad\qquad \psi(\phi^*)$$

$$\Uparrow$$

$$\because \langle \phi | a^+ = \langle \phi | \phi^*$$

$$\therefore \langle \phi | a^+ | \psi \rangle = \phi^* \psi(\phi^*)$$

$$a^+ |\phi\rangle = a^+ e^{\phi a^+}|0\rangle = \frac{d}{d\phi} e^{\phi a^+}|0\rangle$$

$$= \frac{d}{d\phi}|\phi\rangle$$

$\therefore \langle \phi | a = \frac{d}{d\phi^*} \langle \phi | \qquad \left(\langle \phi | a = \langle 0 | e^{\phi^* a} a = \frac{d}{d\phi^*} \langle \phi | \right)$

$$\langle \phi | a | \psi \rangle = \frac{d}{d\phi^*} \langle \phi | \psi \rangle = \frac{d\,\psi(\phi^*)}{d\phi^*}$$

$\Rightarrow \quad a = \frac{d}{d\phi^*}, \quad a^+ = \phi^* \qquad \therefore [a, a^+] = \frac{d\phi^*}{d\phi^*} = 1$

* Completeness relation :

$$\delta(x-x') = \int \frac{dy}{2\pi} e^{iy(x-x')} \Rightarrow z = iy, \ \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} e^{-z(x'-x)} = \delta(x-x') = \delta(x'-x)$$

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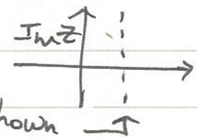
Formally, $\int \frac{dz}{2\pi i} e^{-z(z_1-z_2)} = \delta(z_1-z_2)$

(path )

can be shown

too formal !

$$\therefore \Psi(\phi^*) = \int d\phi'^* \, \delta(\phi'^* - \phi^*) \, \Psi(\phi'^*)$$

$$= \int d\phi'^* \int \frac{dz}{2\pi i} e^{-z(\phi'^* - \phi^*)} \Psi(\phi'^*)$$

no relation

$$\underset{\uparrow}{=} \iint \frac{d\phi'^* d\phi'}{2\pi i} \underbrace{e^{-\phi'(\phi'^* - \phi^*)}}_{\parallel \atop e^{-\phi'\phi'^*} \langle\phi|\phi'\rangle} \Psi(\phi'^*)$$

$\cancel{\phi}$?

$z \to \phi'$

(indept. from ϕ'^* by definition)

$$= \langle\phi| \iint \frac{d\phi'^* d\phi'}{2\pi i} e^{-\phi'\phi'^*} |\phi'\rangle\langle\phi'|\Psi\rangle$$

$$\therefore \mathbb{1} = \iint \frac{d\phi'^* d\phi'}{2\pi i} e^{-\phi'\phi'^*} |\phi'\rangle\langle\phi'|$$

ϕ', ϕ'^* complex conjugate

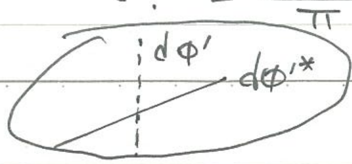
Overcomplete : because # of $|\phi\rangle$ > $|n\rangle$ ($n = 0, 1, 2, \cdots$)

uncountable countable

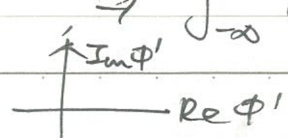
More rigorous proof : $\frac{d\phi'^* d\phi'}{2\pi i} = J \frac{d\mathrm{Re}\phi' \, d\mathrm{Im}\phi'}{2\pi i}$

$$J = \text{Jacobian} = \begin{vmatrix} \frac{d\phi'^*}{d\mathrm{Re}\phi} & \frac{d\phi'^*}{d\mathrm{Im}\phi} \\ \frac{d\phi'}{d\mathrm{Re}\phi} & \frac{d\phi'}{d\mathrm{Im}\phi} \end{vmatrix} = \begin{vmatrix} 1 & -i \\ 1 & i \end{vmatrix} = 2i$$

$$\therefore = \frac{d\mathrm{Re}\phi' \, d\mathrm{Im}\phi'}{\pi} \Rightarrow \int_{-\infty}^{\infty} d\mathrm{Re}\phi' \int_{-\infty}^{\infty} \frac{d\mathrm{Im}\phi'}{\pi} = \int_0^\infty \frac{\rho d\rho}{\pi} \int_0^{2\pi} d\theta$$



