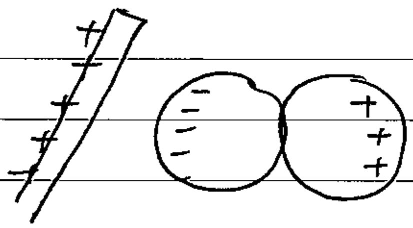
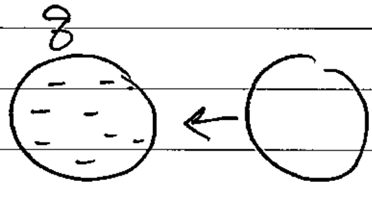
Electrostatics

Coulomb's law

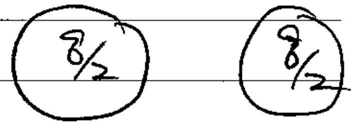
Electric charges were discovered early in human history through rubbing between materials. Later, it's found that by induction (move a charge object near a conductor), one can charge a conductor (materials made by metal).



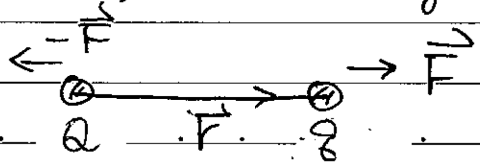
Using method of charging by induction, one can change charges on an object:



touch → separate



Measuring the force, Coulomb found that for static, fixed (point-like) charges separating by $\vec{r}$



$$\vec{F} = k\frac{qQ}{r^2}\hat{r} = k\frac{qQ}{r^3}\vec{r}$$

No.
Date.

Eq. ① is the Coulomb's law and it works only for static charges.

The proportional constant K depends on unit of charges. Charges are later known to be quantized $|q| = N|e|$ with

$|e|$ = charge magnitude of electron charge
$= 1.6 \times 10^{-19}$ Coulomb.

In SI unit, the unit of charge is Coulomb (C) : the amount of charge that is transferred through the cross section of a wire in 1 second when the current in the wire is 1 Ampere.

In this unit, $K \equiv \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9\, Nm^2/C^2$

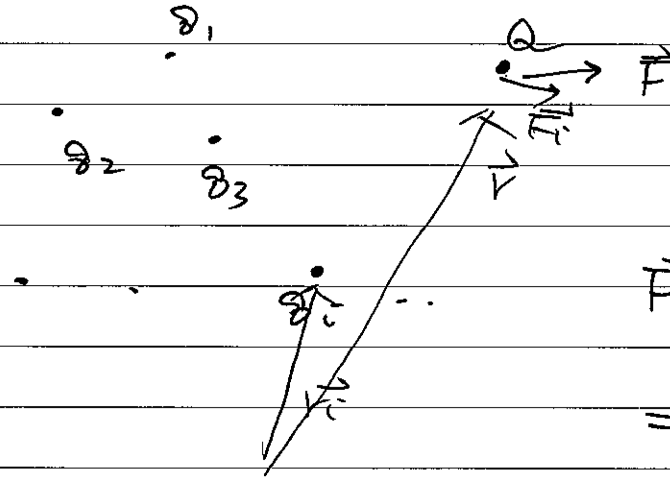
with $\epsilon_0 \equiv$ permittivity of free space
$= 8.85 \times 10^{-12} \frac{C^2}{Nm^2}$

There are other unit systems such as
Gaussian, $K=1$, Heaviside-Lorentz, $K = \frac{1}{4\pi}$
we shall not use them in this (elementary particle physics) course.

Principle of superposition

For static charges, it's found that the interaction between any two charges is not affected by the presence of others.

Therefore, if Q is a test charge, the net force that acts on Q due to $q_1, q_2, \ldots q_i, \ldots$ (source charges) is sum of individual force $\vec{F_i}$ that acts on Q



$$\vec{F} = \vec{F_1} + \vec{F_2} + \cdots$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q_1 Q}{|\vec{r}-\vec{r_1}|^3}(\vec{r}-\vec{r_1}) \quad \text{--- (2)}$$

$$+ \frac{1}{4\pi\epsilon_0}\frac{q_2 Q}{|\vec{r}-\vec{r_2}|^3}(\vec{r}-\vec{r_2})$$

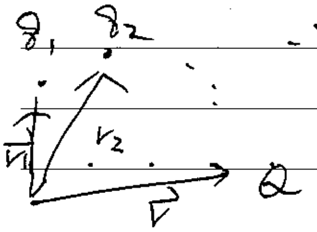
$$+ \cdots + \frac{1}{4\pi\epsilon_0}\frac{q_i Q}{|\vec{r}-\vec{r_i}|^3}(\vec{r}-\vec{r_i}) + \cdots$$

This is the principle of superposition, which is also valid generally when charges are moving and not static.

The principle of superposition, in particular, implies that the system is linear (in q_i) and we shall see that a lot of simplification in describing EM interaction is due to this feature.

Electric field

As we discussed in the introduction, the reality behind the Coulomb's law is that charges has already charge the property ~~the presence of~~ of the space so that when another charge, called test charge Q, is introduced at $\vec{r}$, it is affected by the field $\vec{E}$ at $\vec{r}$ so that a force, $\vec{F}_Q = Q\vec{E}(\vec{r})$, acts on Q.



$$\therefore \vec{E}(\vec{r}) = \frac{\vec{F}_Q(\vec{r})}{Q} = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\vec{r}-\vec{r}_i|^3}(\vec{r}-\vec{r}_i) \quad \text{---(3)}$$

Clearly, indeed such defined $\vec{E}(\vec{r})$ depends only

on source charges and doesn't depend

on Q. $\therefore$ ---- $\vec{E}(\vec{r})$ is consistent with the idea of properties

produced at $\vec{r}$ due to source charges.

And this is possible only when the

superposition principle works.

Continuous charge distribution

In reality, charges may have sizes.

and magnitudes of charges are

quantized $Q = Ne$, i.e., not all values

of Q are allowed.

However, it is still convenient to consider

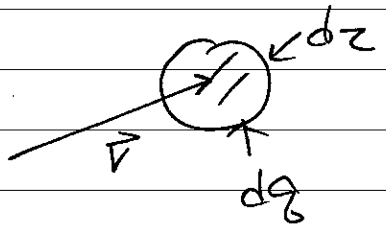
charges as continuous charge distribution.

In this case, one first realizes that each

mathematical space point (dispacement) $\vec{r}$ has

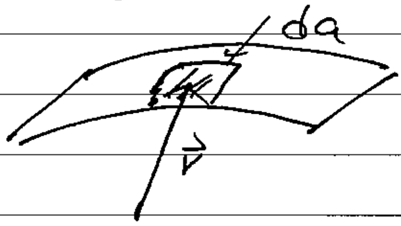
no size. The charge density (3D) at any

point $\vec{r}$ is defined by taking a small volume $d\tau$ centered at $\vec{r}$ and dq is the total charge inside $d\tau$, then



$$\rho(\vec{r}) \equiv \lim_{d\tau \to 0} \frac{dq(\vec{r})}{d\tau}$$

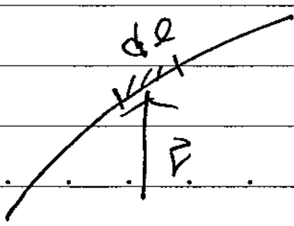
Similarly, if charges distribute only on surfaces, one considers a small area da centered at $\vec{r}$, with the total charge inside da being dq, the surface charge density (charge/area).



$$\sigma(\vec{r}) \equiv \lim_{da \to 0} \frac{dq(r)}{da}$$

(面密度)

