

五. Hamilton's Principle

及 Lagrangian & Hamiltonian Dynamics

歷史上, Lagrange 由 D'Alembert's principle
出發最後推出 Lagrange's equations.

(即運動方程式)

之後, Hamilton 提出 ^以 Variational (變分) 的

觀點 ^上 最後 Hamilton's principle 提出。
^{來看} Lagrange's equation ^的

Hamilton's principle 的優美在於處理的是 scalar
而非牛頓力學的同量 $\vec{F}$, 被認為比牛頓
運動方程式更為基本 (fundamental)。

* Hamilton's principle 與 Lagrange's equations

首先我們由 D'Alembert's principle 出發

$$\sum_{i=1}^N (\vec{F}_i - m_i \vec{a}_i) \cdot \delta \vec{r}_i = 0 \quad \text{--- ①}$$

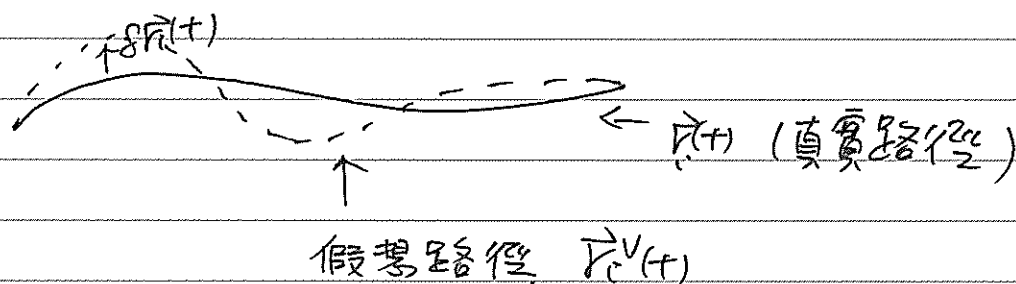
這裏 $\vec{F}_i$ 即為 $\vec{F}_i^{\text{appl}}$.

注意: virtual displacement $\delta \vec{r}_i$ 是

針對某一固定時間 t , 之位移

因此, $\delta \vec{r}_i = \delta \vec{r}_i(t)$, 且 $\vec{F}_i = \vec{F}_i(t)$, $m_i \ddot{\vec{r}}_i(t)$:

當 t 由 t_0 到 $t = \tau_0$ (某時刻), $\delta \vec{r}_i(t)$ 描
述了一個 "virtual path" (假想路徑):



因此, $\delta \vec{r}_i(t)$ 也有對應的速度 $\frac{d}{dt} \delta \vec{r}_i(t)$ 及
加速度 $\frac{d^2}{dt^2} \delta \vec{r}_i(t)$ 等!

$$\text{則 } \because \sum_{i=1}^N m_i \ddot{\vec{r}}_i \cdot \delta \vec{r}_i = \frac{d}{dt} \left(\sum_{i=1}^N m_i \dot{\vec{r}}_i \cdot \delta \vec{r}_i \right) - \sum_{i=1}^N m_i \dot{\vec{r}}_i \cdot \frac{d}{dt} \delta \vec{r}_i(t)$$

$$\text{且 } \frac{d}{dt} \delta \vec{r}_i(t) = \frac{d}{dt} (\vec{r}_i^V(t) - \vec{r}_i(t)) = \dot{\vec{r}}_i^V - \dot{\vec{r}}_i = \delta(\dot{\vec{r}}_i)$$

$$\therefore \dot{\vec{r}}_i \cdot \frac{d}{dt} \delta \vec{r}_i(t) = \dot{\vec{r}}_i \cdot \delta(\dot{\vec{r}}_i) = \frac{1}{2} \delta(\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i)$$

$$\therefore \sum_{i=1}^N m_i \dot{\vec{r}}_i \cdot \delta \vec{r}_i = \frac{d}{dt} \left(\sum_{i=1}^N m_i \dot{\vec{r}}_i \cdot \delta \vec{r}_i \right) - \delta \left(\frac{1}{2} \sum_{i=1}^N m_i \dot{\vec{r}}_i^2 \right)$$

"T = 動能"

另外一子面; $\sum_{i=1}^N \vec{F}_i \cdot \delta \vec{r}_i = \delta W$ $W =$ 外力所作之 虛功

如果外力皆是保守力, $\vec{F}_i = -\nabla_i U(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N)$

$$\text{即 } F_i^x = -\frac{\partial U}{\partial x_i}, F_i^y = -\frac{\partial U}{\partial y_i}, F_i^z = -\frac{\partial U}{\partial z_i}$$

$$\text{則 } \delta W = -\sum_{i=1}^N \nabla_i U(\vec{r}_1, \vec{r}_2, \vec{r}_3, \dots, \vec{r}_N) \cdot \delta \vec{r}_i$$

$$\text{正如 } g(x+\delta x, y+\delta y, z+\delta z) = g(x, y, z)$$

$$+ \frac{\partial}{\partial x} g(x, y, z) \delta x + \frac{\partial}{\partial y} g(x, y, z) \delta y + \frac{\partial}{\partial z} g(x, y, z) \delta z$$

$$= g(x, y, z) + Dg \cdot \delta \vec{r}, \quad \frac{g(x+\delta x, y+\delta y, z+\delta z) - g(x, y, z)}{\delta \vec{r}} = Dg \cdot \delta \vec{r}$$

$$\sum_{i=1}^N \nabla_i U \cdot \delta \vec{r}_i = \delta U$$

$$\therefore \delta W = -\delta U$$

因此, ①式可被寫為

$$\frac{d}{dt} \left(\sum_{i=1}^N m_i \dot{\vec{r}}_i \cdot \delta \vec{r}_i \right) = \delta T + \delta W = \delta T - \delta U \dots \textcircled{2}$$

Hamilton's principle: 如果我們固定 t 由 t_1 到 t_2

且要求 $\vec{r}_i(t_1)$ 及 $\vec{r}_i(t_2)$ 固定, 即 $\delta \vec{r}_i(t_1) = 0, \delta \vec{r}_i(t_2) = 0$

$$\text{則 } \delta \int_{t_1}^{t_2} (T - U) dt = 0 \quad \dots \textcircled{3}$$

(3)式之右式為零是因為 $\int_{t_1}^{t_2} \frac{d}{dt} \left(\sum_{i=1}^N m_i \vec{r}_i \cdot \delta \vec{r}_i \right) dt$

$$= \sum_{i=1}^N \left(m_i \vec{r}_i(t_2) \cdot \delta \vec{r}_i(t_2) - m_i \vec{r}_i(t_1) \cdot \delta \vec{r}_i(t_1) \right) = 0$$

之故！

即在所有可能連接 $\vec{r}(t_1)$ 及 $\vec{r}(t_2)$ 之路徑中，

真實路徑 (牛頓力學所得之路徑) 的
亦為

$$\int_{t_1}^{t_2} (T - U) dt \text{ 值為 } \underline{\underline{\text{極值}}}!$$

事實上，絕大多數物理中之例子，此極值

皆為極小值！

由於 $T - U$ 之出現，我們定 $L = T - U$

為 Lagrangian 或 Lagrangian function.

而 $\int_{t_1}^{t_2} L dt$ 則稱為 action.

& Calculus of Variations

極值原理，即為 (functional analysis) 及 Euler-Lagrangian

方程式：

由求運動方程式之解

將物理問題轉成求極值的問題，在物理中

是一個重要的方法！

其優美在於 不須去處理運動方程式中的量

之複雜特性，因為 L 本身是一個純量！

另外一面，也可能對牛頓力學的起源有進一步的暗示。事實上，現在知道 Hamilton's principle 比牛頓運動方程式要基本：

在 Feynman 對量子力學的路徑積分 formulation 中，每一路徑貢獻 $e^{\frac{i}{\hbar} S}$ ，其中 $\hbar = \frac{h}{2\pi}$ ， h = Planck 常數，

$\hbar$ 很小，故偏離了最小值之路徑

相消的很厲害，造成 只有最小值之貢獻最大！

這些在之後的量子力學課程中會進一步之

說明，因此，Hamilton's principle 有其量子物理之 三點三原！
更基本之

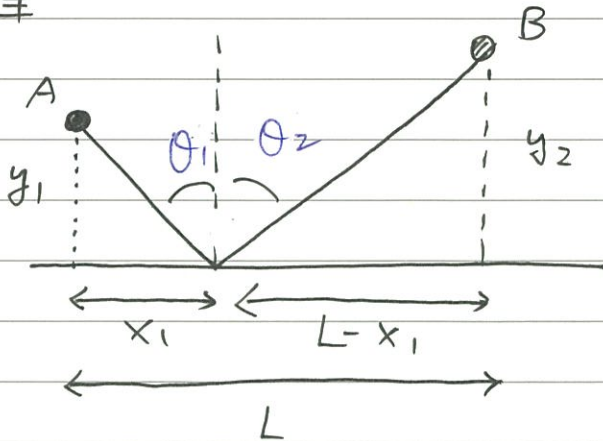
歷史上，除了 Hamilton's principle 外，另一個

將問題轉成求極值之思考方式

著名之 光學上之 Fermat principle.

Fermat's principle:

反射定律



由 $A \rightarrow B$ 所需之時間

$$t = (\sqrt{y_1^2 + x_1^2} + \sqrt{(L-x_1)^2 + y_2^2}) / c$$

將 x_1 視為變數，則 t 之極小值發生在

$$\frac{dt}{dx_1} = 0 \quad \text{可得} \quad \frac{x_1}{\sqrt{x_1^2 + y_1^2}} = \frac{L-x_1}{\sqrt{(L-x_1)^2 + y_2^2}}$$

$$\cos\left(\frac{\pi}{2} - \theta_1\right) = \cos\left(\frac{\pi}{2} - \theta_2\right)$$

換句話說，

故 $\theta_1 = \theta_2$ (此即實驗上所觀測到之反射定律)

由此可知光在走最短之路徑！此即 Fermat's principle,

它可用來證明折射定律 (problem 6-7)

對簡單的問題，並沒有簡化問題，但對於

較複雜之折射問題，如 折射率 是位置之複雜
的函數 ($n = n(r)$)，此方法之使用較簡單。

Function v.s. functional:

將問題取代是一個重要觀念上之突破，

以求某函數之極小，

其影響層面深遠，許多困難的問題都

受其影響。如 cost function 之概念等。

求最佳策略時之

許多問題如果要以傳統方程式方法 formulate

並不容易，以極小化之方式寫下比較容易。

但這裏所要極小化之對象並非極小化

某一函數，而是所謂泛函 (functional)。

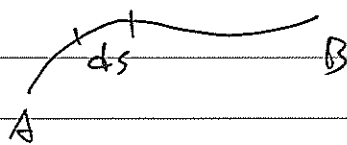
以上述：反身定律來說，如果不是事先假

之例

設為二段直線，也不是單純只是 x_1 之函數。

對折射問題來說，若 n = 折射率

則光在該介質之速率 = $\frac{c}{n}$



故行經 ds 所需時間

$$= \frac{ds}{\frac{c}{n}} = \frac{nds}{c}$$

故由 $A \rightarrow B$ 之時間 = $\int_A^B \frac{1}{c} n ds$

所要極小化 即是 $\int_A^B n ds$

而如果設 $A \rightarrow B$ 之 路徑 $= \vec{r}(s)$

則 $I = \int_A^B n(\vec{r}(s)) ds$ 與 $\vec{r}(s)$; $s=0$ 到 $s=L_{AB}$ 有關
 整個

而非：反射定律之例中一個 x_1 值有關。

所以，所面對的：問題是找到一個函數 $\vec{r}(s)$

使 $I[\vec{r}(s)]$ 最小！

我們稱 I 為：一個 functional，其變數 $\rightarrow$ 記為 $I[\vec{r}(s)]$

不再是一個數，而是一個函數 $\vec{r}(s)$ ！

同理， $S = \int_{t_1}^{t_2} (\pi - U) dt$ 也是一個 functional

其變數為 $\vec{r}(t)$ ， $t_1 \leq t \leq t_2$

因為 I 及 S 皆非函數，我們需要新的數學方法求其極值！

注意：一旦 $\vec{r}(t)$ (或 $\vec{r}(s)$) 給定， $\dot{\vec{r}}(t)$ 即給定

不能自行變動，所以 I 或 S 的變數只

有 $\vec{r}(t)$ ， $t_1 \leq t \leq t_2$ 不須再指定 $\dot{\vec{r}}(t)$ ！

Calculus of variations and Euler-Lagrangian Equation

由上可知，我們所要考慮之 functional 可表示為

$$J = \int_{x_A}^{x_B} f(y(x), y'(x), x) dx \quad \dots (4)$$

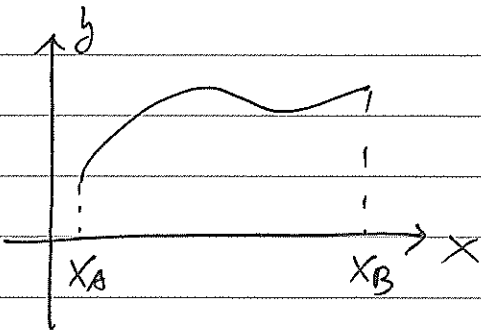
其中 f 之變數為 $y(x)$, $y'(x)$ 及 x

例如 $f = L$ 時

$$x = t, \quad L = \frac{1}{2} m (\dot{x}(t))^2 - U(x(t))$$

對於折身問題, $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (y'(x))^2} dx$
_{2-維}

$$\therefore \int_A^B n(x, y) ds = \int_{x_A}^{x_B} n(x, y(x)) \sqrt{1 + y'^2(x)} dx$$



$\therefore$ (4) 式 可說是物理中所見之一般式。

(4) 式 必須配合一「界條件」, 以前述的「問題」

(Hamilton's principle), $\delta \vec{r}(t_1) = \delta \vec{r}(t_2) = 0$

$\therefore$ 伴隨 (4) 式, 我們要求

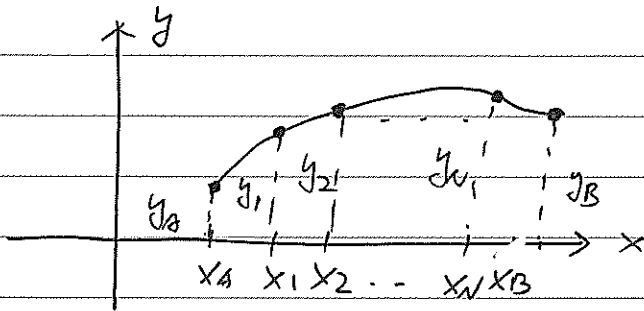
$$\delta y(x_A) = 0, \quad \delta y(x_B) = 0 \quad \dots (5)$$

即 $y(x_A)$ 及 $y(x_B)$ 是固定值

在⑤之限制下，使④ 最小之 $y(x)$ 。

如果我們將 $x_A \leq x \leq x_B$ 分割為

$$x_A < x_1 < x_2 < x_3 < \dots < x_N < x_B$$



則任意曲線 $y(x)$

可視為由

$$(y_A, y_1, y_2, \dots, y_N, y_B)$$

組成

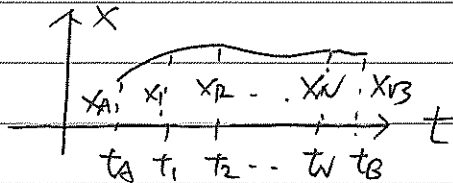
其中 $y_i = y(x_i)$, $x_i - x_{i-1} = \Delta x$

$$J = \Delta x \sum_{i=1}^N f(y(x_i), \frac{y(x_i) - y(x_{i-1})}{\Delta x}, x_i)$$

$= J(y_A, y_1, y_2, \dots, y_N, y_B)$ 為 多變數函數

因此，當 $N \rightarrow \infty$ 時， J 為無限多變數函數

$\therefore$ functional 一個無限多變數的函數
可視為



例如：

$$S = \int_{t_A}^{t_B} \left(\frac{1}{2} m \left(\frac{dx}{dt} \right)^2 - \frac{1}{2} k x^2 \right) dt \quad (U = \frac{1}{2} k x^2)$$

$$= \frac{1}{2} m \left[\frac{(x_A - x_1)^2}{\Delta t^2} + \frac{(x_2 - x_1)^2}{\Delta t^2} + \frac{(x_3 - x_2)^2}{\Delta t^2} + \dots + \frac{(x_B - x_N)^2}{\Delta t^2} \right] \Delta t$$

$- \frac{1}{2} k (x_1^2 + x_2^2 + \dots + x_N^2) \Delta t$ 為 $(x_1, x_2, \dots, x_N)$ 之函數

另一種看法則是將 $y(x)$ 分為“橫向”及“振幅”：

$$y(\alpha, x) = y(0, x) + \alpha \eta(x) \quad \text{--- (6)}$$

α 為額外 ≥ 1 之參數， $\alpha=0$ 時 $y=y(0, x)$ → 描述變化之振幅

$$\text{故 } y(0, x_A) = y_A, \quad y(0, x_B) = y_B$$

$\therefore y(\alpha, x_A) = y_A$ 及 $y(\alpha, x_B) = y_B$ 也成立

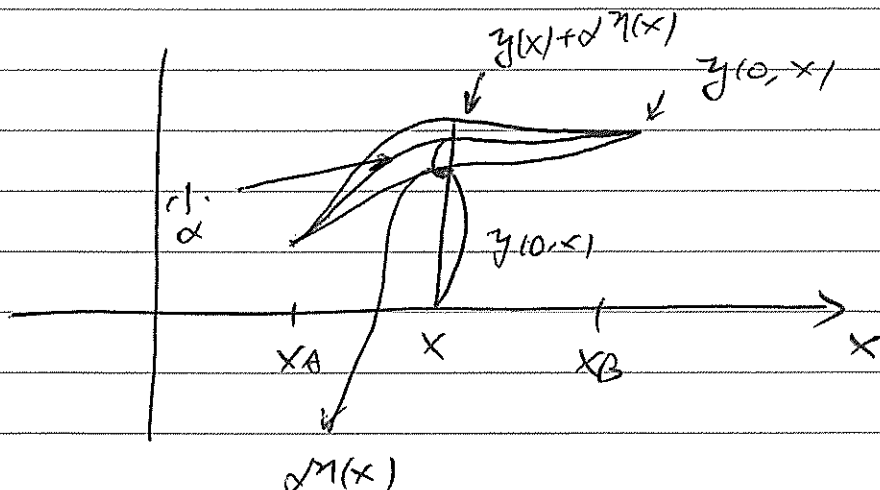
$$\text{故 } \eta(x_A) = 0, \quad \eta(x_B) = 0$$