$\Longrightarrow$



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Check completeness :

$$|\phi\rangle = \sum_n \frac{1}{\sqrt{n!}} \phi^n |n\rangle$$

$$I = \iint \frac{d\phi^* d\phi}{2\pi i} e^{-\phi\phi^*} |\phi\rangle\langle\phi|$$

$$= \sum_{n,m} \frac{1}{\sqrt{n!}}\frac{1}{\sqrt{m!}} \int_0^\infty \frac{1}{\pi}\rho d\rho \int_0^{2\pi} d\theta\, e^{-\rho^2} (\rho e^{i\theta})^n |n\rangle$$

$$\langle m| (\rho e^{-i\theta})^m$$

$$\because \int_0^{2\pi} d\theta\, e^{i\theta(n-m)} = 2\pi \delta_{n,m}$$

$$\therefore I = \sum_n \frac{1}{n!} \underbrace{\int_0^\infty 2\rho d\rho\, e^{-\rho^2} \rho^{2n}} |n\rangle\langle n|$$

$$\| $$

$$\int_0^\infty dz\, e^{-z} z^n , \quad z = \rho^2$$

$$\| $$

$$\left(-\frac{d}{d\alpha}\right)^n \int_0^\infty dz\, e^{-\alpha z} \Big|_{\alpha=1}$$

$$\| $$

$$\left(-\frac{d}{d\alpha}\right)^n \frac{1}{\alpha}\Big|_{\alpha=1} = n!$$

$$\therefore I = \sum_n |n\rangle\langle n| = \mathbb{I}$$

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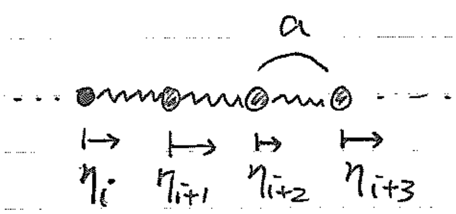
A "Real" example of harmonic oscillators :

phonons :

1D case :



$$H = \frac{1}{2}\sum_i\left[m\,\dot{\eta}_i^2 + K(\eta_{i+1}-\eta_i)^2\right]$$

$$= \frac{1}{2}\sum_i a\left\{\left[\frac{m}{a}\dot{\eta}_i^2\right] + Ka\left(\frac{\eta_{i+1}-\eta_i}{a}\right)^2\right\}$$

$$= \frac{1}{2}\int dx\left(\rho\dot{\eta}^2 + Y\left(\frac{d\eta}{dx}\right)^2\right) \qquad \rho = \frac{m}{a},\ Y = Ka$$

Classically, $L = \frac{1}{2}\int dx\left[\rho\dot{\eta} - Y\left(\frac{d\eta}{dx}\right)^2\right]$

eq. of motion $\rho\frac{\partial^2}{\partial t^2}\eta(x,t) - Y\frac{\partial^2\eta(x,t)}{\partial x^2} = 0$

$$C = \sqrt{\frac{Y}{\rho}}, \qquad \omega_k = CK$$

Suppose total length $= L = Na$, Consider a particular mode k.
(one-mode quantization)

$\eta = \frac{q_K(t)}{\sqrt{L}}\sin Kx$. Assuming the periodic B.C., one has

$K = \frac{2\pi n}{L}$, n = integer.

$$\int dx\,\rho\dot{\eta}^2 = \frac{\rho(\dot{q}_K(t))^2}{L}\int_0^L \sin^2 Kx\,dx = \frac{1}{2}\rho(\dot{q}_K(t))^2$$

$$\int dx\,Y\left(\frac{d\eta}{dx}\right)^2 = \frac{YK^2[q_K(t)]^2}{L}\int_0^L \cos^2 Kx\,dx = \frac{1}{2}YK^2(q_K(t))^2$$

$$\therefore H = \frac{1}{2}\rho\dot{q}_K^2 + \frac{1}{2}\rho\omega_K^2 q_K^2(t) = \frac{1}{2\rho}P_K^2 + \frac{1}{2}\rho\omega_K^2 q_K^2(t)$$

$$= \hbar\omega_K(a_K^+ a_K + \tfrac{1}{2}). \qquad a_K = \sqrt{\frac{\rho\omega_K}{2\hbar}}\left(q_K + \frac{iP_K}{m\omega_K}\right)$$

$$a_K^+ = \sqrt{\frac{\rho\omega_K}{2\hbar}}\left(q_K - \frac{iP_K}{m\omega_K}\right)$$