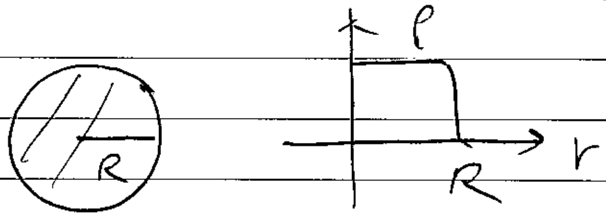
If the charge is spread out along a line, one considers a small segment dl with dq inside, the line charge density λ (charge per unit length)



$$\lambda(\vec{r}) \equiv \lim_{dl \to 0} \frac{dq(r)}{dl}$$

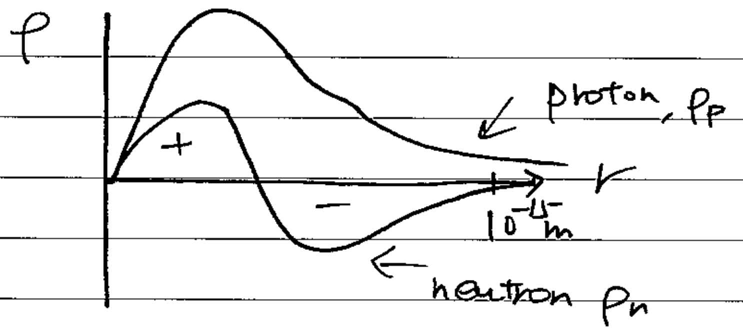
The continuous charge distribution can describe charges even when the charged particle has a finite size & charge is quantized.

In this case, one may have



R = size of charge $\qquad \int \rho \, d\tau = \int dq = Ne$

example : neutron & proton



$\int \rho_n \, d\tau = 0$

$\int \rho_p \, d\tau = |e|$

Point charge & Dirac Delta function

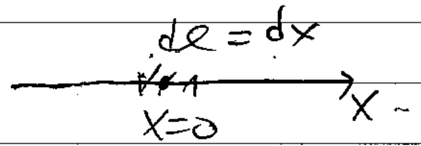
A natural question arises : do all charged particles have finite sizes ?

It turns out that some elementary particles, $r < 10^{-22}$ m appears to be point-like (such as electrons). They may eventually have structures (such as string --) but for practical purposes, it is

Convenient to treat them as <u>idealized mathematical points</u> and are termed as <u>point charges</u>. In order to describe charge densities for these charges, Dirac devised a mathematical object, called Dirac delta function, to the charge density.

To describe point charges with general magnitudes, it's suffice to describe <u>charges with magnitude being 1</u>.

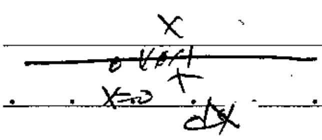
For point charges on 1D, Clearly, Since there is no size, $dq = 1$ for any $d\ell$ including the charge. If the



charge is located at $x=0$, we have

$$\lambda(x=0) = \lim_{d\ell \to 0} \frac{dq}{dx} = \lim_{dx} \frac{1}{dx} = \infty$$

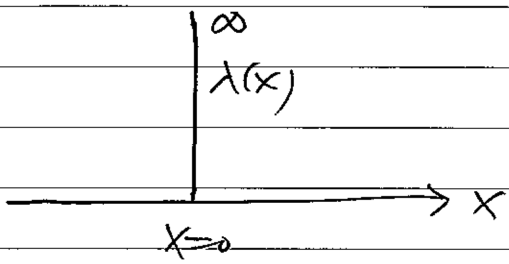
However, at $x \neq 0$, if $d\ell$ ($=dx$) is small enough, $dq = 0$



$$\lambda(x \neq 0) = \lim_{d\ell \to 0} \frac{dq}{dx} = 0$$

Therefore, the charge density $\lambda(x)$ is very singular : $\lambda(x=0)=\infty$

$\lambda(x\neq 0)=0$



However, $\because$ total charge $dq = 1$,

$$\therefore \int dq = \int dl \cdot \lambda(x) = 1$$

$$\therefore \int dx\, \lambda(x) = 1 \qquad (dl = dx)$$

Such function is called one-dimensional Dirac delta function, denoted by $\delta(x)$ or $\delta(x-a)$ if the charge is located at $x=a$.

Clearly, for 1D point charges of magnitude q, $\lambda(x) = q\,\delta(x)$

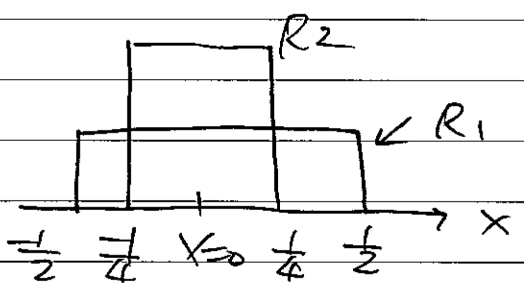
The Dirac delta function is actually not a function at all since its value at $x=0$ (or $x=a$) is not finite.

In the mathematical language, it is known as a generalized function or distribution (linear functionals, see below)

It can be considered as <u>the limit of a sequence of functions</u>

For example, one can consider $\delta(x)$ as limiting of

$$R_n(x) = \begin{cases} n & |x| \le \frac{1}{2n} \\ 0 & |x| > \frac{1}{2n} \end{cases} \quad n \to \infty$$



$$\int R_n(x)\,dx = 1$$

$$\delta(x) = \lim_{n\to\infty} R_n(x)$$

Using $R_n(x)$, one can find

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx \quad \text{for any ordinary function } f(x)$$

$$= f(0) \quad \cdots \text{④}$$

Formally, this is obtained by considering

$\lim_{n\to\infty} \int f(x) R_n(x)\,dx$. Since R_n is non-vanishing only at $x \simeq 0$ $n\to\infty$, $\therefore$ one gets $\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) R_n(x)\,dx = f(0)$

$$\therefore \int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$

Similarly, one can shift $\delta(x)$ to $\delta(x-a)$

and obtains $\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$ -- (5)

Note that δ function is defined to be independent of which sequence being taken under the limit $n\to\infty$, as long as it gives $\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\,R_n(x)\,dx = f(0)$

$\therefore \lim_{n\to\infty} R_n(x)$ is defined as $\delta(x)$.

Mathematically, distribution is defined as a mapping of $f \to R$ for any f

Eq. (5) is such an example : it depends on the whole function $f(x)$ and maps to $f(a)$. This is known as functionals (汎函數) in mathematics example. $\delta(kx) = \frac{1}{|k|}\delta(x)$

consider $I = \int_{-\infty}^{\infty} f(x)\,\delta(kx)\,dx$

change of variable. $y = kx$

$k>0$. $I = \int_{-\infty}^{\infty} f(\frac{y}{k})\,\delta(y)\,\frac{1}{k}\,dy$

$= \frac{1}{k}\int_{-\infty}^{\infty} f(y/k)\,\delta(y)\,dy = \frac{1}{k}f(0)$

$k<0$ $I = \int_{\infty}^{-\infty} f(\frac{y}{k})\,\delta(y)\,\frac{1}{k}\,dy$

$= -\int_{-\infty}^{\infty} \frac{1}{k} f(y/k)\,\delta(y) = -\frac{1}{k}f(0)$.

$\therefore I = \frac{f(0)}{|k|} = \frac{1}{|k|}\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx$

$\therefore \delta(kx) = \frac{1}{|k|}\delta(x)$

<u>Dirac delta function in higher dimensions</u>

Clearly, 1D $\delta(x)$ can be genaralized to 2D & 3D, denoted by $\delta(\vec{r}-\vec{a})$ in general.

Here $\vec{r}$ & $\vec{a}$ are 2D or 3D vectors depending on dimensions involved.