換句話說，在 (6) 式之表示下，我們取

一固定 $y(0, x)$ 作為參考，另其他滿足

$$y(x_A) = y_A \text{ 及 } y(x_B) = y_B \text{ 之 } y(x) \text{ 對 } y(0, x)$$

之偏差以振幅 α 表示，(如下圖)



則 J 為 α 之函數 為 $\eta(x)$ 之 functional!

$$J = J[\alpha, \eta(x)] \quad \text{對 } \alpha, \text{ 可以以函數處理}$$

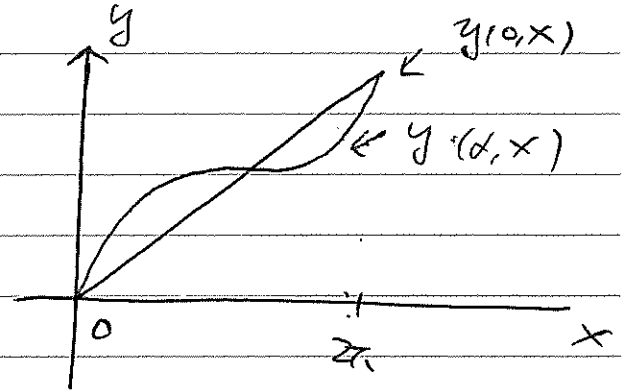
即若 $\frac{dJ}{d\alpha} \Big|_{\alpha=0} = 0$ ----- (6)

$y(0, x)$ 爲了獲得極值之 $y(x)$
起

例: $f = \left(\frac{dy}{dx}\right)^2$

$$y(0, x) = x$$

$$x_A = 0, \quad x_B = 2\pi$$



若 $\eta(x) = \sin x$ 滿足 $\eta(0) = \eta(2\pi) = 0$

則 $y(\alpha, x) = x + \alpha \sin x$ (如上述)

$$\frac{dy(\alpha, x)}{dx} = 1 + \alpha \cos x$$

$\therefore$ 對 $y(\alpha, x)$ 系列而言, $f = (1 + \alpha \cos x)^2$

$$= 1 + 2\alpha \cos x + \alpha^2 \cos^2 x$$

$$\therefore J = \int_0^{2\pi} f = 2\pi + 2\pi \alpha^2$$

$\therefore J(\alpha)$ 總 $> J(\alpha=0)$ $J(\alpha=0)$ 爲這一系列 $\eta(x)$

之最小!

現在

$$J(\alpha) = \int_{x_A}^{x_B} f(\underbrace{y(x) + \alpha \eta(x)}_{y(\alpha, x)}, \underbrace{y'(x) + \alpha \eta'(x)}_{y'(\alpha, x)}, x) dx$$

$$\therefore \frac{\partial J}{\partial \alpha} = \int_{x_A}^{x_B} \left(\underbrace{\frac{\partial f}{\partial y(\alpha, x)}}_{\eta(x)} \underbrace{\frac{\partial y(\alpha, x)}{\partial \alpha}}_{\eta'(x)} + \frac{\partial f}{\partial y'(\alpha, x)} \frac{\partial y'(\alpha, x)}{\partial \alpha} \right) dx$$

$$= \int_{x_A}^{x_B} \left[\frac{\partial f(y, y', x)}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \eta'(x) \right] dx$$

利用部分積分法 (integration by part)

$$\int_{x_A}^{x_B} \frac{\partial f}{\partial y'} \underbrace{\eta'(x)}_{d\eta} dx$$

$$= \left. \frac{\partial f}{\partial y'} \eta(x) \right|_{x=x_A}^{x=x_B} - \int_{x_A}^{x_B} \frac{d}{dx} \left(\frac{\partial f(y(\alpha, x), y'(\alpha, x), x)}{\partial y'} \right) \eta(x) dx$$

$$\because \eta(x_A) = \eta(x_B) = 0$$

$$\therefore \frac{\partial J}{\partial \alpha} \Big|_{\alpha=0} = \int_{x_A}^{x_B} \underbrace{\left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right)}_{\uparrow} \eta(x) dx \dots (6-1)$$

$$\uparrow \text{ 由 (5) } \hat{=} y = y(0, x), \quad y' = y'(0, x)$$

$$\therefore (6) \text{ 式} \Rightarrow \int_{x_A}^{x_B} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \eta(x) dx = 0$$

上式對任意 $\eta(x)$ (只要 $\eta(x_A) = \eta(x_B) = 0$) 都成立!

如我們可以選 $\eta(x)$ 只在某一點 x 非 0, 其他 = 0

故 $\frac{df}{dy} - \frac{d}{dx} \frac{df}{dy'} = 0 \quad \dots (7)$

在極值時

⑦式 即 為 Euler's equation (常用到)

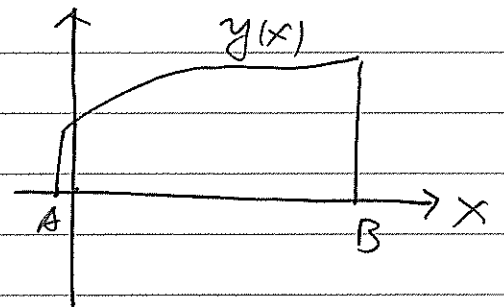
$f=L$ 時, 又稱為 Euler - Lagrangian equation

其解給出 使 J 達成極值!

例: 假設一介質之 $n=n(x)$, 求

光線 $y(x)$ 所滿足

之方程式.



$$J = \int_A^B n(x) ds$$

$$ds = \sqrt{1+y'^2} dx$$

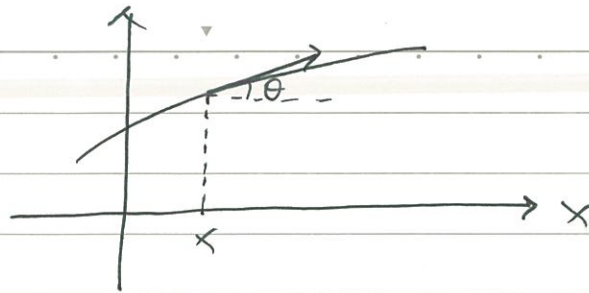
$$\therefore J = \int_{x_0}^{x_B} \underbrace{n(x) \sqrt{1+y'^2}}_f dx$$

Euler equation :

$$\left. \begin{aligned} \frac{df}{dy} &= 0 \\ \frac{df}{dy'} &= \frac{y'}{\sqrt{1+y'^2}} \end{aligned} \right\} \therefore \frac{d}{dx} \left(n(x) \frac{y'}{\sqrt{1+y'^2}} \right) = 0$$

即 $n(x) \frac{y'}{\sqrt{1+y'^2}} = \text{常數} \quad \dots (8)$

物理意義

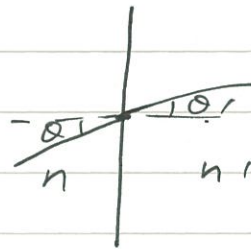


$$y' = \tan \theta, \therefore \sin \theta = \frac{y'}{\sqrt{1+y'^2}}$$

$$\therefore \textcircled{A} \text{式} \Rightarrow n(x) \sin \theta(x) = \text{常數} \quad \text{d.o.f.}$$

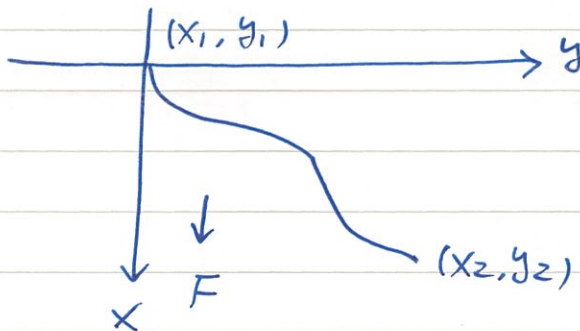
Snell 定律一致:

$$n \sin \theta = n' \sin \theta' !$$



例: Brachistochrone :

$F = \text{constant force field}$



求由 $(x_1, y_1) \xrightarrow{\text{靜止釋放}} (x_2, y_2)$
所花最少時間之
路徑

$$F = -mg$$

$$U = -mgx, \quad \frac{1}{2}mv^2 = mgx \therefore v = \sqrt{2gx}$$

$$t = \int_{(x_1, y_1)}^{(x_2, y_2)} \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gx}} = \int_{x_1=0}^{x_2} \frac{\sqrt{1+(y'(x))^2}}{\sqrt{2gx}} dx$$

(取 $x_1=0$)

由 Euler 方程式

取 $f = \sqrt{\frac{1+y'^2}{x}} \quad \therefore \frac{df}{dy} = 0$

$$\frac{d}{dx} \left(\frac{df}{dy'} \right) = 0 \quad \therefore \frac{df}{dy'} = \text{constant}$$

$$\text{即 } \frac{1}{2} \frac{1}{\sqrt{\frac{1+y'^2}{x}}} \times \frac{2y'}{\sqrt{1+y'^2}} = \text{constant} \equiv \frac{1}{\sqrt{2a}}$$

$$\therefore \frac{y'^2}{x(1+y'^2)} = \frac{1}{2a} \Leftrightarrow \text{Snell's law}$$

$$y'^2 = \frac{x}{2a-x}, \quad y' = \sqrt{\frac{x}{2a-x}} \quad \left(\begin{array}{l} \text{取 } > 0 \\ \text{代表向下} \\ \text{滑} \end{array} \right)$$

$$\int dy = \int \frac{\sqrt{x} dx}{\sqrt{2a-x}} = \int \frac{x dx}{\sqrt{2ax-x^2}}$$

$$\therefore y = \int \frac{x dx}{\sqrt{2ax-x^2}}$$

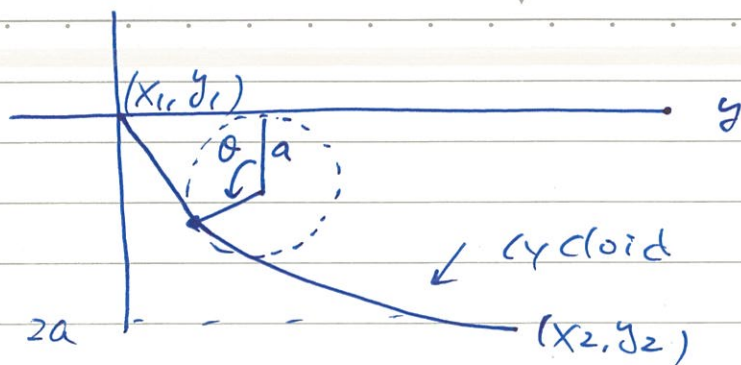
Change of variable: $x = a(1-\cos\theta)$

$$dx = a \sin\theta d\theta$$

$$2ax-x^2 = -(x-a)^2 + a^2 = a^2 - a^2 \cos^2\theta = a^2 \sin^2\theta$$

$$\therefore y = \int a(1-\cos\theta) d\theta = a(\theta - \sin\theta) + \text{const}$$

$\therefore$ 為一 cycloid (見 Homework 4)



Functional δ notation 及 functional derivative

在推导 Euler 方程式时, 我们使用
了 (6-1) 式, 如果将 $d\alpha$ 乘以 (6-1) 式, 则
可得

$$\frac{\delta J}{\delta \alpha} \delta \alpha = \int_{x_0}^{x_B} \left(\frac{\delta f}{\delta y} - \frac{d}{dx} \frac{\delta f}{\delta y'} \right) \eta(x) \delta \alpha \, dx \quad \text{--- (8)}$$

如欲令 $\delta \alpha = 0$

若 $y = y(\alpha, x)$, $y' = y'(\alpha, x)$

$$\therefore J(\alpha + d\alpha) = J(\alpha) + \frac{\delta J}{\delta \alpha} \delta \alpha$$

$$\therefore \frac{\delta J}{\delta \alpha} \delta \alpha = \delta J \quad (\delta J \text{ 之微小变量})$$

$$\text{而 } y(\alpha, x) = y(x) + \alpha \eta(x)$$

$$\begin{aligned} \therefore \delta y &\equiv y(\alpha + \delta \alpha, x) - y(\alpha, x) \\ &= \delta \alpha \eta(x) \end{aligned}$$

故 (8) 式

$$\Rightarrow \delta J = \int_{x_A}^{x_B} \left(\frac{\delta f}{\delta y} - \frac{d}{dx} \frac{\delta f}{\delta y'} \right) \delta y(x) dx \quad \dots (9)$$

其中 $\delta y(x)$ 為 $y(x)$ 之變化 而 J 是 $y(x)$ 之 functional
 δJ 為對應之變化

(9) 式相當於多變數函數之下列等式：

$$g(y_1 + \delta y_1, y_2 + \delta y_2, y_3 + \delta y_3, \dots, y_n + \delta y_n) \\ = g(y_1, y_2, y_3, \dots, y_n) + \delta y_1 \frac{\partial g}{\partial y_1} + \delta y_2 \frac{\partial g}{\partial y_2} + \dots + \delta y_n \frac{\partial g}{\partial y_n}$$

$$\delta g = \sum_i \frac{\partial g}{\partial y_i} \delta y_i \quad \dots (10)$$

比較 (9) 及 (10) 式，可知 $i \Leftrightarrow x$
 (相當於)

因此， $\frac{\delta f}{\delta y} - \frac{d}{dx} \frac{\delta f}{\delta y'}$ (為 $-x$ 函數)

相當於 $\frac{\partial g}{\partial y_i}$!
 (10) 式之

故將 $\frac{\delta f}{\delta y(x)} - \frac{d}{dx} \left(\frac{\delta f}{\delta y'} \right)$ 記為 $\frac{\delta J}{\delta y(x)} \quad \dots (11)$

稱為 J 對 y 之 functional derivative. $\left(\frac{\delta J}{\delta y(x)} \Delta x \Leftrightarrow \frac{\partial g}{\partial y_i} \right)$

數學上，functional derivative 之定義為

$$\frac{\delta J[y(x)]}{\delta y(x)} = \lim_{\varepsilon \rightarrow 0} \frac{J[y(x) + \varepsilon \delta(x-x_0)] - J[y(x)]}{\varepsilon}$$

↓ 以 (9) 式為例, 取 $\delta y(x) = \sum \delta(x-x_0)$

$$\begin{aligned} \therefore J[y(x) + \sum \delta(x-x_0)] - J[y(x)] \\ = \int_{x_0}^{x_B} \left[\frac{\partial f}{\partial y(x)} - \frac{d}{dx} \frac{\partial f}{\partial y'(x)} \right] \sum \delta(x-x_0) dx \\ = \sum \left. \frac{\partial f}{\partial y(x)} - \frac{d}{dx} \frac{\partial f}{\partial y'(x)} \right|_{x=x_0} \end{aligned}$$

$$\therefore \frac{\delta J}{\delta y(x_0)} = \frac{\partial f}{\partial y(x_0)} - \frac{d}{dx} \frac{\partial f}{\partial y'(x)} \Big|_{x=x_0} \text{ 与 (1) 式一致.}$$

例: $I[y] = \int_{x_0}^{x_B} y^2(x) dx$

$$\frac{\delta I}{\delta y(x_0)} = 2y(x_0)$$

例: $I[y(x)] = y(x) = \int_{x_0}^{x_B} y(x') \delta(x-x') dx'$

函数也可被視為 functional

$$\therefore \frac{\delta I}{\delta y(x_0)} = \delta(x-x_0)$$

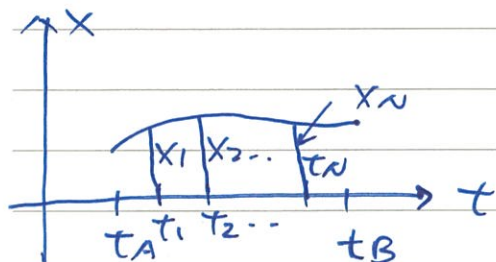
Other generalizations & 2nd form of the Euler Eq.