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In general, $\because \eta^*(x,t) = \eta(x,t)$, $\therefore$ if we expand $\eta = \sum_k \eta_k(t)\, e^{ik\cdot x}$

$$\because \eta^* = \sum_k \eta_k^*\, e^{-ikx} = \underbrace{\sum \eta_{-k}(t)\, e^{-ikx}}_{\eta}$$

$$\therefore \eta_k^* = \eta_{-k}$$

center of mass motion $\eta_0 = \frac{1}{L}\int_0^L \eta\, dx$

$$\therefore \eta = \sum_{k>0} \left(\eta_k e^{ikx} + \eta_{-k} e^{-ikx}\right) + \eta_0$$

$$= \sum_{k>0} \left(\eta_k e^{ikx} + \eta_k^* e^{-ikx}\right) + \eta_0$$

$$= \sum_{k>0} \left[(2\mathrm{Re}\,\eta_k)\cos kx - (2\mathrm{Im}\,\eta_k)\sin kx\right] + \eta_0$$

$$= \sum_{k>0} \underbrace{(2\mathrm{Re}\,\eta_k)}_{q_k(t)} \cos kx + \sum_{k<0} \underbrace{(2\mathrm{Im}\,\eta_k)}_{\tilde{q}_k(t)} \sin kx + \eta_0$$

$$\Rightarrow H = \sum_{k>0} \hbar\omega_k \left(a_k^+ a_k + \tfrac{1}{2}\right) + \sum_{k<0} \hbar\omega_k \left(a_k^+ a_k + 1/2\right)$$

$$= \sum_k \hbar\omega_k \left(a_k^+ a_k + 1/2\right), \quad (\because \omega_{k=0} = 0)$$

Zero point energy $= E_0 = \frac{1}{2}\sum_k \hbar\omega_k$, $\omega_k = c|k|$

$$= \hbar c \sum_{n=1}^{\infty} \frac{2n\pi}{L} \quad \text{diverges !!}$$

need a cutoff: 1st Brillouin $-\frac{\pi}{a} < k \leq \frac{\pi}{a}$

$$\sum_n \Rightarrow \sum_{n=1}^{n_\wedge \sim \frac{L}{2a}}, \quad E_0 \simeq \frac{2\pi}{L}\hbar c\, \frac{1}{2} n_\wedge (n_\wedge + 1)$$

$$\simeq \frac{hc}{2L}\left(\frac{L}{2a}\right)^2 = \frac{hc}{8a^2} L \Rightarrow \frac{E_0}{L} = \frac{hc}{8a^2}$$

Revise presentation of real example. 3.

$$H = \frac{1}{2}\int dx\left[\rho\dot{\eta}^2 + Y\left(\frac{d\eta}{dx}\right)^2\right]$$

$$\eta(x) = \int\frac{dk}{2\pi}\,\eta_k(t)\,e^{ikx} \qquad \because\ \eta^* = \eta \quad \therefore\ \eta_k^*(t) = \eta_{-k}(t)$$

$$\int dx\,\dot{\eta}^2 = \int dx\int\frac{dk}{2\pi}\int\frac{dk'}{2\pi}\,\dot{\eta}_k(t)\,\dot{\eta}_{k'}(t)\,e^{i(k+k')x}$$

$$= \int\frac{dk}{2\pi}\,\dot{\eta}_k(t)\,\dot{\eta}_{-k}(t)$$

Similarly $\int dx\, Y\left(\frac{d\eta}{dx}\right)^2 = \int\frac{dk}{2\pi}\,Yk^2\,\eta_k(t)\,\eta_{-k}(t)$

$$\therefore\quad H = \frac{1}{2}\int\frac{dk}{2\pi}\,\underbrace{\rho|\dot{\eta}_k(t)|^2}_{\frac{P_\eta^2}{\rho}} + Yk^2|\eta_k(t)|^2$$