Eq. (5) is generalized to

$$\int f(\vec{r})\,\delta(\vec{r}-\vec{a})\,d\tau = f(\vec{a}) \quad \text{--- (6)}$$

or $\int f(\vec{r})\,\delta(\vec{r}-\vec{a})\,\underset{\substack{\uparrow \\ \text{area}}}{dA} = f(\vec{a})$

Now, since $d\tau = dx\,dy\,dz$ &

$$\int f(\vec{r})\,\delta(\vec{r}-\vec{a})\,d\tau = \int dx \int dy \int dz\, f(x,y,z)\,\delta(\vec{r}-\vec{a})$$

$$= f(a_x, a_y, a_z), \quad \vec{a} = (a_x, a_y, a_z),$$

$\delta(\vec{r}-\vec{a})$ picks up $x = a_x$, $y = a_y$, & $z = a_z$.

$\therefore$ $f(a_x, a_y, a_z) = \int dx\, f(x, a_y, a_z)\,\delta(x-a_x)$

$$= \int dx\,\delta(x-a_x) \int dy\, f(x,y,a_z)\,\delta(y-a_y)$$

$$= \int dx \int dy\, \delta(x-a_x)\,\delta(y-a_y) \int dz\, f(x,y,z)\,\delta(z-a_z)$$

$\therefore$ $\delta(\vec{r}-\vec{a}) = \delta(x-a_x)\,\delta(y-a_y)\,\delta(z-a_z)$

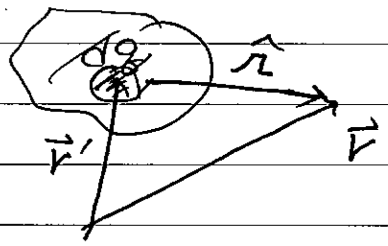
Similarly, for 2D, $\delta(\vec{r}) = \delta(x)\,\delta(y)$

$\delta(\vec{r}-\vec{a}) = \delta(x-a_x)\,\delta(y-a_y)$

Electric field of continuous charge distribution

Given a charge distribution, one can find the corresponding $\vec{E}(\vec{r})$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}^2}\hat{\mathscr{r}}\, dq(\vec{r}\,')$$



$$\begin{aligned} dq(\vec{r}\,') &= \rho(\vec{r}\,')\, d\tau' && 3D \\ &= \sigma(\vec{r}\,')\, da' && 2D \\ &= \lambda(\vec{r}\,')\, dl' && 1D \end{aligned}$$

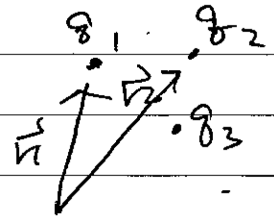
$$\therefore \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}^2}\hat{\mathscr{r}}\,\rho(\vec{r}\,')\, d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{1}{|\vec{r}-\vec{r}\,'|^3}(\vec{r}-\vec{r}\,')\rho(\vec{r}\,')\, d\tau' \quad \text{--- (7)}$$

or.
$$\begin{cases} = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \frac{1}{\mathscr{r}^2}\hat{\mathscr{r}}\,\sigma(\vec{r}\,')\, da' \\ = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \frac{1}{\mathscr{r}^2}\hat{\mathscr{r}}\,\lambda(\vec{r}\,')\, dl' \end{cases}$$

Example:

For point charges, $(q_1, q_2, \ldots)$ $\rho(\vec{r}) = q_1\delta(\vec{r}-\vec{r}_1) + q_2\delta(\vec{r}-\vec{r}_2) + \cdots + q_i\delta(\vec{r}-\vec{r}_i) + \cdots$



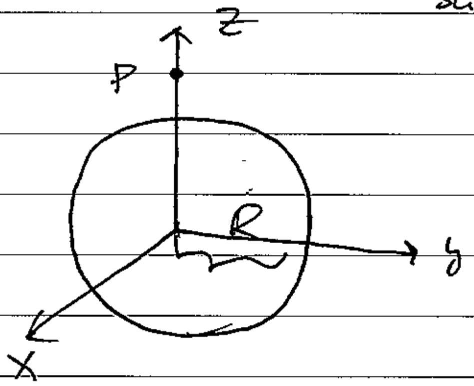
$$= \sum_i q_i\,\delta(\vec{r}-\vec{r}_i)$$

$$\vec{E}(\vec{r}\,') = \frac{1}{4\pi\epsilon_0}\int dz' \frac{1}{|\vec{r}-\vec{r}\,'|^3}(\vec{r}-\vec{r}\,')\left(q_1\delta(\vec{r}\,'-\vec{r}_1)+q_2\delta(\vec{r}\,'-\vec{r}_2) + \cdots\right)$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{|\vec{r}-\vec{r}_1|^3}(\vec{r}-\vec{r}_1) + \frac{q_2}{|\vec{r}-\vec{r}_2|^3}(\vec{r}-\vec{r}_2) + \cdots\right)$$

which recovers eq. ③

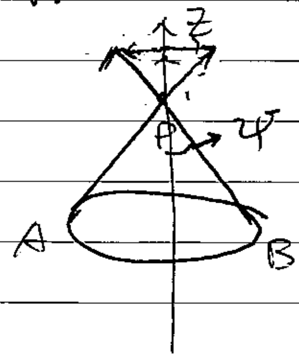
Example Uniformly ^surface^ charged sphere



surface charge density $= \sigma$

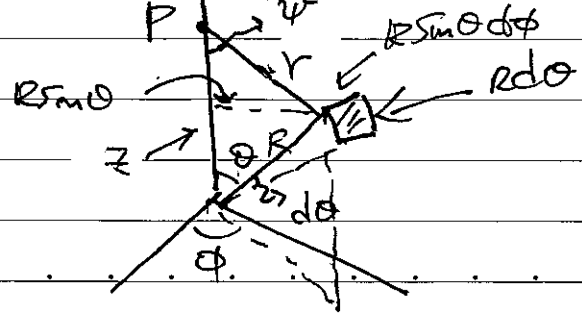
find E_p

Sol: Consider a ring on sphere, clearly



any pair of ^opposite^ points (A & B), their E_x & E_y's cancel against each other

∴ One only needs to consider E_z.



For each point on sphere, it's convenient to use spherical Coordinate (R, θ, ϕ)

The small area $= Rd\theta \cdot R\sin\theta\, d\phi$

$\therefore dq = R^2 \sin\theta\, d\theta\, d\phi \cdot \sigma$

The E_z contributed by dq

$$dE_z = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^3}\cos\psi$$

$$\therefore E_p = \int dE_z = \frac{1}{4\pi\epsilon_0}\int_0^{\pi}\sin\theta\, R^2 d\theta \int_0^{2\pi} d\phi\ \sigma \times \frac{\cos\psi}{r^2}$$

Now, one needs to express r & ψ in terms of ϕ & θ:

$$\because r^2 = R^2 + z^2 - 2Rz\cos\theta$$

$$r\cos\psi = z - R\cos\theta$$

$$\therefore \frac{\cos\psi}{r^2} = \frac{1}{r^3}(z - R\cos\theta) = \frac{z - R\cos\theta}{(R^2+z^2-2Rz\cos\theta)^{3/2}}$$

$$E_p = \frac{\sigma R^2}{4\pi\epsilon_0} \times 2\pi \int_0^{\pi} \frac{(z - R\cos\theta)\sin\theta\, d\theta}{[R^2+z^2-2Rz\cos\theta]^{3/2}}$$

$\uparrow$

$\int_0^{2\pi} d\phi = 2\pi$ Set $\cos\theta = \omega$ $\sin\theta\, d\theta = -d\omega$

$$\therefore \int_0^{\pi} \frac{(z - R\cos\theta)\sin\theta\, d\theta}{[R^2+z^2-2Rz\cos\theta]^{3/2}} = \int_{-1}^{1} \frac{z - R\omega}{[R^2+z^2-2Rz\omega]^{3/2}}\, d\omega$$

$$= \frac{1}{Z^2} \left. \frac{Z\omega - R}{(R^2+Z^2-2RZ\omega)^{1/2}} \right|_{\omega=-1}^{\omega=1}$$

$$= \frac{1}{Z^2}\left(\frac{Z-R}{|Z-R|} + \frac{Z+R}{|Z+R|} \right)$$

$$\therefore E_p = \frac{1}{4\pi\epsilon_0} \frac{2\pi R^2 \sigma}{Z^2} \left\{ \frac{Z-R}{|Z-R|} + \frac{Z+R}{|Z+R|} \right\}$$