當 f 与 x 無關時, Euler Eq. 有比較簡單的形式

此時, 因為 $\frac{df(x, y(x), y'(x))}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial y'} \frac{dy'}{dx}$

例

$$S = \int_{t_A}^{t_B} \underbrace{\left[\frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \right]}_f dt$$



$$S = \frac{1}{2} m \left[\left(\frac{x_A - x_1}{\Delta t} \right)^2 + \left(\frac{x_2 - x_1}{\Delta t} \right)^2 + \dots + \left(\frac{x_B - x_N}{\Delta t} \right)^2 \right] \Delta t$$

$$- \frac{1}{2} k (x_1^2 + x_2^2 + \dots + x_N^2 + x_B^2) \Delta t$$

視 $S \equiv S(x_1, x_2, \dots, x_N)$

$$\text{則 } \frac{\partial S}{\partial x_i} = 0 \quad \because S = \left[\dots + \left(\frac{x_{i+1} - x_i}{\Delta t} \right)^2 + \left(\frac{x_i - x_{i-1}}{\Delta t} \right)^2 + \dots \right] \Delta t$$

$$- \frac{1}{2} k x_i^2 \Delta t$$

$$\therefore \frac{\partial S}{\partial x_i} = m \frac{1}{\Delta t} [2x_i - x_{i-1} - x_{i+1}] - k x_i \Delta t = 0$$

$$\text{Euler eq: } \frac{\partial f}{\partial x(t)} - \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}} \right) = 0 \quad (= \frac{\delta S}{\delta x(t)})$$

$$-kx - m\ddot{x} = 0 \quad \text{与上式一致!}$$

$$(\because \frac{1}{\Delta t^2} (2x_i - x_{i-1} - x_{i+1}) = \frac{-1}{\Delta t^2} [(x_{i+1} - x_i) - (x_i - x_{i-1})] \rightarrow -\ddot{x})$$

自由に見 $\Delta t \frac{\delta S}{\delta x}$ 与 $\frac{\delta S}{\delta x_i}$ 相对应!

$$= \frac{df}{dx} + y' \frac{df}{dy} + y'' \frac{df}{dy}, \quad \dots (12)$$

$$\therefore \frac{d}{dx} \left(y' \frac{df}{dy} \right) = y'' \frac{df}{dy} + y' \frac{d}{dx} \left(\frac{df}{dy} \right)$$

$$\stackrel{(12) \text{式}}{=} \frac{df}{dx} - \frac{df}{dx} - y' \frac{df}{dy} + y' \frac{d}{dx} \left(\frac{df}{dy} \right)$$

$$= \frac{df}{dx} - \frac{df}{dx} - y' \left[\frac{df}{dy} - \frac{d}{dx} \left(\frac{df}{dy} \right) \right] \rightarrow 0$$

$$\therefore \frac{df}{dx} - \frac{d}{dx} \left(f - y' \frac{df}{dy} \right) = 0 \quad \dots (13)$$

This is the 2nd form of Euler eq.

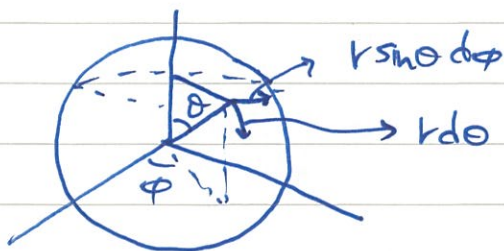
When $\frac{df}{dx} = 0$, $f - y' \frac{df}{dy} = \text{constant}$ ($f = L$ 時)

- 此常數 = -Energy

見最後討論

例 Geodesic on a sphere

Geodesic = shortest path between any two points.



$$ds = \sqrt{(r d\theta)^2 + (r \sin\theta d\phi)^2}$$

$$\therefore S_{1 \rightarrow 2} = r \int_1^2 \sqrt{\left(\frac{d\theta}{d\phi} \right)^2 + \sin^2\theta} d\phi$$

$f(\theta, \theta')$. 將 θ 視為 ϕ 之函數

$\therefore f(\theta) \neq \frac{df}{d\theta}$ $\therefore$ 可用 2nd form of Euler eq.

$$\therefore f - \theta' \frac{df}{d\theta} = \text{常数} = a$$

$$\sqrt{\theta'^2 \sin^2 \theta} - \theta' \frac{\theta'}{\sqrt{\theta'^2 \sin^2 \theta}} = a$$

$$\therefore \frac{\sin^2 \theta}{\sqrt{\theta'^2 \sin^2 \theta}} = a < 1$$

$$\theta'^2 = \frac{\sin^2 \theta (\sin^2 \theta - a^2)}{a^2}$$

$$\frac{d\phi}{d\theta} = \theta'^{-1} = \frac{a \csc \theta}{\sqrt{\sin^2 \theta - a^2}} = \frac{a \csc^2 \theta}{\sqrt{1 - a^2 \csc^2 \theta}}$$

$$\therefore \phi = \int \frac{a \csc^2 \theta d\theta}{\sqrt{1 - a^2 \csc^2 \theta}} \quad \csc^2 \theta = 1 + \cot^2 \theta$$

$$\therefore \sqrt{1 - a^2 \csc^2 \theta} = \sqrt{(1 - a^2) - a^2 \cot^2 \theta}$$

$$= a \sqrt{\underbrace{\frac{1 - a^2}{a^2}}_{\beta^2} - \cot^2 \theta}$$

$$\text{set } \cot \theta = \beta \sin x, \quad d\cot \theta = \beta \cos x dx$$

$\quad \quad \quad - \csc^2 \theta d\theta$

$$\therefore \sqrt{1 - a^2 \csc^2 \theta} = a \beta \cos x$$

$$a \csc^2 \theta d\theta = -a \beta \cos x dx \quad \therefore \phi = \int dx = \sin^{-1} \left(\frac{\cot \theta}{\beta} \right) + \alpha$$

Some
constant

故 $\cot \theta = -\beta \sin(\phi - \alpha)$

$$= -\beta \sin \phi \cos \alpha + \beta \cos \phi \sin \alpha \Rightarrow x/r$$

$$\therefore \underbrace{r \cos \theta}_z = \underbrace{-\beta \cos \alpha}_A \underbrace{r \sin \phi \sin \theta}_y + \underbrace{\beta \sin \alpha}_B \underbrace{r \sin \theta \cos \phi}_x$$

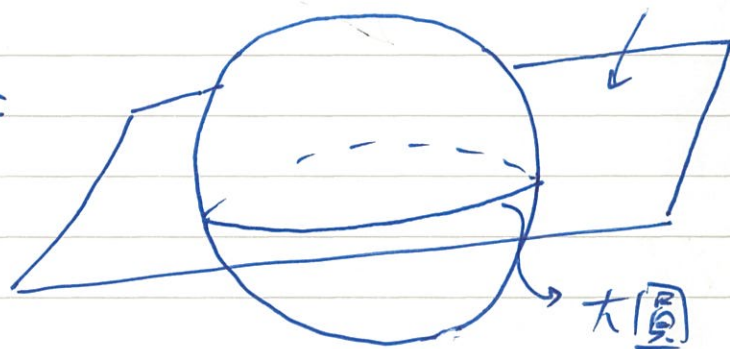
故 geodesic 上之點滿足

$$-Ay + Bx = z \leftarrow \text{過原點之平面}$$

$$\text{及 } x^2 + y^2 + z^2 = r^2$$

$$Ay - Bx = z$$

此即所謂的大圓：



(great circle)

變數之 Euler 方程式

例 1.

$$x = x(t)$$

$$y = y(t)$$

$$z = z(t)$$

$$f = f(x(t), y(t), z(t), x', y', z', t)$$

此時

$$\frac{df}{dx} - \frac{d}{dt} \frac{df}{dx'} = 0$$

$$\frac{df}{dz} - \frac{d}{dt} \frac{df}{dz'} = 0$$

$$\frac{df}{dy} - \frac{d}{dt} \frac{df}{dy'} = 0$$

即分別滿足！

例 2.

$$f = f(x, y, z, p(x, y, z), \partial_x p, \partial_y p, \partial_z p)$$

$$\text{即 } J = \int_V d^3\vec{r} f(\vec{r}, p(\vec{r}), \vec{\partial} p)$$

$$= \int_V \int \int dx dy dz f(\vec{r}, p(\vec{r}), \vec{\partial} p)$$



p 在 V 之表面 (S) 固定.

$$\text{即 } \delta p = 0 \text{ on } S$$

$$\delta J = \int d^3\vec{r} \left[\frac{\delta f}{\delta p} - \frac{\delta}{\delta x} \frac{\delta f}{\delta (\partial_x p)} - \frac{\delta}{\delta y} \frac{\delta f}{\delta (\partial_y p)} - \frac{\delta}{\delta z} \frac{\delta f}{\delta (\partial_z p)} \right] \cdot \delta p$$

$$\text{故 } \frac{\delta f}{\delta p} - \frac{\delta}{\delta x} \left(\frac{\delta f}{\delta (\partial_x p)} \right) - \frac{\delta}{\delta y} \left(\frac{\delta f}{\delta (\partial_y p)} \right) - \frac{\delta}{\delta z} \left(\frac{\delta f}{\delta (\partial_z p)} \right) = 0$$

有变分微分之 Euler 方程式

$$J = \int_{x_0}^{x_B} f(x, y(x), y'(x), y''(x), \dots) dx$$

$$y = y(x) + \alpha \eta(x), \quad \eta(x_0) = \eta'(x_0) = \eta''(x_0) = \dots = 0$$

$$\eta(x_B) = \eta'(x_B) = \eta''(x_B) = \dots = 0$$

$$\frac{dJ}{d\alpha} = \int_{x_0}^{x_B} \left(\frac{\delta f}{\delta y} \eta(x) + \frac{\delta f}{\delta y'} \eta'(x) + \frac{\delta f}{\delta y''} \eta'' + \dots \right) dx$$

--- (14)

部分積分:

$$\begin{aligned}\int_{x_0}^{x_B} \frac{\partial f}{\partial y'} \eta' dx &= \int_{x_0}^{x_B} \frac{\partial f}{\partial y'} d\eta \\ &= \underbrace{\frac{\partial f}{\partial y'} \eta(x)}_0 \bigg|_{x_0}^{x_B} - \int_{x_0}^{x_B} \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \eta(x) dx\end{aligned}$$

$$\begin{aligned}\int_{x_0}^{x_B} \frac{\partial f}{\partial y''} \eta'' dx &= \int \frac{\partial f}{\partial y''} d\eta' \\ &= \underbrace{\frac{\partial f}{\partial y''} \eta'}_0 \bigg|_{x_0}^{x_B} - \int \frac{d}{dx} \left(\frac{\partial f}{\partial y''} \right) \eta'(x) dx\end{aligned}$$

$$= \int_{x_0}^{x_B} \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) \eta(x) dx$$

∴ (14) 式

$$\frac{\delta J}{\delta \alpha} = 0 = \int_{x_0}^{x_B} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial f}{\partial y''} - \frac{d^3}{dx^3} \frac{\partial f}{\partial y'''} + \dots \right) \times \eta(x) dx$$

故 Euler 方程式成為

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) - \frac{d^3}{dx^3} \left(\frac{\partial f}{\partial y'''} \right) + \dots = 0$$

有限制條件之 Euler 方程式

以上求 functional 之極值時, $y(x)$ 都沒有限制。然而我們常遇到的問題是 $y(x)$ 有所限制。例如: $x^2 + y^2 + z^2 = r^2 = \text{常數}$

求 (x_1, y_1, z_1) 到 (x_2, y_2, z_2) 之最短距離及所對應之曲線!

如何得 equations of constraints 包括

進來呢? 方法之一是將 constant 解掉, 如取 $x = r \sin \theta \cos \phi$
 $y = r \sin \theta \sin \phi$
 $z = r \cos \theta$
 令 L 表示為 $L(\theta, \phi, \theta', \phi')$, 然而並不是所有 constant 都能解得掉, 故還要從未解出之常數出發。
 以 $g(y, z, x) = 0$ 為例, $y = y(x), z = z(x) \dots (15)$

$$f = f(y(x), y'(x), z(x), z'(x), x)$$

例如, $J(\alpha) = \int_{x_1}^{x_2} f \, dx$, $y(x, \alpha) = y(x, 0) + \alpha \eta_1(x)$
 $z(x, \alpha) = z(x, 0) + \alpha \eta_2(x)$

則

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left\{ \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right] \eta_1(x) + \left[\frac{\partial f}{\partial z} - \frac{d}{dx} \left(\frac{\partial f}{\partial z'} \right) \right] \eta_2(x) \right\} dx$$

如果沒有 constraint, (16) 式 $\Rightarrow \eta_1$ 及 η_2 獨立

L ... (16)

對而言, 而 (15) 式, $\{y(x, 0), z(x, 0)\}$ 及 $\{y(x, \alpha), z(x, \alpha)\}$ 皆

需滿足, 故 $g(y(x, \alpha), z(x, \alpha), x) - g(y(x, 0), z(x, 0), x) = 0$

$$\frac{\partial g}{\partial y} \eta_1 + \frac{\partial g}{\partial z} \eta_2 = 0 \quad \dots (17)$$

因此 $\eta_2 = -\frac{\frac{\partial g}{\partial y}}{\frac{\partial g}{\partial z}} \eta_1$ 代入 (16) 式

(即 η_1 与 η_2 不再彼此独立)

故

$$\frac{\partial J}{\partial x} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) - \left(\frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z'} \right) \frac{\frac{\partial g}{\partial y}}{\frac{\partial g}{\partial z}} \right] \eta_1(x) dx$$

$$\therefore \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} - \left(\frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z'} \right) \frac{\frac{\partial g}{\partial y}}{\frac{\partial g}{\partial z}} = 0$$

$$\text{即 } \frac{\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'}}{\frac{\partial g}{\partial y}} = \frac{\frac{\partial f}{\partial z} - \frac{d}{dx} \left(\frac{\partial f}{\partial z'} \right)}{\frac{\partial g}{\partial z}} = -\lambda(x)$$

$$\text{故 } \frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \lambda(x) \frac{\partial g}{\partial y} = 0 \quad \dots (18)$$

$$\frac{\partial f}{\partial z} - \frac{d}{dx} \left(\frac{\partial f}{\partial z'} \right) + \lambda(x) \frac{\partial g}{\partial z} = 0 \quad \dots (19)$$

$$g(y, z, x) = 0 \quad \dots (20)$$

(18) - (20) 式是以解出 $y(x)$, $z(x)$ 及 $\lambda(x)$, 为解有

限制条件之 Euler 方程式的修正。

其中 $\lambda(x)$ 称为 Lagrangian undetermined multiplier

當有很多 Constraint 方程式時，

$$q_j = 0 \quad j = 1, 2, 3, \dots, m$$

$$\sum_k \frac{\partial q_j}{\partial g_k} \eta_k = 0 \quad k = 1, 2, 3, \dots, n$$

(只有 $\eta_1, \dots, \eta_{n-m}$ 彼此獨立) $(m < n)$

取 $\lambda_j^{(t)}$ 使得

$$\int_1^2 \sum_{j=1}^m \lambda_j \frac{\partial q_j}{\partial g_k} \eta_k dt = 0$$

$$\text{故 } \int_1^2 dt \sum_k \left(\frac{dL}{dg_k} - \frac{d}{dt} \frac{dL}{dg_k} + \sum_j \lambda_j^{(t)} q_{jk} \right) \eta_k = 0$$

可求 λ_j

for $m+1, \lambda_{n-m+1}^{(t)} + \frac{d}{dt} \left(\frac{dL}{dg_{n-m+1}} \right) = 0$

若取 $\lambda_j^{(t)}$ 滿足 $\lambda_j^{(t)}$ 有解

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

$$\frac{dL}{dg_k} - \frac{d}{dt} \left(\frac{dL}{dg_k} \right) + \sum_j \lambda_j q_{jk} = 0$$

for $k = n-m+1, \dots, n$

$f = n - m = \text{自由度}$, 取 $\eta_1, \dots, \eta_f$ 彼此獨立. ①

$$\text{則 } \int_1^2 \sum_{k=1}^f \left(\frac{dL}{dg_k} - \frac{d}{dt} \frac{dL}{dg_k} + \sum_j \lambda_j^{(t)} q_{jk} \right) \eta_k dt = 0$$

$\therefore \eta_1, \eta_2, \dots, \eta_f$ 彼此獨立

$$\text{故 } \frac{dL}{dg_k} - \frac{d}{dt} \left(\frac{dL}{dg_k} \right) + \sum_j \lambda_j^{(t)} \frac{\partial q_j}{\partial g_k} = 0$$

for $k = 1, \dots, f$ ②

$$\text{①} + \text{②} \Rightarrow \frac{dL}{dg_k} - \frac{d}{dt} \left(\frac{dL}{dg_k} \right) + \sum_j \lambda_j^{(t)} \frac{\partial q_j}{\partial g_k} = 0$$

$\lambda(x)$ 的物理意義為對應到 Constraint force, 之後
在場上

會再討論。一般而言, 每個一個 constraint equation,

$g_j = 0$, 即為一個 $\lambda_j(x)$, 故 Euler 方程式:

$$\frac{df}{dy_i} - \frac{d}{dx} \left(\frac{df}{dy_i'} \right) + \sum_j \lambda_j(x) \frac{dg_j}{dy_i} = 0 \quad \text{--- (21)}$$

(More concisely, $\frac{df}{dy_i} + \sum_j \lambda_j \frac{dg_j}{dy_i} = 0$)

(21) 式必須配合 $g_j = 0$ $j=1, 2, \dots, m$ 才能得解。

$(y_1, \dots, y_n)$ 有 n 個未知, 加上 $\lambda_1, \dots, \lambda_m(x)$, 共有 $n+m$

未知, 以 (21) 式之 n 個方程 + $g_j = 0$ (m 個), 可

以得解。)

除了 holonomic constraint 外, 也有可能是 integral (functional)

Constraint:

$$\int_{x_1}^{x_2} g(y, y', x) dx = \text{常數}$$

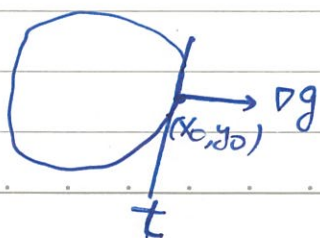
則 $J = \int_{x_1}^{x_2} f(y, y', x) dx$ 之極值必發生

方程式為何?