$$\because\ [\eta_i, \underbrace{m\dot{\eta}_i}_{P_i}] = i\hbar \quad , [\eta_i, m\dot{\eta}_j] = i\hbar\delta_{ij}$$

$$\therefore [\eta(x), \underbrace{\rho\dot{\eta}(x')}_{P}] = \frac{i\hbar}{a}\delta_{x,x'} \rightarrow i\hbar\delta(x-x') \text{ at same time}$$

$$\because\ \eta_k(t) = \int dx\,\eta(x,t)\,e^{-ikx}$$

$$\therefore [\eta_k(t), \rho\dot{\eta}_{k'}(t)] = \int dx\int dx'\,[\underbrace{\eta(x,t), \rho\dot{\eta}(x',t)}_{i\hbar\delta(x-x')}]\,e^{-ikx}e^{-ik'x'}$$

$$= i\hbar\int dx\,e^{-i(k+k')x} = 2\pi i\hbar\,\delta(k+k') \quad \text{— ①}$$

$\therefore$ if $P_k \equiv \frac{\rho \dot{\eta}_k}{\sqrt{2\pi}}$, $x_k = \frac{\eta_k}{\sqrt{2\pi}}$

$[x_k, P_{k'}] = i\hbar\,\delta(k+k')$

$$\therefore H = \int dk \; \frac{1}{2\rho} P_{-k} P_k + \frac{1}{2} \underbrace{Yk^2}_{\rho\omega_k^2} x_k x_{-k}$$

$$P_{-k} = \frac{\rho}{\sqrt{2\pi}} \dot{\eta}_{-k} = \frac{\rho}{\sqrt{2\pi}} \dot{\eta}_k^* \text{ (classically)}$$

$$= P_k^+$$

$$P_k = \int \frac{1}{\sqrt{2\pi}} \rho \dot{\eta}(x,t)\, e^{-ikx}\, dx$$

$$P_{-k} = \int \frac{1}{\sqrt{2\pi}} \rho \dot{\eta}(x,t)\, e^{ikx}\, dx$$

$$= \Big(\int \frac{1}{\sqrt{2\pi}} \underbrace{\rho \dot{\eta}}_{p(x,t)}\, e^{-ikx}\, dx \Big)^+ \qquad p^+(x,t) = p(x,t)$$

i.e. $[x_k^+, P_k^+] = i\hbar$ (1)

$$\therefore H = \sum_k \frac{1}{2\rho} P_k^+ P_k + \frac{1}{2}\rho\omega_k^2 x_k^+ x_k$$

Sum over indept harmonic oscillators.

Harmonic oscillators in other systems

phonons & photons

phonons $\hat{H} = \frac{1}{2m}\sum_k \hat{P}_k^{+} \hat{P}_k + \frac{m}{2}\sum_k \omega_k^2 \hat{x}_k^{+} \hat{x}_k$

$\rightarrow \quad = \sum_k \hbar\omega_k \left(a_k^{+} a_k + 1/2 \right)$ (Homework 6.)

$a_k = \sqrt{\frac{m\omega_k}{\hbar}} \left(x_k + \frac{i P_k}{m\omega_k} \right)$ $\quad [a_k, a_{k'}^{+}] = \delta_{k,k'}$

$a_k^{+} = \sqrt{\frac{m\omega_k}{\hbar}} \left(x_k^{+} - \frac{i P_k^{+}}{m\omega_k} \right)$ $\quad [x_k, P_{k'}^{+}] = i\hbar\,\delta_{k,k'}$

(use the convention : $x_j = \frac{1}{\sqrt{N}} \sum_k x_k e^{-ik\cdot(ja)}$

$P_j = \frac{1}{\sqrt{N}} \sum_k P_k e^{+ik\cdot(ja)}$)

$a_k^{+}|0\rangle$ one phonon with "crystal momentum" $\hbar k$

$\frac{1}{\sqrt{2}}(a_k^{+})^2|0\rangle$ two phonons . . . .