$Z > R.$ $\quad \frac{Z-R}{|Z-R|} = 1, \quad \frac{Z+R}{|Z+R|} = 1$

$$E_p = \frac{1}{4\pi\epsilon_0} \frac{2\pi R^2 \sigma}{Z^2} \times 2 \qquad \vec{E}_p = \frac{1}{4\pi\epsilon_0} \frac{q}{Z^2} \hat{z}$$

$\because 4\pi R^2 \sigma = q$

$= \vec{E}$ field as if q is concentrated at $\vec{r} = 0$

$|Z| < R$ $\quad \frac{Z-R}{|Z-R|} = -1, \quad \frac{Z+R}{|Z+R|} = 1$

$$\vec{E}_p = 0$$

Field lines and Gauss's law

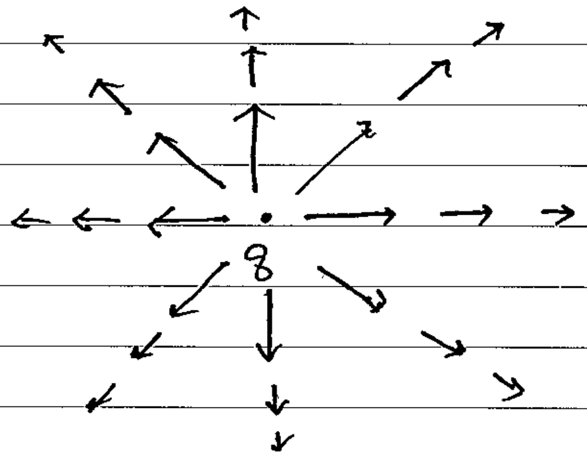
Using integrations to find $\vec{E}$ field, as shown in the above, is generally quite involved mathematically and may not be easily done. Therefore, it would be desired to have more friendly methods to compute $\vec{E}$.

For this purpose, it is found that plotting out $\vec{E}(\vec{r})$ would be useful.

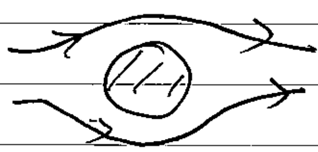
Consider the $\vec{E}$ field due to a charge q at $\vec{r}=0$,

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

At each point, one can draw an arrow in proportional to $\vec{E}$, and obtains

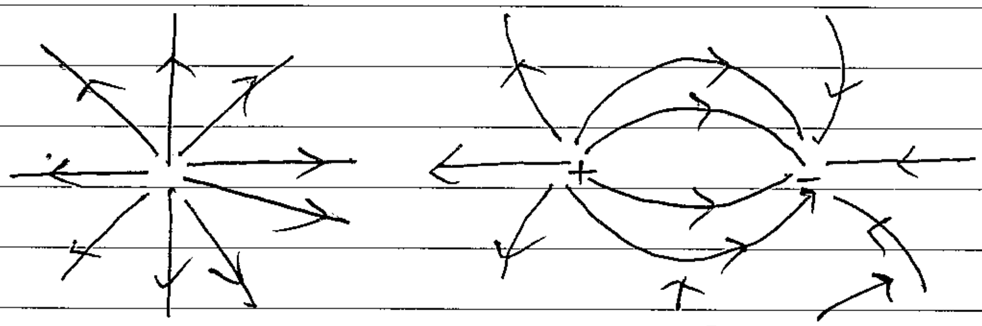


Clearly, one sees that similar to flow of water, these arrows represent some kind of flow.

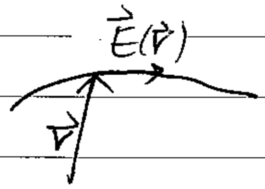


flow of water

To mimic the flow of water, it is useful to connect all arrows to make it continuous. However, in order to be able to represent the strength of E, in the place with strong E, the density of lines increases and is in proportional to $|\vec{E}|$. Therefore, one gets



Formally, the curve of each field line is determined by $d\vec{r} \parallel \vec{E}(\vec{r})$



i.e. $$\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z} \quad \text{--- (8)}$$

which can be used to solve the equation for each curve.

One sees that field lines start from positive charges and end at negative charges.

Field lines don't intersect with other as at each point, there is only one direction of $\vec{E}(\vec{r})$!

To characterize density of field lines, it's useful to define the flux of $\vec{E}$

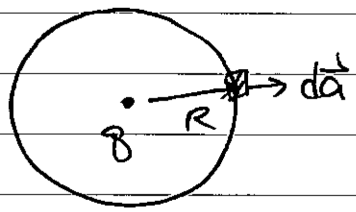
through a surface S by

$$\Phi_E \equiv \int_S \vec{E}\cdot d\vec{a} \quad \cdots\cdots (9)$$

which measures number of field lines that pass through S.

The flow picture apparently suggests that if one computes the total flux of a closed surface surrounding the charge, it measures the total charge inside the surface. — essence of the Gauss's law.

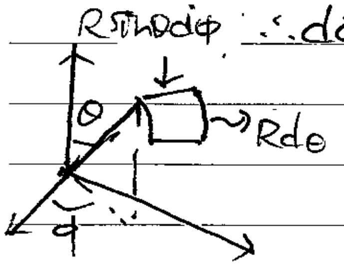
Take a point charge as an example, if the surface is a spherical surface of radius R, one gets the total flux



~~one~~

$$\Phi_E = \frac{1}{4\pi\epsilon_0}\oint \frac{q}{R^2}\hat{R}\cdot d\vec{a}$$

$\therefore d\vec{a} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{R}$, $\sin\theta\, d\theta\, d\phi \equiv d\Omega$ solid angle



$$\therefore \hat{R}\cdot d\vec{a} = R^2 \sin\theta\, d\theta\, d\phi$$

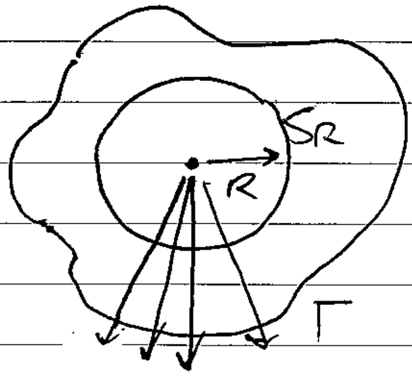
$$\Phi_E = \frac{q}{4\pi\epsilon_0}\underbrace{\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sin\theta}_{4\pi \text{ (area of unit sphere)}} = q/\varepsilon_0$$

$\therefore \quad \oint_{S_R} \vec{E} \cdot d\vec{a} = q/\varepsilon_0 \quad \text{- - -} \; (10)$

which is independent of R !

S_R = spherical surface of radius R
centered at q

Now, Eq. (10) is also valid for any other surfaces that enclose q because for any surfaces Γ, one can find a spherical surface that is either enclosed by Γ or encloses Γ.



From the flow picture, it's clear that what leaves . . also enters Γ.

There is no sink/source between S_R & Γ.

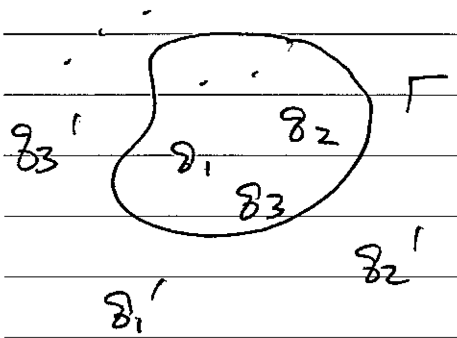
$\therefore \oint_{\Gamma} \vec{E} \cdot d\vec{a} = \oint_{S_R} \vec{E} \cdot d\vec{a}$

Using eq. (10), one obtains

$$\oint_{\Gamma} \vec{E} \cdot d\vec{a} = q/\varepsilon_0 \quad \text{- - -} \; (11)$$

for any Γ that encloses q !