要了解此問題, 先考慮一比較簡單的問題:

求在 $g(x, y) = 0$ 之條件下, $f(x, y)$ 之極小值

設極值發生在 (x_0, y_0) , 則在 (x_0, y_0) 處 Dg 垂直之
平面 (如線上) 上 f 之
變化



$$\therefore \frac{df}{dt} \Big|_{(x_0, y_0)} = 0$$

因為 (x_0, y_0) 為滿足 $g=0$, f 之極值

換句話說, $\delta J = 0$, 即 $\delta f \parallel \delta g$

$$\delta f + \lambda \delta g = 0$$

λ 為一常數, 為 Lagrange's multiplier.

同理, 我們要求 $\delta J \parallel \delta \int_{x_1}^{x_2} g(y, y', x) dx$

$$\text{即 } \delta J + \lambda \delta \int_{x_1}^{x_2} g(y, y', x) dx = 0$$

$$\therefore \delta J = \int_{x_1}^{x_2} \left[\frac{\delta f}{\delta y} - \frac{d}{dx} \left(\frac{\delta f}{\delta y'} \right) \right] \eta(x) dx$$

$$\delta \int_{x_1}^{x_2} g(y, y', x) dx = \int_{x_1}^{x_2} \left[\frac{\delta g}{\delta y} - \frac{d}{dx} \left(\frac{\delta g}{\delta y'} \right) \right] \alpha \eta(x) dx$$

$$\therefore \int_{x_1}^{x_2} \left\{ \left[\frac{\delta f}{\delta y} - \frac{d}{dx} \left(\frac{\delta f}{\delta y'} \right) \right] + \lambda \left[\frac{\delta g}{\delta y} - \frac{d}{dx} \left(\frac{\delta g}{\delta y'} \right) \right] \right\} \eta(x) dx = 0$$

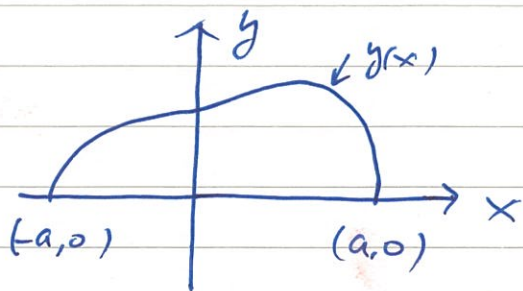
$$\text{故 } \frac{\delta f}{\delta y} - \frac{d}{dx} \left(\frac{\delta f}{\delta y'} \right) + \lambda \left[\frac{\delta g}{\delta y} - \frac{d}{dx} \left(\frac{\delta g}{\delta y'} \right) \right] = 0. \quad \dots (22)$$

注意: 此處 $\lambda = \text{常數}$ 與 x 無關。

例: 求總長 L 通過 $(-a, 0)$ 及 $(a, 0)$ 之

曲線與 x 軸所夾之最大面積之曲線

方程式



$$L = \int_{-a}^a \sqrt{1+y'^2} dx$$

$$J = \int_{-a}^a y dx = \text{面積}$$

$$f = y, \quad g = \sqrt{1+y'^2}$$

$$\frac{\partial f}{\partial y} = 1, \quad \frac{\partial f}{\partial y'} = 0, \quad \frac{\partial g}{\partial y} = 0, \quad \frac{\partial g}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$\therefore (22) \text{式} \Rightarrow 1 - \lambda \frac{d}{dx} \left(\frac{y'}{\sqrt{1+y'^2}} \right) = 0$$

$$\therefore \frac{d}{dx} \left(\frac{y'}{\sqrt{1+y'^2}} \right) = \frac{1}{\lambda}$$

$$\frac{y'}{\sqrt{1+y'^2}} = \frac{x}{\lambda} - \frac{C_1}{\lambda} \Rightarrow \frac{\lambda y'}{\sqrt{1+y'^2}} = x - C_1$$

$$y' = \pm \frac{x - C_1}{\sqrt{\lambda^2 - (x - C_1)^2}} \quad \therefore dy = \pm \frac{x - C_1}{\sqrt{\lambda^2 - (x - C_1)^2}} dx$$

$$y = \mp \sqrt{\lambda^2 - (x - C_1)^2} + C_2$$

$\therefore (x - C_1)^2 + (y - C_2)^2 = \lambda^2$, $y(x)$ 為  的一部分
當 $l = \pi a$ 時

$\therefore \pm(a, 0)$ 及 $(-a, 0)$ $\therefore C_1 = C_2 = 0$, $\pi\lambda = l$, $\lambda = l/\pi = a$
曲線為一半圓

若 $l \neq \pi a$, 則只有 $C_1 = 0$, $r = \sqrt{C_2^2 + a^2} = \lambda$

由 $2r \tan^{-1} \frac{a}{C_2} = l$ 決定 C_2

Generalized Coordinates and generalized forces

除了是 scalar 之優點外, 使用 Lagrangian 解力學問題另一個優點是 L 只牽涉到 一次微分, 而牛頓 $F=ma$ 則涉及了 二次微分。

因此, Hamilton principle 對解非 Cartesian 座標之問題較容易。

一般而言, 如果 $(q_1, q_2, \dots, q_s)$ 為 generalized coordinates (非 Cartesian), 則 $x_i = x_i(q_1, q_2, \dots, q_s, t)$

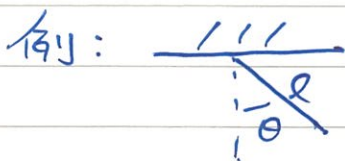
$$\text{則 } L = \frac{1}{2} m \sum_i \dot{x}_i^2 - U(x_1, \dots, x_n)$$

$$\rightarrow L = T(q_i, \dot{q}_i, t) - U(q_1, \dots, q_s, t)$$

Hamilton principle 成為

$$\delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt = 0$$

$$\therefore \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \quad \dots (23)$$



$$T = \frac{1}{2} m \left(\frac{ds}{dt} \right)^2 = \frac{1}{2} m (l \dot{\theta})^2$$

$$U = mgl(1 - \cos \theta)$$

$$\therefore L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - \cos \theta), \quad q = \theta, \quad \frac{\partial L}{\partial \theta} = -mgl \sin \theta$$

$$\frac{d}{dt} \frac{dL}{d\dot{\theta}} = m\ell^2 \ddot{\theta}$$

$$\therefore -m\ell^2 \ddot{\theta} - mg\ell \sin\theta = 0$$

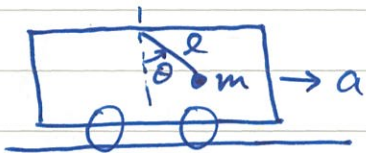
$$\ddot{\theta} + \frac{g}{\ell} \sin\theta = 0 \quad \text{与使牛顿方程式所得}$$

運動

之方程式一致，但不須牽涉 $\vec{v}$ 在極座標之

$$\text{拆解：} \quad \vec{v} = (\dot{r} - r\dot{\theta}^2) \hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \hat{e}_\theta$$

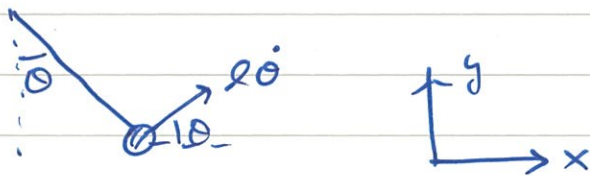
例：



求：作等加速運動車中單擺
之振盪頻率。

設車之初速為 v_0

則 m 之速度為 $(v_0 + at + \ell\dot{\theta}\cos\theta, \ell\dot{\theta}\sin\theta)$



$$\therefore T = \frac{1}{2} m (v_0 + at + \ell\dot{\theta}\cos\theta)^2 + \frac{1}{2} m \ell^2 \dot{\theta}^2 \sin^2\theta$$

$$\therefore U = -mg\ell \cos\theta$$

$$\therefore L = \frac{1}{2} m (v_0 + at + \ell\dot{\theta}\cos\theta)^2 + \frac{1}{2} m \ell^2 \dot{\theta}^2 \sin^2\theta + mg\ell \cos\theta$$

$$\begin{aligned} \frac{dL}{d\dot{\theta}} &= -mg\ell \sin\theta + m\ell^2 \dot{\theta}^2 \sin\theta \cos\theta + m(v_0 + at + \ell\dot{\theta}\cos\theta) \\ &= -mg\ell \sin\theta - m\ell \dot{\theta} \sin\theta (v_0 + at) \quad \times (-\ell\dot{\theta}\sin\theta) \end{aligned}$$

$$\frac{dL}{dt} = m l^2 \dot{\theta} \sin^2 \theta + m (v_0 + at + l \dot{\theta} \cos \theta) \times l \cos \theta$$

$$= m l^2 \dot{\theta} + m (v_0 + at) l \cos \theta$$

$$\therefore \frac{d}{dt} \left(\frac{dL}{d\dot{\theta}} \right) = m l^2 \ddot{\theta} + m (v_0 + at) l \sin \theta \dot{\theta}$$

$$+ m a l \cos \theta \quad \dots (25)$$

$$\text{将 (24) 及 (25) 代入 } \frac{dL}{d\theta} - \frac{d}{dt} \left(\frac{dL}{d\dot{\theta}} \right) = 0$$

$$\text{可得 } -m g l \sin \theta - m l^2 \ddot{\theta} - m a l \cos \theta = 0$$

$$\therefore \ddot{\theta} = -\frac{g}{l} \sin \theta - \frac{a}{l} \cos \theta \quad \dots (26)$$

$$\text{平衡时 } \ddot{\theta} = 0 \quad \therefore g \sin \theta_0 + a \cos \theta_0 = 0$$

$$\tan \theta_0 = -a/g, \quad \sin \theta_0 = -\frac{a}{\sqrt{a^2 + g^2}}$$

$$\cos \theta_0 = \frac{g}{\sqrt{a^2 + g^2}}$$

$$\text{令 } \theta = \theta_0 + \eta, \quad (26) \text{ 变为}$$

$$\Rightarrow \ddot{\eta} = -\frac{g}{l} \sin(\theta_0 + \eta) - \frac{a}{l} \cos(\theta_0 + \eta)$$

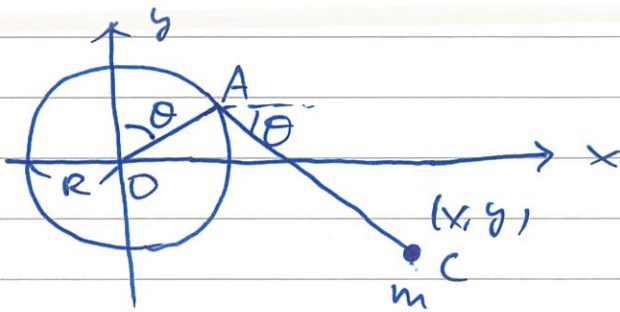
$$= -\frac{g}{l} (\underbrace{\sin \theta_0}_{1} \underbrace{\cos \eta}_{\downarrow \eta} + \underbrace{\cos \theta_0}_{\downarrow \eta} \underbrace{\sin \eta}_{\uparrow \eta}) - \frac{a}{l} (\underbrace{\cos \theta_0}_{\downarrow \eta} \underbrace{\cos \eta}_{\uparrow \eta} - \underbrace{\sin \theta_0}_{\downarrow \eta} \underbrace{\sin \eta}_{\uparrow \eta})$$

$$= -\frac{g}{l} \sin \theta_0 - \frac{a}{l} \cos \theta_0$$

$$- \left(\frac{g}{l} \cos \theta_0 - \frac{a}{l} \sin \theta_0 \right) \eta = -\frac{\sqrt{g^2 + a^2}}{l} \eta$$

$$\therefore \omega = \frac{\sqrt{g^2 + a^2}}{l}$$

例 (problem 7-12)



求小振幅之頻率。

$$\overline{AC} = l - R\theta \quad \therefore x = \overline{AC} \cos \theta + \overline{OA} \sin \theta$$

$$= (l - R\theta) \cos \theta + R \sin \theta$$

$$y = R \cos \theta - \overline{AC} \sin \theta$$

$$= R \cos \theta - (l - R\theta) \sin \theta$$

$$\therefore \ddot{x} = (-R\ddot{\theta}) \cos \theta + (l - R\theta) (-\sin \theta \dot{\theta}) + R \cos \theta \dot{\theta}^2$$

$$= R\ddot{\theta} \sin \theta - l \dot{\theta} \sin \theta$$

$$\ddot{y} = -R \sin \theta \ddot{\theta} - (-R\ddot{\theta}) \sin \theta - (l - R\theta) \dot{\theta} \cos \theta$$

$$= R\ddot{\theta} \cos \theta - l \dot{\theta} \cos \theta$$

$$\therefore T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m (l^2 \dot{\theta}^2 + R^2 \dot{\theta}^2 - 2Rl \dot{\theta}^2 \theta)$$

$$U = mgy = mg(R \cos \theta - (l - R\theta) \sin \theta)$$

$$\therefore L = T - U = \frac{1}{2} m (l^2 \dot{\theta}^2 + R^2 \dot{\theta}^2 - 2Rl \dot{\theta}^2 \theta) - mg[R \cos \theta - (l - R\theta) \sin \theta]$$

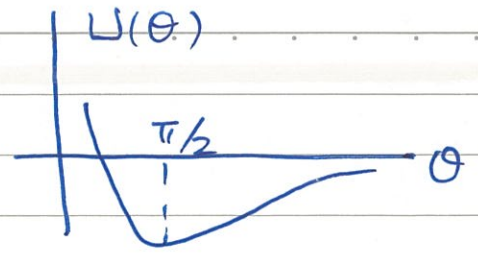
平衡時 $U'(\theta_0) = 0 \quad \therefore -R \sin \theta_0 - (l - R\theta_0) \cos \theta_0 + R \sin \theta_0 = 0$

$$\therefore (l - R\theta_0) \cos \theta_0 = 0 \quad \therefore \theta_0 = \pi/2$$

To $O(\eta^2)$, $\theta = \theta_0 + \eta$

$$U = U(\pi/2) + \frac{1}{2} \eta^2 U''(\pi/2)$$

$$= mg(l - \frac{\pi}{2}R) + \frac{1}{2} \eta^2 \times mg(l - \frac{\pi}{2}R)$$



$$T = \frac{1}{2} m l^2 \dot{\eta}^2 + \frac{1}{2} m R^2 (\frac{\pi}{2})^2 \dot{\eta}^2 - 2 R l \dot{\eta}^2 \times \frac{\pi}{2} \times m$$

$$= \frac{1}{2} m (l - \frac{\pi}{2}R)^2 \dot{\eta}^2$$

$$\frac{\partial L}{\partial \eta} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\eta}} \right) = 0$$

$$\Rightarrow m (l - \frac{\pi}{2}R)^2 \ddot{\eta} + mg (l - \frac{\pi}{2}R) \eta = 0$$

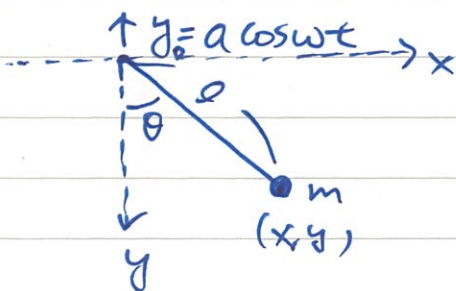
$$\therefore \omega = \sqrt{\frac{g}{l - \frac{\pi}{2}R}}$$

例: pivot 作簡諧運動之單擺 (fig 4-17,

為 Chaotic 運動之簡諧運動)

可能 有

相當於盪鞦韆 (swing) 時
上下移動質心之效果



$$x = l \sin \theta$$

$$y = y_0 - l \cos \theta$$

$$= a \cos \omega t - l \cos \theta$$

$$\dot{x} = l \dot{\theta} \cos \theta$$

$$\dot{y} = -a \omega \sin \omega t + l \dot{\theta} \sin \theta$$

$$\therefore T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2} m [l^2 \dot{\theta}^2 - 2la\omega \dot{\theta} \sin \theta \sin \omega t + a^2 \omega^2 \sin^2 \omega t]$$

$$U = mgy = mg(a \cos \omega t - l \cos \theta)$$

$$L = \frac{1}{2} m [a^2 \omega^2 \sin^2 \omega t - 2la\omega \sin \omega t \dot{\theta} \sin \theta + l^2 \dot{\theta}^2]$$

$$- mg(a \cos \omega t - l \cos \theta)$$

$$\frac{\partial L}{\partial \theta} = -m la \omega \sin \omega t \dot{\theta} \cos \theta$$

$$- mgl \sin \theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = -ml\omega^a \sin \omega t \sin \theta + ml^2 \dot{\theta}$$

$$\therefore \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = -ml\omega^a \cos \omega t \sin \theta - ml\omega^a \sin \omega t \cos \theta \dot{\theta} + ml^2 \ddot{\theta}$$

$$= \frac{\partial L}{\partial \theta} = -mla\omega \sin \omega t \dot{\theta} \cos \theta - mgl \sin \theta$$

$$\therefore \ddot{\theta} + \left(\frac{g}{l} - \frac{a}{l} \omega^2 \cos \omega t \right) \sin \theta = 0$$

$\therefore$ the oscillation of pivot changes $\frac{g}{l}$

to $\frac{g}{l} - \frac{a}{l} \omega^2 \cos \omega t$!

Generalized momentum; 由 generalize coordinate,

可對應 - generalize momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$$

例如在單擺的例子, $q = \theta$, $\therefore p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta}$
 $= I \omega = \text{角動量}$

Generalized force

將 $L = T(\dot{q}_i, q_i) - U(q_i)$

代入 (23) 式, 並且注意在使用 generalized coordinates 時, T 可與 q_i 有關! 但 U 與 q_i 無關

故 (23) 式

$$\Rightarrow \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = - \frac{\partial U}{\partial q_i} \dots (27)$$

當 $q_i = x_i$ (Cartesian coordinate)

(27) 式: $m \ddot{x}_i = - \frac{\partial U}{\partial x_i} = F_i$ recovers Newton's eq.