$a_k^{+} a_{k'}^{+}|0\rangle$ two " $\hbar k, \hbar k'$

Photons : $H = \frac{1}{2}\int d^3\vec{r} \left(\vec{E}^2 + \vec{B}^2 \right)$ $\quad \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \vec{B} = \vec{\nabla}\times\vec{A}$

$A_k(t) \leftrightarrow x_k$

$E_k(t) \leftrightarrow \dot{x}_k$

For phonons

$$a_k = \sqrt{\frac{m\omega_k}{\hbar}}\left(x_k + \frac{ip_k}{m\omega_k}\right)$$

$$a_k^+ = \sqrt{\frac{m\omega_k}{\hbar}}\left(\overset{\overset{x_k^+}{\shortparallel}}{x_{-k}} - \frac{i\overset{\overset{p_k^+}{\shortparallel}}{p_{-k}}}{m\omega_k}\right) \qquad \omega_{-k}=\omega_k$$

$$\therefore\ x_k = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega_k}}\left(a_k + a_{-k}^+\right)$$

$$\therefore\ x(t) = \frac{1}{\sqrt{N}}\sum_k x_k e^{ik\cdot ja}$$

$$= \frac{1}{2\sqrt{N}}\sqrt{\frac{\hbar}{m\omega_k}}\sum_k a_k(t) e^{ik\cdot ja} + a_{-k}^+(t) e^{ik\cdot ja}$$

$$= \frac{1}{2\sqrt{N}}\sqrt{\frac{\hbar}{m\omega_k}}\sum_k \underset{\downarrow\ a_k e^{-i\omega_k t}}{a_k(t)} e^{ik\cdot ja} + \underset{\downarrow\ a_k^+ e^{i\omega_k t}}{a_k^+(t)} e^{-ik\cdot ja}$$

$$= \frac{1}{2\sqrt{N}}\sqrt{\frac{\hbar}{m\omega_k}}\sum_k a_k e^{i(k\cdot ja - \omega_k t)} + a_k^+ e^{-i(k\cdot ja - \omega_k t)}$$

$$\Rightarrow \vec{A}(r,t) \propto \sum \sqrt{\frac{\hbar}{\omega_k}}\Big[a_{ks}\,\hat{e}_s(\vec{k})\, e^{i(\vec{k}\cdot\vec{r} - \omega_k t)} + a_{ks}^+\,\underset{\uparrow\ \text{polarization}}{\hat{e}_s^{*}(\vec{k})}\, e^{-i(\vec{k}\cdot\vec{r} - \omega_k t)}\Big]$$

$$\sum_s \hat{e}_{si}(\vec{k})\,\hat{e}_{sj}^{*}(\vec{k}) = \delta_{ij} - \hat{k}_i\hat{k}_j$$

$$\vec{E}(\vec{r},t) = -\frac{1}{c}\frac{d\vec{A}}{dt} \propto \sum_k i\sqrt{\hbar\omega_k}\Big[a_{ks}\,\hat{e}_s(\vec{k})\, e^{i(\vec{k}\cdot\vec{r} - \omega_k t)} - a_{ks}^+\,\hat{e}_s^{*}\, e^{-i(\vec{k}\cdot\vec{r} - \omega_k t)}\Big]$$

$$\vec{B}(r,t) \propto i\sum_k \sqrt{\hbar\omega_k}\, \hat{k}\times\left(\hat{e}_s(k) a_{ks} e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} - \hat{e}_s^*(k) a_{ks}^+ e^{-i(\ \ )}\right]$$

phonon 3

$$H = \sum_{ks} \hbar\omega_k \left(a_{ks}^+ a_{ks} + 1/2\right)$$

Simplied discussion:

$$\hat{E}(t) = i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\left(a e^{-i\omega t} - a^+ e^{i\omega t}\right)$$

Vacuum:

$$\langle 0|E(t)|0\rangle = 0$$

$$\langle 0|B(t)|0\rangle = 0$$

$$\therefore \langle 0|(a e^{-i\omega t} - a^+ e^{i\omega t})^2|0\rangle$$

$$= \langle 0| -aa^+ - a^+a|0\rangle = \langle 0|-2a^+a - 1|0\rangle$$

$$= -1$$

$$\therefore \langle 0|E^2|0\rangle = \frac{\hbar\omega}{2\epsilon_0 V} \neq 0$$

Classical fields: $a|\lambda\rangle = \lambda|\lambda\rangle$ coherent state

$$\hat{E}(t)|\lambda\rangle = E|\lambda\rangle$$

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