Eq. (11) can be easily generalized to the case with many charges. Suppose Γ encloses $q_1, q_2, q_3, \ldots$. By the principle of superposition, the electric field $\vec{E}$ on Γ is the summation of $\vec{E}_i$ from q_i (inside) and $\vec{E}_i'$ from q_i' (outside).



$$\vec{E} = (\vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_n \cdots) + (\vec{E}_1' + \vec{E}_2' + \cdots + \vec{E}_n' + \cdots)$$

$$\therefore \oint_\Gamma \vec{E} \cdot d\vec{a} = \sum_i \oint_\Gamma \vec{E}_i \cdot d\vec{a} + \sum_i \oint_\Gamma \vec{E}_i' \cdot d\vec{a}$$

$$\because \oint \vec{E}_i' \cdot d\vec{a} = 0$$

$$\oint \vec{E}_i \cdot d\vec{a} = q_i/\varepsilon_0$$

$$\therefore \oint_\Gamma \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} \sum_i q_i = \frac{1}{\varepsilon_0} Q_{\text{enclosed}} \quad (12)$$

Eq. (12) is the integral form of the Gauss's law.

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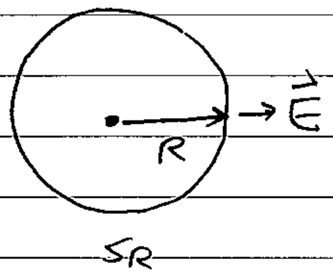
Application of Gauss's law.

The Gauss's law (when combined with symmetries of space) is equivalent to the Coulomb law.

For instance, for a given point charge q at $\vec{r}=0$, the isotropy of space implies that the electric field $\vec{E}$ must align with $\hat{R}$, and depends only on R.



Therefore, from the Gauss's law $\oint_{S_R} \vec{E}\cdot d\vec{a} = \frac{q}{\varepsilon_0}$ and $\vec{E} = E(R)\hat{R}$,

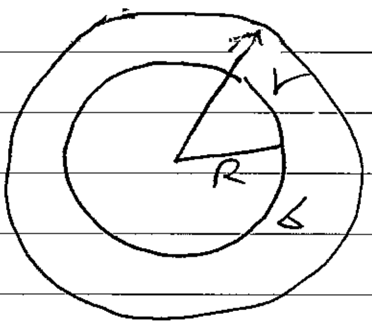
one gets $\oint_{S_R} \vec{E}\cdot d\vec{a} = \oint_{S_R} E(R)\, R^2 \sin\theta\, d\theta\, d\phi$

$$= R^2 E(R)\cdot 4\pi = \frac{q}{\varepsilon_0}$$

$$\therefore E(R) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}, \quad \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\hat{R}$$

One recovers the Coulomb law.

Similarly, using the Gauss's law, one can easily find the problem of E field due to a uniformly charged spherical surface:



for $r > R$, one consider a concentric spherical surface of radius r.

Again, the electric field at $\vec{r}$ must be parallell to $\hat{r}$, $\vec{E} = E(r)\hat{r}$.

Since $\oint_{S_r} \vec{E} \cdot d\vec{a} = \oint_{S_r} E(r) \cdot r^2 \sin\theta d\theta d\phi$

$$= r^2 E(r) \cdot 4\pi = \frac{1}{\epsilon_0} Q_{enclosed}$$

$$= \frac{1}{\epsilon_0} \times \underbrace{4\pi R^2 \sigma}_{q}$$

$$\therefore E(r) = \frac{R^2\sigma}{\epsilon_0} \frac{1}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}.$$

while for $r < R$, $\because Q_{enclosed} = 0$

$$\therefore E(r) = 0$$

One recovers previous results by direct integration.

As we can see, if the problem, for

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evaluating $\vec{E}$ field has symmetries, one can use the Gauss's law to compute $\vec{E}$ and the calculation effort can be greatly reduced.

There are three kinds of symmetry that work.

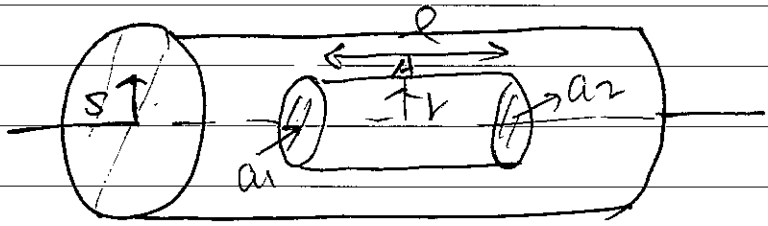
(i) Spherical symmetry (Gauss surface = Concentric sphere)

(ii) Cylindrical symmetry (Gauss surface = coaxial cylinder)

Example

A long cylinder carries a charge density $\rho = ks$. s = distance from the axis, k = some constant.

Find the electric field inside this cylinder.



For a point A deep inside the cylinder, the electric field is along the direction of $\hat{r}$.

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This is true as long as A is far from two ends of the cylinder.

$\therefore \quad \vec{E} = E(r)\,\hat{r}$

Consider a closed cylindrical surface shown in the figure. According to the Gauss's law, one has

$$\oint \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enclosed}$$

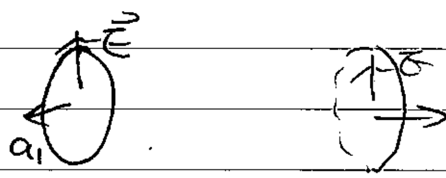
$$Q_{enclosed} = \int \rho \, dz$$

$$= \int k s' \cdot s' ds' \underbrace{d\phi}_{2\pi} \underbrace{dz}_{l}$$

$$= 2\pi l k \int_0^r s'^2 ds'$$

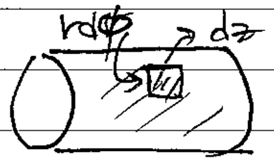
$$= \frac{2\pi l k}{3} r^3$$

On the other hand, $\because \int_{\vec{a}_1} \vec{E}\cdot d\vec{a} = 0 \quad \vec{E} \perp \vec{a}_1$



$$\int_{\vec{a}_2} \vec{E}\cdot d\vec{a} = 0 \quad \vec{E} \perp \vec{a}_2$$

for lateral surface



$$\int \vec{E}\cdot d\vec{a} = \int E(r)\, r d\phi\, dz = E(r)\cdot 2\pi r l$$

$$\therefore\ 2\pi r E(r)\, l = \frac{1}{\epsilon_0} Q_{enclosed}$$

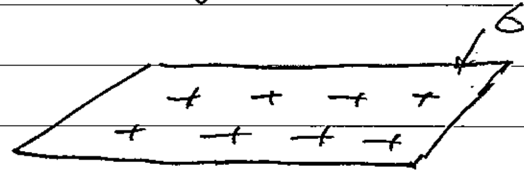
$$= \frac{1}{\epsilon_0} \frac{2\pi l k}{3} r^3$$

$$\therefore\ E(r) = \frac{1}{3\epsilon_0} k r^2$$

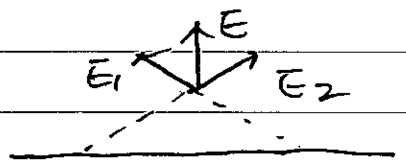
$$\therefore\ \vec{E} = \frac{1}{3\epsilon_0} k r^2 \hat{r}$$

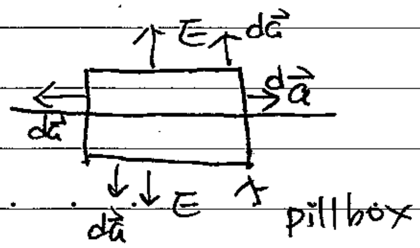
(iii) Plane symmetry (Use pill box as the Gauss surface)

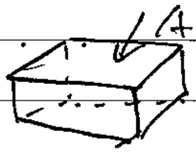
Example: Find the electric field of an infinite plane with a uniform surface charge density σ



Sol. First, from consideration of symmetries, the electric field E must be perpendicular to the plane. Therefore, one considers a closed surface shown in the left figure: a pillbox







pill box

If the area of top and bottom surfaces for the pill box is A, then

$$\oint \vec{E} \cdot d\vec{a} = 2EA$$

The lateral surfaces don't contribute as $\vec{E} \perp d\vec{a}$ for these surfaces.