$\therefore$ 右邊是力的推廣, 我們因此定義 $Q_i = - \frac{\partial U}{\partial q_i}$ 為 generalized force. $\because U = U(x_1, x_2, \dots, x_N)$, $x_i = x_i(q_1, q_2, \dots, q_s)$

$$\therefore \frac{\partial U}{\partial q_i} = \sum_j \frac{\partial U}{\partial x_j} \frac{\partial x_j}{\partial q_i}$$

$$Q_i = - \sum_j \frac{\partial U}{\partial x_j} \frac{\partial x_j}{\partial q_i} = \sum_j F_j \frac{\partial x_j}{\partial q_i}$$

Force of constraint & Lagrange's multiplier

如前所述，當有 constraint 時，不論是 holonomic
(此時有 constraint 方程式， $g_j(q_1, \dots, q_n, t) = 0$) 或
是 nonholonomic ^{都不一定} 能將 constraint 解開，得

$q_1, q_2, \dots, q_n$ 表示成 $\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_p$ 之獨立變數。

因此，我們必須有由 $q_1, \dots, q_n$ 出發的手法，此
手法即為 method of Lagrange's multiplier.

此手法能處理最廣的 constraint 方程式為

$$\sum_k a_{jk} dq_k + a_j dt = 0 \quad j=1, 2, \dots, m$$

$$\text{即} \quad \sum_k a_{jk} \dot{q}_k + a_j = 0 \quad \dots \quad (28)$$

$$\text{則虛位移滿足} \quad \sum_k a_{jk} \delta q_k = 0 \quad \dots \quad (29)$$

(28) 式相當於 $\sum_k \frac{da_j}{dq_k} \delta q_k = 0$ (一般來說 a_{jk} 與 q_i 有
關，(29) 式不一定可以積分
為 $g_j(q_1, \dots, q_n) = 0$)

故由 (21) 式

$$\text{可知} \quad \frac{dL}{dq_k} - \frac{d}{dt} \left(\frac{dL}{d\dot{q}_k} \right) + \sum_j \lambda_j(t) a_{j-k} = 0 \quad \dots \quad (30)$$

$$\text{即} \quad \frac{d}{dt} \left(\frac{dL}{d\dot{q}_k} \right) - \frac{dL}{dq_k} = \sum_j \lambda_j(t) a_{j-k}$$

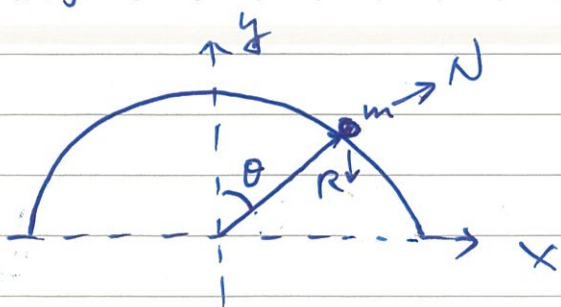
$$\frac{d}{dt} \left(\frac{d\tau}{d\dot{q}_k} \right) - \frac{d\tau}{dq_k} = \underbrace{-\frac{dU}{dq_k}}_{F_k} + \sum_j \lambda_j a_{j-k}$$

與 (21) 相比可知，constraint (λ_j) 引入新力 $\checkmark = \sum_j \lambda_j a_{j-k} \quad \frac{\partial Q_k}{\partial q_k}$

因此，很明顯的， λ_j 與 constraint force 的組合習性相同！
線性

例

在光滑半球上之運動



$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) \quad U = mgr \cos \theta$$

$$\therefore L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - mgr \cos \theta$$

Equation of constraint : $r = R$, $a_r = 1$, $a_\theta = 0$

Lagrange's equations :

$$\frac{\partial L}{\partial r} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) + \lambda a_r = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = 0$$

$$\Rightarrow m R \dot{\theta}^2 - mg \cos \theta + \lambda = 0$$

$$m R^2 \ddot{\theta} - mg R \sin \theta = 0$$

$$\therefore \ddot{\theta} = \frac{d\dot{\theta}}{d\theta} \dot{\theta} \quad \therefore \text{第 2 式} \Rightarrow \dot{\theta} \frac{d\dot{\theta}}{d\theta} = \frac{g}{R} \sin \theta, \quad d\dot{\theta}^2 = \frac{2g}{R} \sin \theta d\theta$$

$$\text{積分可得} \quad \dot{\theta}^2 = -\frac{2g}{R} \cos \theta + \frac{2g}{R} \quad (\text{取 } \theta=0 \text{ 時 } \dot{\theta}=0)$$

$$\text{代入第一式} \Rightarrow \lambda = mg \cos \theta - m R \left(-\frac{2g}{R} \cos \theta + \frac{2g}{R} \right) \\ = mg (3 \cos \theta - 2)$$

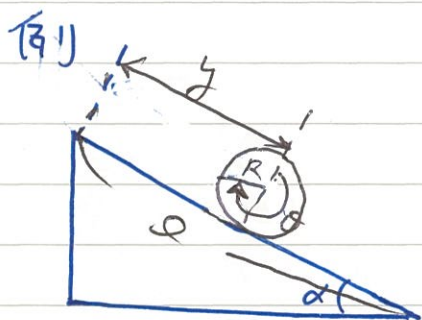
由力直接分析: $mg \cos \theta - N = m \frac{v^2}{R}$

$$\frac{1}{2}mv^2 = mg(R - R \cos \theta)$$

$$\therefore mg \cos \theta - N = 2mg(1 - \cos \theta)$$

$$\therefore N = 3mg \cos \theta - 2mg = \lambda$$

由此可得証 λ 即为 Constraint 所造成之力



$$T = \frac{1}{2} M \dot{y}^2 + \frac{1}{2} I \dot{\theta}^2$$

$$= \frac{1}{2} M \dot{y}^2 + \frac{1}{4} M R^2 \dot{\theta}^2$$

$$(I = MR^2/2)$$

$$U = Mg(l - y) \sin \alpha$$

constraint: $y = R\theta \quad f = y - R\theta = 0$

$$\frac{\partial L}{\partial y} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) + \lambda \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow Mg \sin \alpha - M \ddot{y} + \lambda = 0 \quad \dots (31)$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) + \lambda \frac{\partial f}{\partial \theta} = 0$$

$$\therefore -\frac{1}{2} MR^2 \ddot{\theta} - \lambda R = 0 \quad \dots (32)$$

由 $y=R\theta \therefore \ddot{\theta} = \frac{\ddot{y}}{R}$ 代入 (32)

式 (31) 可得 $\lambda = -\frac{1}{2}M\ddot{y}$ 代入 (31) 式

$\therefore \ddot{y} = \frac{2g}{3} \sin \alpha$ 故 $\lambda = -\frac{Mg \sin \alpha}{3}$

$\ddot{\theta} = \frac{2g \sin \alpha}{3R}$ $\downarrow a_{iy}$

Constraint force: $Q_y = \lambda \frac{\partial F}{\partial y} = \lambda = -\frac{Mg \sin \alpha}{3}$

$Q_\theta = \lambda \frac{\partial F}{\partial \theta} = -\lambda R = \frac{MgR \sin \alpha}{3}$
 $\uparrow a_{i\theta}$

Check:



$Mg \sin \alpha - f_s = M\ddot{y} = \frac{2M}{3}g \sin \alpha$

$\therefore f_s = \frac{1}{3}Mg \sin \alpha$
 $= |Q_y|$

而 Q_θ 只是 f_s 所產生之力矩

$(f_s R = I\alpha = \frac{I}{R}a, \therefore f_s = \frac{1}{2}Ma, a = \ddot{y})$

以上問題當然可以一開始即將 $y=R\theta$ 代入 L

則 $L = \frac{3}{4}M\dot{y}^2 + Mg(y-l)\sin \alpha$

而得到 $\frac{3}{2}M\ddot{y} = Mg \sin \alpha$

方法雖然比較簡單，但無法得到 force of constraint.

Conservation theorems 与 Noether 定理

(i) 能量守恒

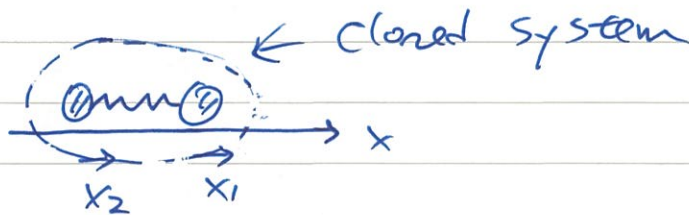
一個封閉系統 (closed system), 因為与外界隔絕, $\therefore L(q, \dot{q}, t) = L(q, \dot{q})$ 沒

有所謂的 explicit time dependence (time is homogenous)

$$\text{即 } \frac{dL}{dt} = 0$$

沒有那一時刻的特別意義不同)

如:



$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$U = \frac{1}{2} K (x_1 - x_2)^2$$

$\therefore L = T - U$ 与 t 之相關只是

透過 $x_1(t)$ 及 $x_2(t)$ 与 t 之相關!

反之, 若有^{外界}雷射照射彈簧致使 $K = K_0 + \alpha t$

則 L 除了由 x_1 及 x_2 得到与 t 之相關, 還通过

$K(t)$ 也有 t 相關, 此時 $\frac{dL}{dt} \neq 0$

由 Euler 方程式之 2nd form 可知, 此時

$$L - \sum_j \dot{q}_j \frac{dL}{d\dot{q}_j} = \text{常数}$$

常 $q_j = \text{Cartesian 座標時} = x_j$

$$L = \frac{1}{2} \sum_j m_j \dot{x}_j^2 - U(x_1, x_2, \dots, x_N)$$

$$\frac{dL}{d\dot{x}_j} = m_j \dot{x}_j$$

$$\therefore \sum_j \dot{x}_j \frac{dL}{d\dot{x}_j} = \sum_j m_j \dot{x}_j^2 = 2T$$

$$\therefore L - \sum_j q_j \frac{dL}{dq_j} = T - U - 2T = -(T+U) \equiv -H$$

H 稱為 Hamiltonian

在此上之情形為 Energy!
即

$$\therefore \frac{dL}{dt} = 0 \Rightarrow \text{Energy 守恆}$$

$$\therefore \frac{dL}{dt} = 0 \text{ 令 } L(q_j, \dot{q}_j, t+\delta t) = L(q_j, \dot{q}_j, t) \text{ 一樣}$$

時間變換之平移

$\therefore$ 又被述說為「若系統時間平移不變或具有

時間平移對稱

則能量守恆」

以上是在 Cartesian 座標且 U 與 $\dot{x}_j$ 無關， H 才為 Energy。
使用

一般而言, 由 Cartesian 座標, 我們可以作座標轉換到一般座標:

$$x_{\alpha,i} = x_{\alpha,i}(q_1, q_2, \dots, q_s, t)$$

$$\text{則 } \frac{dx_{\alpha,i}}{dt} = \dot{x}_{\alpha,i} = \sum_{j=1}^s \frac{dx_{\alpha,i}}{dq_j} \dot{q}_j + \frac{dx_{\alpha,i}}{dt}$$

$$\text{因此, } \dot{x}_{\alpha,i}^2 = \sum_{j,k} \frac{dx_{\alpha,i}}{dq_j} \frac{dx_{\alpha,i}}{dq_k} \dot{q}_j \dot{q}_k + 2 \sum_j \frac{dx_{\alpha,i}}{dq_j} \frac{dx_{\alpha,i}}{dt} \dot{q}_j + \left(\frac{dx_{\alpha,i}}{dt}\right)^2$$

$\therefore T$ 有下式

$$T = \sum_{j,k} a_{j,k} \dot{q}_j \dot{q}_k + \sum_j b_j \dot{q}_j + c$$

$$c = c(q_1, \dots, q_s, t)$$

$$\text{且 } U = U(q_1, q_2, \dots, q_s, t)$$

$$\text{故 故 Energy} = T + U$$

$$\text{但 } \therefore L = T - U$$

$$\frac{dL}{d\dot{q}_j} = \sum_k a_{j,k} \dot{q}_k + b_j$$

$$\therefore \sum_j \dot{q}_j \frac{dL}{d\dot{q}_j} = 2 \sum_{j,k} a_{j,k} \dot{q}_j \dot{q}_k + \sum_j b_j \dot{q}_j = 2T \dots (33)$$

$$\text{故 } L - \sum_j \dot{q}_j \frac{dL}{d\dot{q}_j} = -H = T - U - 2T = -(T + U)$$

故 $H = \text{Energy}$!

只有當 $b_j = 0$, $C = 0$, 即 $\frac{dx_i}{dt} = 0$ (轉換到 t 沒有直接相關, 稱為 scleronomic)

此時 $T = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k$, $U = U(q_1, q_2, \dots, q_s)$

(28)式 $\Rightarrow \sum_j \dot{q}_j \frac{dL}{d\dot{q}_j} = 2T$ (34)

$L - \sum_j \dot{q}_j \frac{dL}{d\dot{q}_j} = (T - U) - 2T = -(T + U) = -E = -H$

此時 $\hat{H} = \text{Energy}$. (34)式即

$\sum_j \dot{q}_j \frac{dT}{d\dot{q}_j} = 2T$, 為 Euler's theorem 之特例:

$f(y_k)$ 是 y_k 之 homogeneous 函數 (degree n)

則 $\sum_k y_k \frac{df}{dy_k} = n f$

(T : $n=2$)

$\therefore$ 對於一般座標, Hamiltonian 為 Energy, 只有

當 T 為 homogeneous function of degree 2 $\dot{q}_j$ 之

才成立!

(ii) 動量守恒

對稱 - 座標

若系統是平移對稱的 (homogeneous in space),

即 $\vec{r} \rightarrow \vec{r} + \delta \vec{r}$ (原點平移 $\delta \vec{r}$)L 不變, 則 $L(r_1, \dots, \vec{r}_1 + \delta \vec{r}_1, \dots, \vec{r}_i)$ $= L(r_1, \dots, \vec{r}_1, \dots, \vec{r}_i)$

$$\therefore \frac{dL}{dx_i} = 0$$

$$\frac{d}{dt} \left(\frac{dL}{dx_i} \right) = 0$$

$$\frac{dL}{dx_i} = \text{常數}$$

$$\frac{dL}{dx_i} = \frac{dT}{dx_i} = m \dot{x}_i = p_i = \text{常數}$$

 $\therefore$ 對 x_i 之平移不變, 則 p_i 是守恒。若整體對原點平移不變, 即 $x_i \rightarrow x_i + \delta x$ (即質心, 平移 δx)

$$L(x_1 + \delta x, x_2 + \delta x, \dots, x_N + \delta x, \dot{x}_1, \dot{x}_2, \dots, \dot{x}_N)$$

$$= L(x_1, x_2, \dots, x_1, x_2, x_3, \dots, x_N)$$

$$\text{則 } \sum_i \frac{dL}{dx_i} = 0 \quad \therefore \sum_i \frac{dL}{dx_i} = \sum_i \underbrace{\frac{d}{dt} \left(\frac{dL}{dx_i} \right)}_{m \dot{x}_i}$$

$$\therefore \underbrace{\sum_i m \dot{x}_i}_{\text{系統之總動量}} = \text{常數}$$

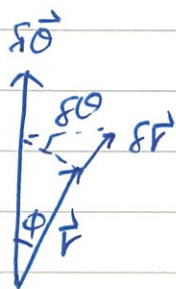
系統之總動量 守恒

(iii) 角動量守恆

如果所考慮的系統為均勻性 (isotropic) 的

則此系統之性質與其轉向 (orientation) 無關。

特別是對某一軸作無限小旋轉, L 不變:



$$\delta \vec{r} = \delta \vec{\theta} \times \vec{r}$$

$$|\delta \vec{r}| = r \sin \phi \delta \theta$$

取 $\delta \vec{\theta}$ 為轉軸 -- (35)
之方向

同樣的旋轉亦造成 $\dot{\vec{r}}$ 之變化



$$\delta \dot{\vec{r}} = \delta \vec{\theta} \times \dot{\vec{r}}$$

(36)

$$\therefore L(\vec{r} + \delta \vec{r}, \dot{\vec{r}} + \delta \dot{\vec{r}}) = L(\vec{r}, \dot{\vec{r}}), \quad \delta L = 0$$

$$\text{可得 } \delta L = \sum_i \frac{\partial L}{\partial x_i} \delta x_i + \sum_i \frac{\partial L}{\partial \dot{x}_i} \delta \dot{x}_i = 0 \quad \text{--- (37)}$$

$$\because \frac{\partial L}{\partial \dot{x}_i} = p_i, \quad \frac{\partial L}{\partial x_i} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) = \dot{p}_i$$

$$\therefore \text{(37) 式} \Rightarrow \sum_i \dot{p}_i \delta x_i + p_i \delta \dot{x}_i = 0$$

$$\text{即 } \dot{\vec{p}} \cdot \delta \vec{r} + \vec{p} \cdot \delta \dot{\vec{r}} = 0 \quad \text{--- (38)}$$

利用 (35) 及 (36) 式

$$(38) \text{ 式} \Rightarrow \vec{p} \cdot \delta \vec{O} \times \vec{r} + \vec{p} \cdot \delta \vec{O} \times \dot{\vec{r}} = 0$$

$$\therefore \vec{r} \times \dot{\vec{p}} \cdot \delta \vec{O} + \dot{\vec{r}} \times \vec{p} \cdot \delta \vec{O} = 0$$