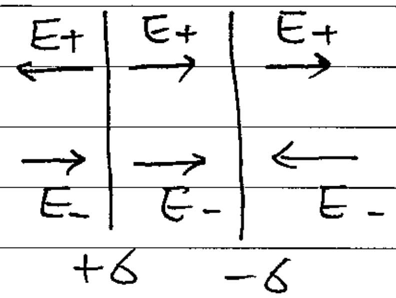
$$\because \oint \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enclosed} = \frac{1}{\varepsilon_0} A\sigma$$

$$\therefore 2EA = \frac{1}{\varepsilon_0} A\sigma, \quad E = \frac{\sigma}{2\varepsilon_0}$$

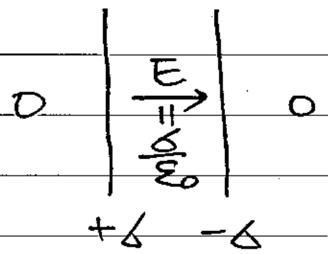
In general, even though the usage of the Gauss's law is restricted to cases with symmetries, using the principle of superpositions, one can evaluate $\vec{E}$ of systems with combined objects of different symmetries.

example. two infinite parallel planes carry equal but opposite uniform charge density $\pm\sigma$. Find $\vec{E}$ in all space points.

For $+\sigma$ plane, one has $E_+ = \frac{\sigma}{2\epsilon_0}$ pointing outwards, while for $-\sigma$ plane, $E = E_- = \frac{\sigma}{2\epsilon_0}$ point inwards as follows



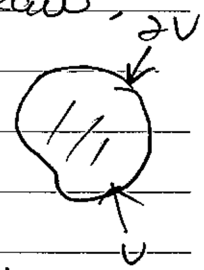
Using the principle of superposition, $E = E_+ + E_-$, one obtains



Differential form of the Gauss's law

From the integral form of the Gauss's law, one has

$$\oint_{\partial V} \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enclosed}.$$



On the other hand, if $\rho = \rho(\vec{r})$ is the charge density at $\vec{r}$, $Q_{enclosed} = \int_V \rho(\vec{r})\, d\tau$ -- (13)

From the divergence theorem, one has

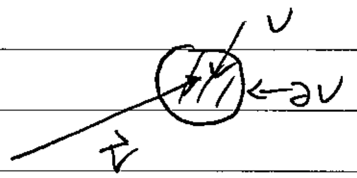
$$\oint_{\partial V} \vec{E} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{E} \, d\tau \quad \text{--- (14)}$$

Combining eqs. (13) and (14), we obtain

$$\int_V \vec{\nabla} \cdot \vec{E} \, d\tau = \int_V \frac{1}{\varepsilon_0} \rho(\vec{r}) \, d\tau .$$

$$\text{L.} \quad (15)$$

Eq (15) is valid for any V. If we choose V surrounding $\vec{r}$ and take $V \to 0$, one gets



$$\vec{\nabla} \cdot \vec{E} \cdot \cancel{V} = \frac{1}{\varepsilon_0} \rho(\vec{r}) \cancel{V}$$

i.e.

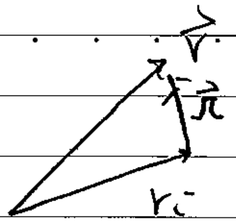
$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho(\vec{r}) \quad \text{--- (16)}$$

This is the Gauss's law in differential form.

Divergence of E

As we discussed, for a point charge q located at $\vec{r}_i$, the charge density is

$$\rho(\vec{r}) = q_i \, \delta(\vec{r} - \vec{r}_i) \quad \text{--- (17)}$$



Since the electric field due to q_i at $\vec r$ is

$$\vec E = \frac{1}{4\pi\epsilon_0}\frac{q_i}{|\vec r-\vec r_i|^3}(\vec r-\vec r_i)$$

$$= \frac{q_i}{4\pi\epsilon_0}\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \quad \cdots (18)$$

Combining eqs. (17), (18) and (16), one obtains.

$$\frac{q_i}{4\pi\epsilon_0}\vec\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{q_i}{\varepsilon_0}\delta(\vec{\mathscr{r}})$$

Therefore, $\vec\nabla\cdot\dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 4\pi\,\delta(\vec{\mathscr{r}})$ --- (19)

where $\delta(\vec{\mathscr{r}}) = \delta^3(\vec r-\vec r_i)$ and we shall omit "3" and understand it means 3D Dirac delta function. acting on $\vec r$!

More precisely, $\vec\nabla\cdot\dfrac{(\vec r-\vec r_i)}{4\pi|\vec r-\vec r_i|^3} = \delta(\vec r-\vec r_i)$ --- (20)

The function $\frac{1}{4\pi}\frac{\vec r-\vec r_i}{|\vec r-\vec r_i|^3}$ that determines the electric field due to a unit charge is known as the Green's function. $G(\vec r,\vec r_0) = \dfrac{\vec r-\vec r_i}{4\pi|\vec r-\vec r_i|^3}$

which can be also viewed as a description of how $\vec{E}$ field propagates from $\vec{r_0}$ to $\vec{r}$ from a unit charge.

Due to the principle of superposition, once G is known, one knows $\vec{E}(\vec{r})$ from the knowledge of charge density:

Eq. 7

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\rho(\vec{r}')\,d\tau'$$

$$= \int G(r,\vec{r}')\frac{\rho(\vec{r}')}{\varepsilon_0}\,d\tau' \quad \cdots \text{(21)}$$

It's easy to check that $\vec{E}$ in eq. (21) is indeed a solution to eq. (6):

$$\vec{\nabla}\cdot\vec{E} = \int\left[\vec{\nabla}\cdot G(r,\vec{r}')\right]\frac{\rho(\vec{r}')}{\varepsilon_0}\,d\tau'$$

$$= \int \delta(\vec{r}-\vec{r}')\frac{\rho(\vec{r}')}{\varepsilon_0}\,d\tau' = \frac{1}{\varepsilon_0}\rho(\vec{r}).$$

Eq. (21) states the total E field at $\vec{r}$ is summation of contributions from all charges at $\vec{r}$. The propagation of $\vec{E}$ from charges to $\vec{r}$ is G which does not depend on charge density only.

~~depends on positions $\vec{r}$ and $\vec{r_i}$.~~

The response ($\vec{E}$) at $\vec{r}$ is linear in charge density. This feature is generally true for all linear systems.

Example: Direct check $\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 4\pi\delta(\vec{\mathscr{r}})$.

$\vec{\mathscr{r}} \neq 0$ $\quad \vec{\nabla}\cdot\frac{\vec{\mathscr{r}}}{\mathscr{r}^2} = 0$

Take $\vec{r_i} = 0$ $\quad \therefore \vec{\nabla}\cdot\frac{\vec{r}}{r^3} = 0$

$$\vec{\nabla}\cdot\frac{\vec{r}}{r^3} = \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{\partial x}{\partial x}\frac{1}{r^3} + x\frac{\partial \frac{1}{r^3}}{\partial x}$$

$$= \frac{1}{r^3} - x\cdot\frac{3/2}{r^5}\frac{\partial x^2}{\partial x} = \frac{1}{r^3} - \frac{3x^2}{r^5}$$

$$\therefore \vec{\nabla}\cdot\frac{\vec{r}}{r^3} = \frac{3}{r^3} - \frac{3}{r^5}(x^2+y^2+z^2) = 0 \quad \text{if } r \neq 0$$

(One can also calculate it using spherical coordinates)

This derivation is not valid for $r=0$ as $\frac{\vec{r}}{r^3}$ is not defined at $r=0$.

For $r=0$, one needs to replace $\frac{\vec{r}}{r^3}$ by a sequence of functions that approach $\frac{\vec{r}}{r^3}$.

We consider $\frac{\vec{r}}{r^3} \rightarrow \frac{\vec{r}}{(r^2+a^2)^{3/2}}$ and take $a\to 0$ at the end.

$$\vec{\nabla}\cdot\frac{\vec{r}}{(r^2+a^2)^{3/2}} = \frac{\partial x}{\partial x}\frac{1}{(r^2+a^2)^{3/2}} + \frac{\partial y}{\partial y}\frac{1}{(r^2+a^2)^{3/2}}$$

$$+\frac{\partial z}{\partial z}\frac{1}{(r^2+a^2)^{3/2}}$$

$$+\, x\frac{\partial}{\partial x}\left(\frac{1}{(r^2+a^2)^{3/2}}\right) + \cdots$$

$$= \frac{3}{(r^2+a^2)^{3/2}} - \left[x\,\frac{3}{2}\frac{1}{(r^2+a^2)^{5/2}}\frac{\partial r^2}{\partial x} + y\,\frac{3}{2}\frac{1}{(r^2+a^2)^{5/2}}\frac{\partial r^2}{\partial y}\right.$$

$$\left. + z\,\tfrac{3}{2}\frac{1}{(r^2+a^2)^{5/2}}\frac{\partial r^2}{\partial z}\right]$$

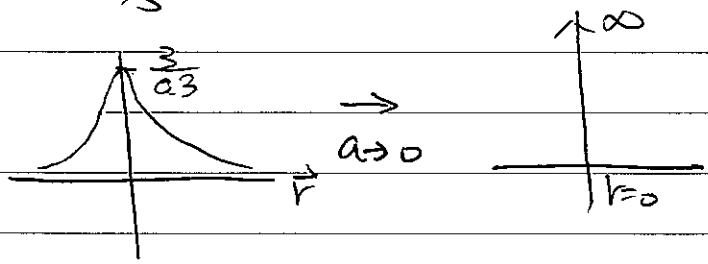
$$= \frac{3}{(r^2+a^2)^{3/2}} - \frac{3r^2}{(r^2+a^2)^{5/2}} = \frac{3a^2}{(r^2+a^2)^{5/2}} \equiv f(a)$$

(one can also use the spherical coordinate to calculate it : $\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)$)

$$\because \int d\tau\,\frac{3a^2}{(r^2+a^2)^{5/2}} = 4\pi\int_0^\infty r^2dr\,\frac{3a^2}{(r^2+a^2)^{5/2}}$$

$$\underset{x=r/a}{\uparrow}\!\!= 12\pi\underbrace{\int_0^\infty \frac{x^2dx}{(x^2+1)^{5/2}}}_{=\,1/3} = 4\pi$$

and $f(a)\to$

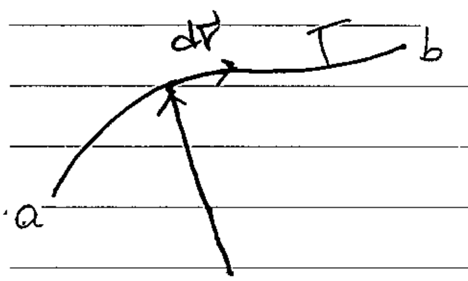


$$\therefore \lim_{a\to 0} f(a) = 4\pi\,\delta(\vec{r}) \quad \therefore\ \vec{\nabla}\cdot\frac{\vec{r}}{r^3} = 4\pi\,\delta(\vec{r})$$

$\nabla \times \vec{E}$ and path dependance of work

One of important quantity for the force is the work done by the force.

When the force is a vector field, the work done is generally path dependent.



As shown on the left figure, if one applies a force on a charged particle q to counter the electric force $q\vec{E}$ and moves q from a to b along Γ, the work that needs to be done is

$$\omega = \int_{a,\,\Gamma}^{b} \vec{F} \cdot d\vec{\ell} = -q \int_{a,\,\Gamma}^{b} \vec{E} \cdot d\vec{\ell} \quad \cdots (22)$$

From elementary consideration on conservation of energy, the work must be due to either (i) change of Kinetic energy or (ii) change of configuration.

Since the change of kinetic energy is zero, the work done is attributed to the change of configuration (as particle is moved from a to b).

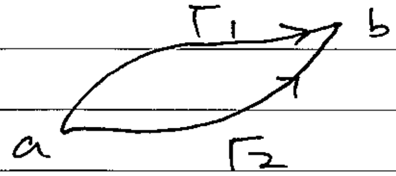
This is how the "potential energy" is introduced as one argues that the work done goes to change of the potential energy U.

$$W = U_b - U_a \quad \cdots (23)$$

However, the potential energy U is only a function of position. Hence this can be achieved only when W is path independent.

Note that U is proportional to the charge q and hence $U \equiv qV$. V is the potential $\cdots$ (24)

The path independence implies



$$\int_{a,\,\Gamma_1}^{b} \vec{E}\cdot d\vec{\ell} = \int_{a,\,\Gamma_2}^{b} \vec{E}\cdot d\vec{\ell} \quad \cdots (25)$$

(Such force is termed conservative).

The question is how to judge eq. (25). First, if eq (25) is correct, the integral is independent of Γ and can only depend on end points a and b, ∵ For fixed a,

∴ $\int_a^b \vec{E}\cdot d\vec{l}$ path-independent

$$\Rightarrow \int_a^b \vec{E}\cdot d\vec{l} = \text{a function of } b$$

Let $\int_a^b \vec{E}\cdot d\vec{l} = f(\vec{r}_b)$

$$\therefore \int_a^{b+db} \vec{E}\cdot d\vec{l} = f(\vec{r}_b + d\vec{r}_b)$$

$$f(\vec{r}_b + d\vec{r}_b) - f(\vec{r}_b) = \int_{\vec{r}_b}^{\vec{r}_b + d\vec{r}_b} \vec{E}\cdot d\vec{l}$$

taking $d\vec{r}_b \to 0$, ∵ $\int_{\vec{r}_b}^{\vec{r}_b + d\vec{r}_b} \vec{E}\cdot d\vec{l} = \vec{E}(\vec{r}_b)\cdot d\vec{r}_b$

$$f(\vec{r}_b + d\vec{r}_b) - f(\vec{r}_b) = \nabla f(\vec{r}_b)\cdot d\vec{r}_b$$

$$\therefore \nabla f(\vec{r}_b)\cdot d\vec{r}_b = \vec{E}(\vec{r}_b)\cdot d\vec{r}_b$$

$\therefore \vec{E} = \nabla f$, in fact, $f = -V$

$\therefore \int_a^b \vec{E}\cdot d\vec{l}$ path-independency $\Rightarrow \vec{E} = -\nabla V$ --- (26)

i.e. $\vec{E}$ = gradiant of (− potential)

So that $\vec{E}\cdot d\vec{l} = -\nabla V\cdot d\vec{l} = -dV$ is

a total differential --- (27)

Therefore, to check whether work done by a force is path independence, one can check whether the force can be written as a gradient : $\vec{F} = -\nabla U$

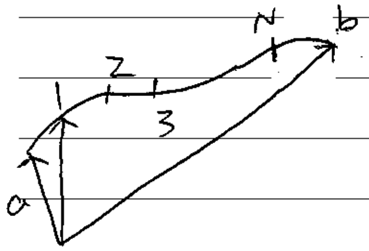
or $\vec{E} = -\nabla V$

Note that conversely, if $\vec{E} = -\nabla V$

$$\vec{E}\cdot d\vec{r} = -\vec{\nabla} V \cdot d\vec{r} = -dU \quad (= U(\vec{r}+d\vec{r}) - U(r))$$

$$\therefore \int_a^b \vec{E}\cdot d\vec{r} = -\int_a^b dU$$

$$= -(V_a - \cancel{V_1} + \cancel{V_2} - \cancel{V_1} + \cancel{V_3} - \cancel{V_2} + \cdots + V_b - V_N)$$



$= V_b - V_a$ is path independent

$$\therefore \vec{E} = -\nabla U \Leftrightarrow \int_a^b \vec{E}\cdot d\vec{\ell}$$

path - independent

L..(28)

Generally, however, it is difficult to check if $\vec{E} = -\nabla V$, therefore, one needs other more convenient criteria.