上式對任意 $\delta \vec{O}$ 皆成立

$$\text{故 } \vec{r} \times \dot{\vec{p}} + \dot{\vec{r}} \times \vec{p} = 0$$

$$\therefore \frac{d}{dt} (\vec{r} \times \vec{p}) = 0$$

$$\therefore \vec{r} \times \vec{p} = \text{角動量} = \text{常數}$$

$\therefore$ 我們得知：若系統旋轉不變則角動量守恆。

以上的結果是所謂 Noether 定理的展現：

當系統有某-對稱性 (symmetry)

則有對應之物理量不變 (invariance) 或守恆。

如：平移對稱 $\Rightarrow$ 動量守恆

旋轉 " " $\Rightarrow$ 角動量守恆

時間平移對稱 $\Rightarrow$ 能量守恆

} 共 7 個守恆量！

且以上守恆與粒子之間有無交互作用無關！

Hamiltonian dynamics 及其應用

由 generalized momentum 之定義

$p_j = \frac{\partial L}{\partial \dot{q}_j}$ 可將 Euler-Lagrangian 方程式

改寫為
$$\dot{q}_j = \frac{\partial L}{\partial p_j} \quad \dots (39)$$

則
$$H = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = \sum_j p_j \dot{q}_j - L \quad \dots (40)$$

在 (39) 式中, 我們視 L 為 $q_j, \dot{q}_j$ 及 t 之函數

$$L = L(q_j, \dot{q}_j, t) \quad \dots (41)$$

我們可將 p_k, q_k 視為已知而解 $\dot{q}_j$ 得到為 q_k 及 p_k, t 之函數

$$\dot{q}_j = \dot{q}_j(q_k, p_k, t)$$

再代入 (40) 式, 可得 H 視為 q_k, p_k 及 t 之函數

$$H = H(q_j, p_j, t) \quad \dots (42) \quad q, \dot{q} \rightarrow q, p$$

注意: 此時 $q_j = q_j(t), p_j = p_j(t)$ (Legendre transformation)

這樣作之優勢是運動方程式成為一階微分方程式:

$$dH = \sum_k \left(\frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial p_k} dp_k \right) + \frac{\partial H}{\partial t} dt \quad \dots (43)$$

(將 H 視為 p_k, q_k, t 之函數)

$$dH = H(q_k(t+dt), p_k(t+dt), t+dt) - H(q_k(t), p_k(t), t)$$

但由 (40) 式可得

$$dH = \sum_k \left(dp_k \dot{q}_k + p_k \cancel{d\dot{q}_k} - \frac{\partial L}{\partial q_k} dq_k - \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k \right) - \frac{\partial L}{\partial t} dt$$

上式得 q_k 視為 q, p, t 之函數

$$\text{而 } dq_k = q_k(t+dt) - q_k(t)$$

$$\therefore p_k = \frac{dL}{dq_k} \therefore \text{2nd 及 4th 項消去}$$

$$\therefore dH = \sum_k \left(\dot{q}_k \underbrace{dp_k}_{\frac{dL}{dq_k}} - \dot{p}_k dq_k \right) - \frac{dL}{dt} dt \quad \text{--- (45)}$$

比較 (43) 及 (45) 式，因為 q_k 及 p_k 被視為 H 中

為獨立變數，故 dq_k, dp_k 及 dt 之係數相等。因此可得到

$$\dot{p}_k = - \frac{dH}{dq_k} (q_k, p_k, t) \quad \text{--- (46)}$$

$$\dot{q}_k = \frac{dH}{dp_k} (q_k, p_k, t) \quad \text{--- (47)}$$

$$\frac{dH}{dt} = - \frac{dL}{dt} \quad \text{--- (48)}$$

to be Hamilton's equation of motion (或又稱 canonical equation of motion)

前者 Euler-Lagrangian 方程式不同處為

一階微分方程式 (將 q_j 換為 q_k 及 p_k) 而後者為二階微分方程式。

將 (46) 及 (47) 代入 (43) 式可得

$$\frac{dH}{dt} = \sum_k \left(\frac{dH}{dq_k} \frac{dq_k}{dt} - \frac{dH}{dp_k} \frac{dp_k}{dt} \right) + \frac{dH}{dt} \therefore \text{對 } H \text{ 而言, 全微分} = \text{偏微分!}$$

若 $\frac{dH}{dt} = 0$, $\frac{dH}{dt} = 0 \Rightarrow H$ 守恆!

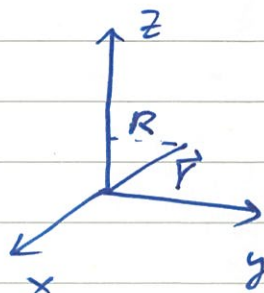
例：一質點被限制在 $x^2 + y^2 = R^2$ 之圓柱面上

運動，並受 $\vec{F} = -k\vec{r}$ 之力，求 Hamilton's method 中

之運動方程式

$$U = \frac{1}{2} k r^2 = \frac{1}{2} k (x^2 + y^2 + z^2)$$

$$= \frac{1}{2} k (R^2 + z^2)$$



$$v^2 = \dot{r}^2 + (R\dot{\theta})^2 + \dot{z}^2 = (R\dot{\theta})^2 + \dot{z}^2$$

$$\therefore T = \frac{1}{2} m (R^2 \dot{\theta}^2 + \dot{z}^2)$$

$$\therefore L = \frac{1}{2} m (R^2 \dot{\theta}^2 + \dot{z}^2) - \frac{1}{2} k (R^2 + z^2)$$

$$\left. \begin{aligned} p_{\theta} &= \frac{dL}{d\dot{\theta}} = mR^2 \dot{\theta} \\ p_z &= \frac{dL}{d\dot{z}} = m\dot{z} \end{aligned} \right\} \text{generalized momenta}$$

$$H = T + U, \therefore T = \frac{1}{2} m R^2 \left[\frac{p_{\theta}}{mR^2} \right]^2 + \frac{1}{2} m \left(\frac{p_z}{m} \right)^2$$

$$\therefore H = H(z, p_{\theta}, p_z) = \frac{p_{\theta}^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{1}{2} k z^2$$

Canonical eq. of motion

$$\dot{p}_{\theta} = -\frac{dH}{d\theta} = 0$$

$$\dot{\theta} = \frac{dH}{dp_{\theta}} = \frac{p_{\theta}}{mR^2}$$

$$\dot{p}_z = -\frac{dH}{dz} = -kz$$

$$\dot{z} = \frac{dH}{dp_z} = \frac{p_z}{m}$$

$$\because \dot{P}_\theta = 0 \quad \therefore P_\theta = \text{常數} = mR^2 \dot{\theta}$$

$$\therefore \dot{\theta} = \frac{P_\theta}{mR^2}, \quad \theta(t) = \theta_0 + \frac{P_\theta}{mR^2} t$$

$$P_z = m\dot{z} = -kz \quad \therefore \ddot{z} = -\omega^2 z, \quad \omega = \sqrt{\frac{k}{m}}$$

故在 z 方向作 simple harmonic motion!

由 Lagrangian formulation 到 Hamiltonian dynamics,

我們... 的未知由 $(q_1, q_2, \dots, q_f)$ (f 個未知)

轉變為 $(q_1, q_2, \dots, q_f, p_1, p_2, \dots, p_f)$ ($2f$ 個未知)

方程式則由二階到一階。最簡單的例子為

$$L = \frac{1}{2} m \dot{x}^2 - U(x) \Rightarrow m\ddot{x} = -\frac{\partial U}{\partial x}$$

而在 Hamiltonian dynamics 中

$$H = \frac{p^2}{2m} + U(x), \quad \dot{p} = -\frac{\partial U}{\partial x}$$

$$\dot{x} = \frac{\partial U}{\partial p} = \frac{p}{m}$$

若將 p 消去, 得 $m\ddot{x} = -\frac{\partial U}{\partial x}$, 故二者一樣。

用 Hamiltonian dynamics 並非將問題簡化。

使用 Hamiltonian dynamics 卻在許多觀念上, formulation 不過

或擴展到其他方法上, 有其優美

$$\xi_{\pm} = \pm \sqrt{mk} i X + P$$

$$\therefore \sqrt{mk} i X(t) + P(t) = e^{i\omega t} (\sqrt{mk} i X_0 + P_0) \quad (4)$$

$$-\sqrt{mk} i X(t) + P(t) = e^{-i\omega t} (-\sqrt{mk} i X_0 + P_0) \quad (5)$$

由 $\frac{(4)+(5)}{2}$ 可得 $P(t) = \cos \omega t P_0 - \sqrt{mk} \sin \omega t X_0$
 $\therefore V(t) = \cos \omega t V_0 - \sqrt{\frac{k}{m}} \sin \omega t X_0$

$\frac{(4)-(5)}{2i\sqrt{mk}}$ 可得 $X(t) = \sqrt{\frac{m}{k}} \sin \omega t V_0 + \cos \omega t X_0$

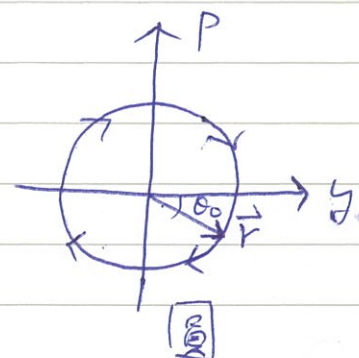
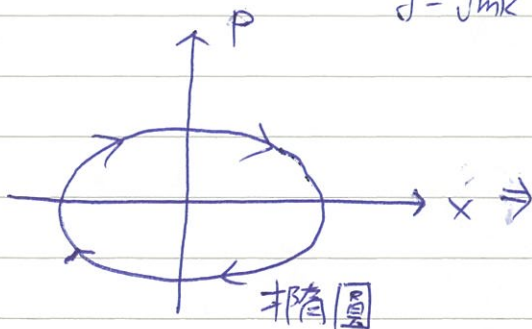
而得 $X(t)$ 及 $V(t)$!

註：也可取 $\xi_{\pm} \rightarrow a_{\pm} = \sqrt{\frac{m\omega}{2}} \left(X \pm i \frac{P}{\sqrt{mk}} \right)$

$\xi_{\pm}$ 之選取亦可由 $E = \frac{p^2}{2m} + \frac{1}{2} k x^2$

之相圖得知： $= \frac{1}{2m} (p^2 + mkx^2) \equiv \frac{1}{2m} (p^2 + y^2)$

$$y = \sqrt{mk} x$$



①及② $\Rightarrow \dot{y} = \omega p \quad \vec{r} = \vec{r} \times \vec{\omega}, \quad \vec{\omega} = (0, 0, \omega)$
 $\dot{p} = -\omega y \quad \therefore$ 在 py 圖上作

$p + iy = [\cos(\theta_0 + \omega t) + i \sin(\theta_0 + \omega t)] r$ 圖中運動
 $= r e^{i(\theta_0 + \omega t)} = (p_0 + iy_0) (\cos \omega t + i \sin \omega t)$

例: 1/1 Hamiltonian dynamics 解

簡諧運動

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad - \quad (1)$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -kx \quad - \quad (2)$$

①及②可以寫為 $\frac{d}{dt} \begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} \quad - \quad (3)$

由此可以看出, $\dot{x}$ 與 p 有關, 而 $\dot{p}$ 與 x 有關

這表現在③式中之矩陣 $\begin{pmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{pmatrix}$ 非對角化上。
問題困難在於

如果可以找到 x 及 p 之線性組合 ξ_1 及 ξ_2 使

$$\text{得 } \dot{\xi}_1 = A\xi_1, \quad \dot{\xi}_2 = B\xi_2 \quad \text{則}$$

問題被解決了。

因此, 我們嘗試 $\xi = \alpha x + \beta p$

由 ① $\times \alpha + ② \times \beta$ 可得 $\dot{\xi} = \frac{\alpha}{m}p - k\beta x$

欲使右式正比於 ξ , $\therefore$ 要求 $\frac{\alpha}{\beta} = \frac{-k\beta}{\alpha}$

故 $\alpha^2 + mk\beta^2 = 0$, 取 $\beta=1$, 則 $\alpha = \pm\sqrt{mk}i$

$$\therefore \dot{\xi}_{\pm} = \pm\sqrt{\frac{k}{m}}i p - kx = \pm\sqrt{\frac{k}{m}}i \underbrace{(p - \sqrt{mk}i x)}_{\xi_{\pm}}$$

$$\therefore \dot{\xi}_{\pm} = \pm i\omega \xi_{\pm}, \quad \xi_{\pm} = e^{\pm i\omega t} \xi_0$$

(稱為 cyclic)

No.
Date

5-46

如 H 含某 q_k 無關, 則其

"Conjugate momentum" $p_k = \text{常數}$

因為 $\dot{p}_k = -\frac{\partial H}{\partial q_k} = 0$ 之故

而在 Lagrangian 中, 若 $\frac{\partial L}{\partial q_k} = 0$ q_k 仍在 L 中!

一旦 q_k 是 cyclic, 則 $p_k = \alpha_k = \text{常數}$ (由起始條件決定)

$$\dot{q}_k = \frac{\partial H}{\partial p_k} = \frac{\partial H}{\partial \alpha_k} \text{ 與 } q_k \text{ 無關}$$

即 $q_k(t)$ 由 $H(q_1, q_2, \dots, q_{k-1}, q_{k+1}, \dots, q_f, p_1, p_2, \dots, \alpha_k, \dots, p_f, t)$

決定 (由其他變數決定)。

一旦 $\int q_1, q_2, \dots, q_{k-1}, q_{k+1}, \dots, q_f$ 決定了 則 $q_k(t) = \int \frac{\partial H}{\partial \alpha_k} dt$
 $p_1, p_2, \dots, p_{k-1}, p_{k+1}, \dots, p_f, \alpha_k$

而且 $q_1, q_2, \dots, q_{k-1}, q_{k+1}, \dots, q_f$ 與 q_k 無關!
解

因此, 如果能找到適當之座標使得 $q_1, q_2, \dots, q_f$ 皆為 cyclic, 則問題完全得到解決!

Modified Hamilton's principle

Hamiltonian dynamics 也可由

$$\int_{t_1}^{t_2} \left(\sum_j p_j \dot{q}_j - H \right) dt \text{ 得到}$$

但中須將 q_j 及 p_j 視為獨立變數：

$$\delta \int_{t_1}^{t_2} (\sum_j p_j \dot{q}_j - H) dt$$

$$= \int_{t_1}^{t_2} \sum_j (p_j \delta \dot{q}_j + \dot{q}_j \delta p_j - \frac{\partial H}{\partial q_j} \delta q_j - \frac{\partial H}{\partial p_j} \delta p_j) dt = 0$$

L - (49)

$$\int_{t_1}^{t_2} p_j \delta \dot{q}_j dt = p_j \delta q_j \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{q}_j p_j dt$$

$$\therefore (49) \Rightarrow \int_{t_1}^{t_2} \sum_j \left\{ \left(\dot{q}_j - \frac{\partial H}{\partial p_j} \right) \delta p_j - \left(p_j + \frac{\partial H}{\partial q_j} \right) \delta q_j \right\} dt = 0$$

$\therefore \delta p_j$ 及 δq_j 獨立

$$\therefore \dot{q}_j = \frac{\partial H}{\partial p_j} \quad p_j = - \frac{\partial H}{\partial q_j} \quad \dots (50)$$

Phase space and Liouville's theorem

由 $(q_1, q_2, \dots, q_f, p_1, p_2, \dots, p_f)$ 定出之 $2S$ 維度
的空間稱為相空間 (phase space)

而由 $(q_1, q_2, \dots, q_f)$ 定出之 S 維度的 Configuration
space 不同。

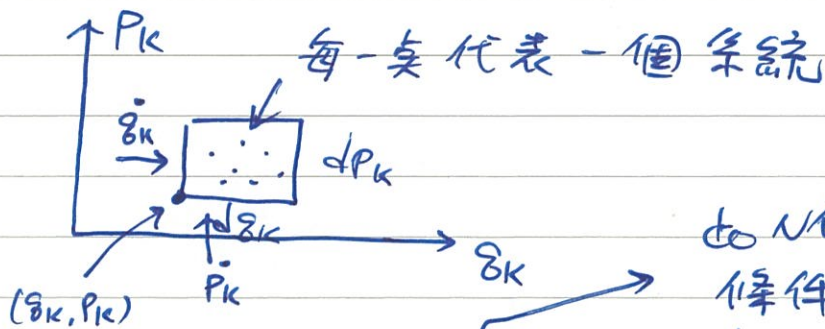
最重要的是若考慮一組由 H 主宰 ^{動力} (dynamics)
會 (跟著粒子走)

的系統，其在相空間之密度 ρ 不變

則可證明

(Liouville's theorem)

而以 Configuration space 來看，則並無，
 此特性。這表示 Hamiltonian dynamics 是
 討論 統計力學 較方便的教學 formulation.



考慮 N 個系統，則

在 $dv = dp_1 dp_2 \dots dp_f dq_1 \dots dq_f$ 中有

而 N 個系統可能為巨觀
 條件下所對應之可能的系統
 如：壓力、體積給定下，氣體
 的微觀狀態並不唯一

$$\rho(p_1 p_2 \dots p_f, q_1 \dots q_f) dv = N$$

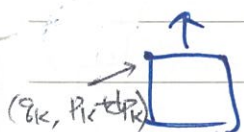
以上圖來說，由左邊流入右邊之系統數（系統數）
 單位時間

$$\text{為 } \rho \dot{q}_k dp_k \quad \dots (51)$$

由下邊流入為 $\rho \dot{p}_k dq_k$

$$\therefore \text{共流入 } \rho (\dot{q}_k dp_k + \dot{p}_k dq_k) \quad \dots (52)$$

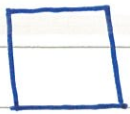
上式取值在 (q_k, p_k) ，同理，對流出為：



由上流出為 $\rho \dot{q}_k dq_k$ at $p_k + dp_k$

$$= \rho \dot{p}_k dq_k + \frac{d}{dp_k} (\rho \dot{p}_k) dp_k dq_k \quad \dots (53)$$

evaluated at (p_k, q_k)



$(g_k + dg_k, p_k)$

由右邊流出為

$$= \rho \dot{g}_k dP_k \text{ at } g_k + dg_k$$

$$= \rho \dot{g}_k dP_k + \frac{d(\rho \dot{g}_k)}{dg_k} dg_k dP_k \dots (54)$$



evaluated at (p_k, g_k)

結合 (52), (53) 及 (54) 式,

流入 $\square$ 中之淨流量

$$= \rho (\dot{g}_k dP_k + \dot{P}_k dg_k)$$

$$- \rho \dot{P}_k dg_k - \frac{d}{dP_k} (\rho \dot{P}_k) dg_k dP_k$$

$$- \rho \dot{g}_k dP_k - \frac{d}{dg_k} (\rho \dot{g}_k) dg_k dP_k$$

$$= - \left[\frac{d(\rho \dot{P}_k)}{dP_k} + \frac{d(\rho \dot{g}_k)}{dg_k} \right] dg_k dP_k$$

此量即為單位時間內在 $dg_k dP_k$ 中流失之熱能

$$= \frac{dP}{dt} dg_k dP_k \quad (\text{注意: } P_k \text{ 及 } g_k \text{ 固定, } \therefore \text{是 } \rho \text{ 的偏微分})$$

$$\text{故 } \frac{dP}{dt} dg_k dP_k = \sum_k \left[\frac{d(\rho \dot{P}_k)}{dP_k} + \frac{d(\rho \dot{g}_k)}{dg_k} \right] dg_k dP_k$$

$$\therefore \frac{dP}{dt} + \sum_k \frac{d}{dP_k} (\rho \dot{P}_k) + \frac{d}{dg_k} (\rho \dot{g}_k) = 0 \dots (15)$$

此式事實上為所謂之 Continuity equation: $\frac{dP}{dt} + \nabla \cdot \vec{J} = 0$
之特例

其中 $\vec{j} = \rho \vec{v}$ 為對應 ρ 之流量密度

對於力學系統而言，(5)式可以進一步簡化：

$$\text{首先 } \frac{d}{dt}(\rho \dot{p}_k) = \frac{d\rho}{dt} \dot{p}_k + \rho \frac{d\dot{p}_k}{dt}$$

$$\frac{d}{dt}(\rho \dot{q}_k) = \frac{d\rho}{dt} \dot{q}_k + \rho \frac{d\dot{q}_k}{dt}$$

$$\therefore \dot{p}_k = -\frac{\partial H}{\partial \dot{q}_k}, \quad \therefore \frac{d\dot{p}_k}{dt} = -\frac{\partial^2 H}{\partial p_k \partial \dot{q}_k}$$

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \therefore \frac{d\dot{q}_k}{dt} = +\frac{\partial^2 H}{\partial p_k \partial \dot{q}_k}$$

$$\therefore \frac{d\dot{p}_k}{dt} + \frac{d\dot{q}_k}{dt} = 0$$

故 (5) 式

$$\Rightarrow \frac{d\rho}{dt} + \sum_K \left(\frac{d\rho}{d\dot{q}_k} \frac{d\dot{q}_k}{dt} + \frac{d\rho}{dp_k} \frac{dp_k}{dt} \right) = 0 \quad \dots (56)$$

$$(56) \text{ 式表示 } \frac{d\rho}{dt} = 0 \quad \dots (57)$$

$$\therefore \rho = \rho(q_1, q_2, \dots, q_f, p_1, p_2, \dots, p_f, t)$$

其中 t 之關係 來自 $q_1, \dots, q_f, p_1, \dots, p_f$ 及 t

$$\text{故 } \frac{d\rho}{dt} = \frac{d\rho}{dt} + \sum_K \frac{d\rho}{d\dot{q}_k} \dot{q}_k + \frac{d\rho}{dp_k} \dot{p}_k$$

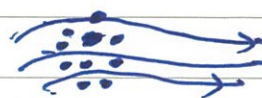
(5) 式即為 Liouville's theorem.

其意義是：隨著粒子之動力 (dynamics),

如果跟隨這一羣系統點

在 phase space 中流動

其密度不變！



換句話說，系統點羣是不可被壓縮的！

（如水流一樣）

(incompressible)

注意：在固定的相空間點 $(q_1, \dots, q_f, p_1, \dots, p_f)$ 來看，

ρ 是會變化的， $\therefore \dot{\rho} \neq 0$

Liouville's theorem 只在 phase space 成立，

在 configuration space 並沒有對應的

定理，這是何以需要 Hamiltonian dynamics 之處！
使用

Poincaré recurrence Theorem

Liouville's theorem 最重要的應用之一是 Poincaré recurrence theorem, 它指出任何相空間中運動, ^{有限}

經過足夠長的時間後, 系統一定回到十分接近 ^{之狀態} 起始狀態.

Theorem: Let g be a volume-conserving, continuous, one-to-one map from a bounded region $D \in \mathbb{R}^n$ onto itself, i.e. $gD = D$. Then in any open set $U \subset D$, there exists a point P , which returns to U under some iteration of the action of g , i.e. $\exists n \rightarrow, g^n P \in U$.

物理上, 這定理並不難理解。假設, 我們要系統回到距起始狀態 $< \varepsilon$, 則取一相空間半徑 R_0 之

R_0 半徑 $= \varepsilon$.

(4)

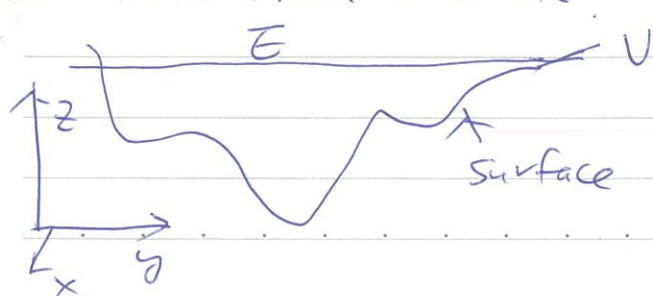
起始
狀態

如考慮一個在某一曲面上運動 (在重力下)

之質點, $t=0$. $(X(0), Y(0), P_X(0), P_Y(0))$ 為

起始點, 則 R_0 是以此點為球心之

4維半徑為 ε 之球



$$E = -mgz(X, Y) + \frac{1}{2m}(P_X^2 + P_Y^2)$$

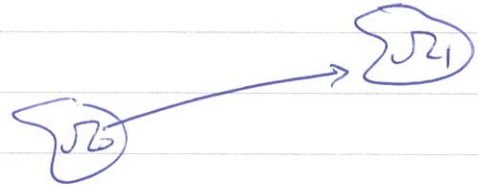
質點 (相空間) 能去的是有限 $ES \leq R_0$

∴ 相空間件積不變,

∴ $t=T$ 時 $\Omega_0 \rightarrow \Omega_1$, $\nu(\Omega_1) = \nu(\Omega_0)$ 件積不變

Ω_1 形狀改變但件積不變

考慮 $t=KT$



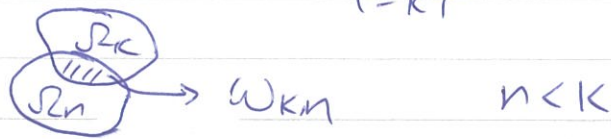
假設 $\Omega_0, \Omega_1, \dots, \Omega_K$ 皆不重疊,

則當 K 夠大 $\nu(\Omega_0) + \dots + \nu(\Omega_K) > \nu$.

由相空間 $< \nu$ 矛盾,

故至少有二值, 設為 $K \neq n$, 重疊

$t=KT$



此時 W_{Kn} 中的點 $\in \Omega_K$ and $\in \Omega_n$.

現在, 考慮反轉時間 t , 經過 nT , 則 $\Omega_K \rightarrow \Omega_{K-n}$

∴ $\Omega_n \rightarrow \Omega_0$.

此時 W_{Kn} 屬於 Ω_K 之 $W_K \rightarrow W_{K-n}$

" " Ω_n 之 $W_n \rightarrow W_0$

∴ 反轉, 給起起始條件, 是唯一定, ∴ $W_0 = W_{K-n}$

且 $\in \Omega_0$ 且 $\in \Omega_{K-n}$. 即至少 Ω_{K-n} 與 Ω_0 重疊

所以, 系統回到接近原起始狀態 Ω_0 內!

Canonical transformation

Motivation

給定一個 Hamiltonian, $H = H(q_i, p_i, t)$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

當 q_i 為 cyclic, 即 $\frac{\partial H}{\partial q_i} = 0 \Rightarrow \dot{p}_i = 0$

$\therefore p_i = \beta_i = \text{const}$. 因此如果 H 有 cyclic 座標 (q_i)
所對應之動力可被簡化
問題

問題是 H 是否有 cyclic 座標 與所選取之座標系統
有關: 如只有在極座標 (r, θ) , 才看出 向心力
之位能 $V = V(r)$ 與 θ 無關.

因此, 如果可以找到新的座標 Q_i, P_i

$$Q_i = Q_i(p_i, q_i, t)$$

$$P_i = P_i(p_i, q_i, t)$$

使得 所有 Q_i 為 cyclic, 則所有 P_i 皆為常數.

$$P_i = \beta_i, \quad \text{則} \quad H = H(P_i) = H(\beta_i)$$

$$\dot{Q}_i = \frac{\partial H}{\partial P_i} = \omega_i = \text{const}$$

$\therefore$ 整個系統可被解出 $Q_i = \omega_i t + \omega_0$

$$P_i = \beta_i = \text{const}$$

以上的推導假設了 Q_i, P_i 滿足

同樣形式的 Hamiltonian equations

即
$$\bar{H} = \bar{H}(P_i, Q_i, t)$$

$$\dot{Q}_i = \frac{\partial \bar{H}}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial \bar{H}}{\partial Q_i}$$

這樣的轉換

$$\begin{aligned} Q_i &= Q_i(P_i, Q_i, t) \\ P_i &= P_i(P_i, Q_i, t) \end{aligned} \quad (\text{point transformation})$$

使得存在新的 Hamiltonian $\bar{H}(P_i, Q_i, t)$

使得
$$\dot{P}_i = -\frac{\partial \bar{H}}{\partial Q_i}, \quad \dot{Q}_i = \frac{\partial \bar{H}}{\partial P_i}$$

稱為 canonical transformation.

Generating function

在新的座標中, Hamilton principle 仍成立.

$$\therefore \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt = 0$$

$$\text{且 } \delta \int_{t_1}^{t_2} \bar{L}(Q_i, \dot{Q}_i, t) dt = 0$$

$$\text{故 } \delta \int_{t_1}^{t_2} (L - \bar{L}) dt = 0 \quad \dots \textcircled{1}$$

當 $L - \bar{L} = \frac{dF}{dt}, \quad F(Q_i, P_i, P_i, Q_i, t)$

①式 = $\delta(F(t_2) - F(t_1)) = 0$ 可成立 $\uparrow$ $4n+1$ 個變數

致

因為 $\left. \begin{aligned} Q_i &= Q_i(p_i, q_i, t) \\ p_i &= p_i(p_i, q_i, t) \end{aligned} \right\}$ 共有 $2n$ 個方程式

F 中 $4n+1$ 個變數 另有 $2n+1$ 個獨立變數，為 generating function，
有 4 種情形：

$$F_1 = F(q_i, Q_i, t), \quad F_2 = F(q_i, p_i, t)$$

$$F_3 = F(p_i, Q_i, t), \quad F_4 = F(p_i, p_i, t)$$

這 4 種情形，彼此可用 Legendre transformation
互換，可依實際需求選擇。

$$\therefore L = \left[+ \frac{dF}{dt} \right], \quad L = \sum_i p_i \dot{q}_i - H$$

$$\downarrow$$

$$\therefore \sum_i p_i \dot{q}_i - H = \sum_i p_i \dot{Q}_i - \bar{H} + \frac{dF}{dt} \quad \dots (2)$$

$$\text{若 } F = F_1(q_i, Q_i, t), \quad \frac{dF_1}{dt} = \sum \frac{\partial F_1}{\partial q_i} \dot{q}_i + \frac{\partial F_1}{\partial Q_i} \dot{Q}_i + \frac{\partial F_1}{\partial t}$$

$$(2) \text{ 式 } \sum_i p_i \dot{q}_i - \sum_i p_i \dot{Q}_i - H + \bar{H} = \sum_i \frac{\partial F_1}{\partial q_i} \dot{q}_i + \frac{\partial F_1}{\partial Q_i} \dot{Q}_i + \frac{\partial F_1}{\partial t}$$

$$\text{故 } p_i = \frac{\partial F_1(q_j, Q_j, t)}{\partial q_i}$$

$$p_i = - \frac{\partial F_1(q_j, Q_j, t)}{\partial Q_j}$$

$$\bar{H} = H + \frac{\partial F_1}{\partial t}(q_j, Q_j, t) \quad \dots (3)$$

即 $p_i, \bar{p}_i$ 可由 F_1 決定 (generate)，故 F_1 為 ^稱 generating function.

另一個常用的轉換為 $F_2 \equiv S(q_i, p_i, t)$ DATE

可由 F_1 經 Legendre transformation 得到

$$S(q_i, p_i, t) = \sum_i p_i q_i + F_1(q_i, Q_i, t)$$

↑

由 $p_i = -\frac{\partial F_1}{\partial Q_i}$ 得 Q_i

視為 p_i, q_i 函數

由 eq. (2) 可得

$$\sum_i p_i \dot{q}_i - H = \sum_i p_i \dot{Q}_i - \bar{H} + \frac{dF_1}{dt}$$

$$= \sum_i p_i \dot{Q}_i - \bar{H} + \frac{d}{dt} (S(q_i, p_i, t) - \sum_i p_i Q_i)$$

$$\text{故 } \sum_i p_i \dot{q}_i - \sum_i p_i \dot{Q}_i - H + \bar{H} = \frac{d}{dt} (S - \sum_i p_i Q_i)$$

$$= \sum_i \frac{\partial S}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial S}{\partial p_i} \dot{p}_i + \frac{\partial S}{\partial t} - \sum_i \dot{p}_i Q_i - \sum_i \dot{q}_i p_i$$

$$\therefore \sum_i p_i \dot{q}_i + \sum_i \dot{p}_i Q_i \overset{-H+\bar{H}}{=} \sum_i \frac{\partial S}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial S}{\partial p_i} \dot{p}_i + \frac{\partial S}{\partial t}$$

$$\text{故可得 } p_i = \frac{\partial S(q_i, p_i, t)}{\partial q_i}, \quad Q_i = \frac{\partial S(q_i, p_i, t)}{\partial p_i}$$

$$\bar{H}(p_i, Q_i, t) = H(p_i, q_i, t) + \frac{\partial S(q_i, p_i, t)}{\partial t}$$

同理, F_3 & F_4 所對應之轉換也可以建構。

Note that: 我們要求 $\delta q_i, \delta p_i$ 在 t_1 & $t_2 = 0$

$\therefore \delta F_1|_{t_1, t_2} = 0$ 同理可以將 p_i 視為 q_i, Q_i 的函數

$$\therefore \delta p_i = \frac{\partial p_i}{\partial q_i} \delta q_i + \frac{\partial p_i}{\partial Q_i} \delta Q_i = 0(t_1, t_2), \text{ 故 } \delta S|_{t_1, t_2} = 0$$

其他函轉換也滿足 $\delta F_3 = 0, \delta F_4 = 0$.

$$F_3(p_i, q_i, t) = F_1(q_i, p_i, t) - q_i p_i, \quad q_i = \frac{\partial F_3}{\partial p_i}, \quad p_i = \frac{\partial F_3}{\partial q_i}$$

$$F_4(p_i, p_i, t) = F_3(p_i, q_i, t) + p_i q_i, \quad q_i = -\frac{\partial F_4}{\partial p_i}, \quad p_i = \frac{\partial F_4}{\partial p_i}$$

例

$$F_1(q_i, p_i, t) = \sum_i q_i p_i$$

$$\text{則 (3) 式} \Rightarrow p_i = \frac{\partial F_1}{\partial q_i} = q_i$$

$$p_i = -\frac{\partial F_1}{\partial p_i} = -q_i$$

因此 $p_i = -q_i$, $q_i = p_i$ 交換動量與位置，顯示

在 Hamiltonian 架構下，動量與位置扮演等價的角色。

例. 簡諧振盪

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2$$

$$\text{設 } F_1(q, p) = \frac{m}{2} \omega^2 q^2 \cot Q, \quad \text{求 } H(p, Q)$$

$$p = \frac{\partial F_1}{\partial q} = m \omega^2 q \cot Q, \quad p = -\frac{\partial F_1}{\partial Q} = \frac{m}{2} \frac{\omega^2 q^2}{\sin^2 Q}$$

$$\therefore q = \sqrt{\frac{2}{m \omega^2}} p \sin Q, \quad \text{代入 } p = m \omega^2 q \cot Q$$

$$= \sqrt{2 m \omega^2} p \cos Q$$

$$\therefore \bar{H} = H + \frac{\partial F_1}{\partial t} \rightarrow 0$$

$$= \frac{1}{2m} (2m \omega p) \cos^2 Q + \frac{1}{2} m \omega^2 \frac{2p}{m \omega} \sin^2 Q$$

$$= p \omega (\cos^2 Q + \sin^2 Q) = \omega p$$

$$\therefore Q \text{ is cyclic in } \bar{H}, \quad \text{故 } \dot{p} = -\frac{\partial \bar{H}}{\partial Q} = 0, \quad p = \text{constant}$$

$$E = \omega p = \text{const.} \quad \therefore \dot{Q} = \frac{\partial \bar{H}}{\partial p} = \omega \quad \therefore Q = \omega t + \phi$$

故 $q = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \phi)$ 可以被完全解出。

例 時間演化

如果取 $S(q_i, p_i, \epsilon) = \sum_i q_i p_i + \epsilon G(q_i, p_i)$

ϵ 為一參數，則

$$P_i = \frac{\partial S}{\partial q_i} = p_i + \epsilon \frac{\partial G}{\partial q_i}$$

$$Q_i = \frac{\partial S}{\partial p_i} = q_i + \epsilon \frac{\partial G}{\partial p_i}$$

因此，當 $\epsilon = 0$ 時， $P_i = p_i$ ， $Q_i = q_i$

S 所產生的為 identity transformation.

若取 $\epsilon = dt$ ($t = \text{時間}$)， $G(q_i, p_i) = H(q_i, p_i) = H(q_i, p_i) + o(\epsilon)$
 $dt = \epsilon \rightarrow 0$

則 $Q_i = q_i + dt \frac{\partial}{\partial p_i} H(q_i, p_i) + o(dt^2)$

$$P_i = p_i - dt \frac{\partial}{\partial q_i} H(q_i, p_i) + o(dt^2)$$

因為 $\frac{\partial H}{\partial p_i} = \dot{q}_i$ ， $\frac{\partial H}{\partial q_i} = -\dot{p}_i$

可得 $Q_i = q_i + dt \dot{q}_i = q_i(t+dt)$

$$P_i = p_i + dt \dot{p}_i = p_i(t+dt)$$

故 $H(p_i, q_i)$ generates time evolution of $p_i(t)$ & $q_i(t)$.

Hamilton-Jacobi theory

給定一個力學系統，此系統可以完全解

決：如果可以找到一個座標系統 Q_i, P_i

使得 $\underbrace{P_i = p_i(t=0)}_{\text{所有的}}$ ， $\underbrace{Q_i = q_i(t=0)}_{\text{起始條件}}$ 為常數

則 Q_i, P_i 與 p_i, q_i 間之轉換即為 ^{解之}座標與起

始條件之關係： $q_i = q_i(P_i, Q_i, t) = q_i(q_{i0}, p_{i0}, t)$

$$p_i = p_i(P_i, Q_i, t) = p_i(q_{i0}, p_{i0}, t)$$

而在 Q_i, P_i 座標

$$\dot{Q}_i = 0 = \frac{dH}{dP_i} \quad \text{--- (1)}$$

$$\dot{P}_i = 0 = -\frac{dH}{dQ_i} \quad \text{--- (2)}$$

最簡單達到方程式 (1) 與 (2) 之轉換為選取

$$H = 0$$

此時，因為 $H(Q_1, Q_2, \dots, Q_N, P_1, \dots, P_N, t)$

$$= H(q_1, q_2, \dots, q_N, p_1, \dots, p_N, t) + \frac{dF}{dt} \quad \text{--- (3)}$$

$$\text{故 } H(q_1, q_2, \dots, q_N, p_1, \dots, p_N, t) + \frac{dF}{dt} = 0$$

由此可求 ^{所需要} generating function F .

歷史上，Jacobi 選取 $F = F_2 = S(q_i, p_i, t)$ ，因為 $p_i = \frac{\partial S}{\partial q_i}$

Eq. ③ 成為一個一階微分方程式，比較方便求解：

$$H(q_1, q_2, \dots, q_N, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \dots, \frac{\partial S}{\partial q_N}, t) + \frac{\partial S}{\partial t} = 0 \quad \text{--- (4)}$$

此為 Hamilton-Jacobi equation.

解 S 稱 Hamilton's principal function.

Eq. ④ 為 $N+1$ 個獨立變數，故有 $N+1$ 個一階微分方程式

積分常數待定：可取為 P_k , $k=1, \dots, N$ (remember: S 為 $P_1, \dots, P_N$ 的函數)

及 $S + \text{constant}$ 之常數。

解 Hamilton-Jacobi 方程式的解原運動方程式等價。

$$\begin{aligned} \text{而 } \frac{dS}{dt} &= \sum_k \frac{\partial S}{\partial q_k} \dot{q}_k + \frac{\partial S}{\partial t} \\ &= \sum_k P_k \dot{q}_k - H = L \end{aligned}$$

$\therefore S = \int L dt$ 稱為 action. 但 $S = S(q_i, P_i, t)$

欲求 S , 必須知道 $q_1(t), q_2(t), \dots$, 如 $L = L(q_1(t), \dots, q_N(t), \dot{q}_1(t), \dots, \dot{q}_N(t), t)$ 才知道, $\int L dt$ 才能計算, 而 $q_i(t)$ 由以下決定:

$$\frac{\partial S}{\partial P_i} = q_i \quad (q_i = 0, q_i = \text{常數}) = \text{常數}, \quad P_i = \alpha_i = \text{常數} \\ = \beta_i \quad (\dot{P}_i = 0)$$

其中 α_k 為積分常數，可以由起始條件

定之：

$$\beta_k = \frac{\partial S(q_1, q_2, \dots, q_N, \alpha_1, \alpha_2, \dots, \alpha_N, t)}{\partial \alpha_k} \Big|_{t=0}$$

$$p_k(0) = \frac{\partial S(q_1, \dots, q_N, \alpha_1, \dots, \alpha_N, t)}{\partial q_k} \Big|_{t=0}$$

由 $2N$ 個方程式可以解出 α_k, β_k (以 $q_k(0)$ 及 $p_k(0)$ 表示)，再由 $S = \int L dt$ 解出 $S = S(q_i, \alpha_i, t)$ 。

例：簡諧振盪

$$H = \frac{p^2}{2m} + \frac{1}{2} K q^2, \quad S = S(q, p, t)$$

$$p = \frac{\partial S}{\partial q}$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left(\frac{\partial S}{\partial q} \right)^2 + \frac{1}{2} K q^2 = 0 \quad \dots (5)$$

此方程式可以分離變數法解之。

$$S = S_1(t) + S_2(q)$$

$$\text{Eq. (5)} \Rightarrow -\dot{S}_1(t) - \frac{1}{2m} \left(\frac{dS_2}{dq} \right)^2 + \frac{1}{2} K q^2 = 0$$

$$\therefore S_1(t) = -\alpha t, \quad \frac{dS_2(q)}{dq} = \sqrt{2m\alpha - mKq^2}$$

$$\therefore S(q, \alpha, t) = \sqrt{mK} \int \sqrt{\frac{2\alpha}{K} - q^2} dq - \alpha t$$

$$\alpha = \frac{\partial S}{\partial \alpha} = \text{常數} \equiv \beta = \frac{\sqrt{mK}}{K} \int \frac{dq}{\sqrt{\frac{2\alpha}{K} - q^2}} - t$$

$$\text{令 } q = \sqrt{\frac{2\alpha}{K}} \sin \theta \quad \dots = \sqrt{\frac{m}{K}} \sin^{-1} \left(\sqrt{\frac{K}{2\alpha}} q \right) - t$$

故解為: $q = \sqrt{\frac{2\alpha}{k}} \sin \omega(t + \frac{\alpha}{\beta})$

由解比較可知, $\alpha = \text{energy} = E$



β 則與初始條件有關

$$\frac{\partial S}{\partial q} = p$$

$$\alpha = \frac{p^2}{2m} + \frac{1}{2}kq^2 = E.$$

分離變數法

由以上 ^解 Hamilton-Jacobi 方程式, ^{之形式可知} 如果 H 與時間無關

可以設 $S = S_0(q, p) - Et$

$$\text{則 Eq. (4)} \Rightarrow H(q_1, q_2, \dots, q_N, \frac{\partial S_0}{\partial q_1}, \dots, \frac{\partial S_0}{\partial q_N}) = E \quad (6)$$

問題變成解 S_0 .

對於具有相加性之 Hamiltonian (如 non-interacting particles)

$$H(q_1, q_2, \dots, q_N, p_1, \dots, p_N) = H_1(q_1, p_1) + \dots + H_N(q_N, p_N).$$

此時可以用分離變數法, 令

$$S_0(q_1, \dots, q_N, p_1, \dots, p_N) = S_1(q_1, p_1) + \dots + S_N(q_N, p_N)$$

$$\text{而得 } H_1(q_1, \frac{\partial S_1}{\partial q_1}) + H_2(q_2, \frac{\partial S_2}{\partial q_2}) + \dots + H_N(q_N, \frac{\partial S_N}{\partial q_N}) = E$$

$$\text{故解為 } H_1(q_1, \frac{\partial S_1}{\partial q_1}) = \alpha_1, H_2(q_2, \frac{\partial S_2}{\partial q_2}) = \alpha_2, \dots, H_N(q_N, \frac{\partial S_N}{\partial q_N}) = \alpha_N$$

$$\text{而 } \alpha_1 + \alpha_2 + \dots + \alpha_N = E$$

此時每一對 (q_k, p_k) 与其他独立, 可以
个别解决而得到整个系统的解。

例:

$$H = \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) + mgz$$

重力作用下的运动

$$\text{Hamilton-Jacobi eq. : } \frac{1}{2m} \left[\left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} \right)^2 + \left(\frac{\partial S}{\partial z} \right)^2 \right] + mgz + \frac{\partial S}{\partial t} = 0$$

設

$$S = W_1(x) + W_2(y) + W_3(z) - Et$$

$$\frac{1}{2m} \left[\left(\frac{dW_1}{dx} \right)^2 + \left(\frac{dW_2}{dy} \right)^2 + \left(\frac{dW_3}{dz} \right)^2 \right] + mgz = E$$

$$\text{因故 } \frac{1}{2m} \left(\frac{dW_1}{dx} \right)^2 = \alpha_1$$

$$\frac{1}{2m} \left(\frac{dW_2}{dy} \right)^2 = \alpha_2$$

$$\frac{1}{2m} \left(\frac{dW_3}{dz} \right)^2 + mgz = \alpha_3$$

$$\text{且 } \alpha_1 + \alpha_2 + \alpha_3 = E$$

$$\therefore W_1(x) = \pm \sqrt{2m\alpha_1} x, \quad W_2(y) = \pm \sqrt{2m\alpha_2} y$$

$$W_3(z) = \pm \sqrt{\frac{8}{9mg^2}} (\alpha_3 - mgz)^{3/2}$$

($W_{1,2,3}$ 可以再加上常数, 但此常数属于 $S \rightarrow S + S_0$
 S 本身就可以再加上任一常数 S_0 , 不影响所有结果, 可以设为 0)

$\alpha_1, \alpha_2, \alpha_3$ 為 3 個積分常數, 可定為 DATE

momentum.

$$S = S(X, y, z, \alpha_1, \alpha_2, \alpha_3, t)$$

$$\frac{\partial S}{\partial \alpha_1} = \theta_1 = \beta_1 = \pm X \sqrt{\frac{m}{2\alpha_1}} - t$$

$$\left(S = W_1 + W_2 + W_3 - Et = W_1 + W_2 + W_3 - (\alpha_1 + \alpha_2 + \alpha_3)t \right)$$

$$\frac{\partial S}{\partial \alpha_2} = \pm y \sqrt{\frac{m}{2\alpha_2}} - t = \beta_2 = \theta_2$$

$$\frac{\partial S}{\partial \alpha_3} = \theta_3 = \beta_3 = \pm \sqrt{\frac{2(\alpha_3 - mgy)}{mg^2}} - t$$

因故, $X = \pm \sqrt{\frac{2\alpha_1}{m}} (\beta_1 + t)$

$$y = \pm \sqrt{\frac{2\alpha_2}{m}} (\beta_2 + t)$$

$$z = \frac{\alpha_3}{mg} - \frac{g}{2} (\beta_3 + t)^2$$

由上述可知: $2\alpha_1 = P_x^2, 2\alpha_2 = P_y^2,$

$$\beta_1 = \frac{\pm X_0}{v_{0x}}, \beta_2 = \pm \frac{y_0}{v_{0y}}$$

$$z = \frac{\alpha_3}{mg} - \frac{g}{2} \beta_3^2 - g\beta_3 t - \frac{g}{2} t^2 = z_0 + v_{0z} t - \frac{1}{2} g t^2$$

$$\beta_3 = -\frac{v_{0z}}{g}, \quad \frac{\alpha_3}{mg} - \frac{g}{2} \beta_3^2 = \frac{\alpha_3}{mg} - \frac{1}{2} \frac{v_{0z}^2}{g} = z_0$$

$$\alpha_3 = mg z_0 + \frac{1}{2} m v_{0z}^2 = E_z(t=0)$$

Action - Angle variables

由以上分離變數可得

$$E = \alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_n$$

$$S = \sum_i S_i(q_i, \alpha_i)$$

$$\therefore \frac{\partial S(q_i, P_i, t)}{\partial P_i} = q_i$$

$$\frac{\partial S(q_i, P_i, t)}{\partial q_i} = P_i$$

$$\therefore H = \sum_k H_k(q_k, P_k) \quad \therefore \text{每 } (q_i, P_i) \text{ 与其他 } (q_k, P_k)$$

不耦合，可以一對一對的考慮。

考慮週期運動 (即 P_i, q_i 為一個一給週期運動)
(P_i, q_i) 為

(將 label i 移除)

如果能作 Canonical transformation 到 J, ϕ

使得 $H = H(J)$ 与 ϕ 無關，則

$$\dot{J} = -\frac{\partial H}{\partial \phi} = 0, \quad J = \text{constant} \quad H = H(J) = \text{constant}$$

$$\text{且} \quad \dot{\phi} = \frac{\partial H}{\partial J} = \text{constant} \equiv \nu$$

$$\phi = \nu(t - t_0)$$

J 為 action variable, 而 ϕ 則是 angle variable.

如何找 J, ϕ ?

由 $S_i = S_i(q_i, \alpha_i)$, 可知若是 $J = \oint p dq$
(phase integral for periodic motion).

則因 $p = \frac{dS}{dq}(q, \alpha)$

$$J = \oint \frac{dS}{dq}(q, \alpha) dq = J(\alpha)$$

由此可解出 $\alpha = \alpha(J)$

故 $S(q, \alpha) \rightarrow S(q, J)$, J 可視為 p , $\dot{\phi} = \frac{\partial H}{\partial J} = \text{常數}$
 $= v = \text{頻率}$

1. 亦可以作 Canonical transformation (即 $F_1(q, \phi, t)$)
 第 1 型

$\bar{W}(q, \phi)$ 且因為 $\frac{\partial \bar{W}}{\partial q} = p$, $\frac{\partial \bar{W}}{\partial \phi} = -J$,
 將 q 換為 ϕ $\uparrow$
 p

$$\therefore d\bar{W}(q, \phi) = p dq - J d\phi$$

p, q 是週期性的, 故 J, ϕ 亦是

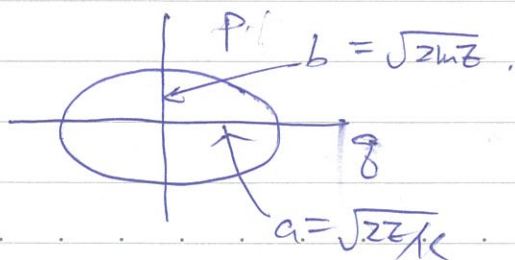
$$\oint d\bar{W}(q, \phi) = 0 = \oint p dq - \oint J d\phi$$

$J = \text{常數} \therefore \oint J = \oint p dq - \text{致}$

例: $H = \frac{p^2}{2m} + \frac{1}{2}kq^2$ (簡諧振盪)

以 J, ϕ 為變數

$$\therefore \frac{p^2}{2mE} + \frac{q^2}{23/k} = 1, \quad E = H$$



$$J = \oint p dq = \pi ab = 2\pi E \sqrt{\frac{m}{k}}$$

$$\therefore E = H = \frac{J}{2\pi} \sqrt{\frac{k}{m}}$$

$$v = \frac{dH}{dJ} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

generating function

$$\bar{W}(q, \phi) = \int p dq - \int J d\phi$$

$$= \int \sqrt{2mE - m\kappa q^2} dq - J\phi \quad (J = \text{const.})$$

$$= \sqrt{2mE} \int \sqrt{1 - \frac{m\kappa q^2}{2E}} dq - J\phi$$

$$\text{set } z = \frac{\omega q \sqrt{m}}{\sqrt{2E}} \quad \int \sqrt{1-z^2} dz = \frac{1}{2} (\sin^{-1} z + z \sqrt{1-z^2})$$

$$J = 2\pi E/\omega$$

$$\text{則 } \bar{W} = \frac{E}{\omega} [\sin^{-1} z + z \sqrt{1-z^2}] - 2\pi \frac{E}{\omega} \phi$$

$$= \frac{E}{\omega} [\sin^{-1} z - 2\pi \phi + z \sqrt{1-z^2}]$$

$\phi d\bar{W} = 0 \quad \therefore \bar{W}$ 中須具週期性。

可以取 $\sin^{-1} z = 2\pi \phi$ 定出所需之 generating function

$$\bar{W} \quad \therefore z = \sin 2\pi \phi$$

$$\text{令 } Q = 2\pi \phi \quad \text{則 } \bar{W} = \frac{E}{\omega} \sin Q \cos Q$$

$$\sin Q = \frac{\sqrt{m}}{\sqrt{2E}} \omega q$$

$$E = \left(\frac{\omega q}{\sin Q} \right)^2 \times \frac{m}{2}$$

$$\bar{W}(q, Q) = \frac{1}{2} m \omega q^2 \cot Q \quad \text{即 } \text{如前之例子中}$$

$$= F_1(q, Q)!$$