For this purpose, one first realizes that if $\int_a^b \vec{E}\cdot d\vec{\ell}$ is path - independent, eq. (25) is satisfied.

Furthermore, since $\int_{a,\Gamma}^{b} \vec{E}\cdot d\vec{\ell} = -\int_{b,\Gamma}^{a} \vec{E}\cdot d\vec{\ell}$ for any Γ

$(b \to a,\ d\vec{\ell} \to -d\vec{\ell})$.

eq.(25) is equivalently as:

$$\underset{\Gamma_1}{\int_a^b} \vec{E}\cdot d\vec{l} = \underset{\Gamma_2}{\int_a^b} \vec{E}\cdot d\vec{l} = -\underset{\Gamma_1}{\int_b^a} \vec{E}\cdot d\vec{l}$$

$$\therefore \left(\underset{\Gamma_2}{\int_a^b} + \underset{\Gamma_1}{\int_b^a}\right) \vec{E}\cdot d\vec{l} = 0$$

$$\text{i.e.}, \quad \oint_{\Gamma_1+\Gamma_2} \vec{E}\cdot d\vec{l} = 0$$

Therefore, if $\int_\Gamma \vec{E}\cdot d\vec{l}$ is path-independent for any $\Gamma \Rightarrow \oint_C \vec{E}\cdot d\vec{l} = 0$ for any closed curve C.

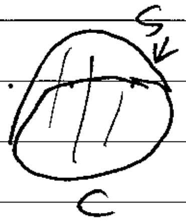
Conversely, if $\oint_C \vec{E}\cdot d\vec{l} = 0$ for any closed, $\underset{\Gamma}{\int_a^b} \vec{E}\cdot d\vec{l}$ is path-independent.

$\therefore \int_\Gamma \vec{E}\cdot d\vec{l}$ path-independent for any Γ

$\Leftrightarrow \oint_C \vec{E}\cdot d\vec{l} = 0$ for any closed curve C --- (29)

Now, according to the Stoke's theorem,

$$\oint_C \vec{E}\cdot d\vec{l} = \oint_S \nabla\times\vec{E}\cdot d\vec{s}.$$



By taking $S \to 0$, one gets

$$\vec{S}\cdot\vec{\nabla}\times\vec{E} = 0$$

$$\therefore \vec{\nabla}\times\vec{E} = 0$$

Conversely, if $\nabla\times\vec{E}=0$, one has

$$\oint_C \vec{E}\cdot d\vec{l} = 0 \text{ for any } C.$$

$$\therefore \vec{\nabla}\times\vec{E} = 0 \Leftrightarrow \oint_C \vec{E}\cdot d\vec{l} = 0 \quad \text{-- (30)}$$

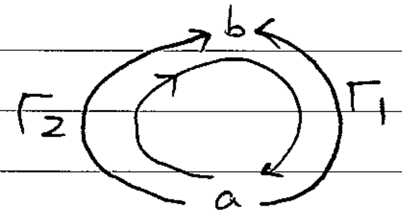
Combining eqs. (28), (29) and (30), one obtains that in a region R (simply-connected, see below), the followings are equivalent:

$$\boxed{\begin{aligned} &\int_{a\,\Gamma}^{b} \vec{E}\cdot d\vec{l} = \text{path-independent for any } \Gamma\in R \Leftrightarrow \vec{E} = -\nabla V \text{ in } R \\ &\Leftrightarrow \oint_C \vec{E}\cdot d\vec{l} = 0 \text{ for any } C\in R \Leftrightarrow \vec{\nabla}\times\vec{E} = 0 \text{ in } R. \quad \text{-- (31)} \end{aligned}}$$

<u>Physical picture</u>: There is a physical picture behind eq. (31): $\vec{\nabla}\times\vec{E}$ measures "vortex" flow in $\vec{E}$. When $\vec{\nabla}\times\vec{E}\neq 0$, there is a vortex flow. Which is the reason why

different paths need different work

as some paths are against flow, the other paths are not as shown in below:



Γ_1 is against flow $\therefore \int_{\Gamma_1} \neq \int_{\Gamma_2}$!

Therefore, if there is no vortex, works along different paths are the same.

Note that the statement

$$\vec{\nabla}\times\vec{E} = 0 \iff \int_a^b \vec{E}\cdot d\vec{l} \text{ is path-independent}$$

is only valid when the region R that $\nabla\times\vec{E}=0$ is simply-connected, i.e., there is no hole in R !

In general, when the region R has a hole, i.e., not simply-connected, $\vec{\nabla}\times\vec{E}=0$ in R does not imply $\int_a^b \vec{E}\cdot d\vec{l}$ is path-independent !

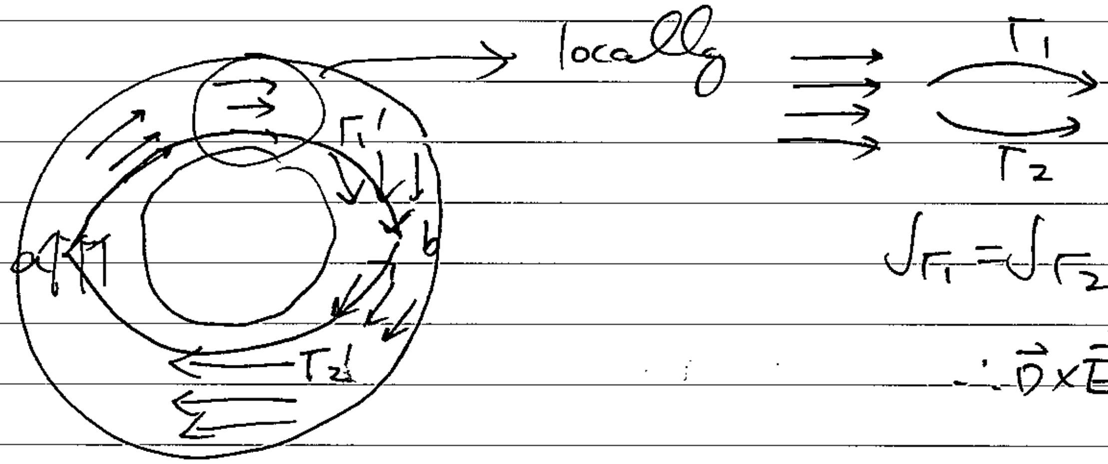
A good example is to consider the case when there is a hole, it could happen that locally $\int \vec{E}\cdot d\vec{l}$ is path-independent

so that $\oint_{\partial S} \vec{E}\cdot d\vec{l} = \int_S \nabla\times\vec{E}\cdot d\vec{a} = 0$

$\therefore \vec{\nabla}\times\vec{E} = 0$

but globally, $\int_{a\,\Gamma_1'}^{b} \vec{E}\cdot d\vec{l} \neq \int_{a\,\Gamma_2'}^{b} \vec{E}\cdot d\vec{l}$

see below:



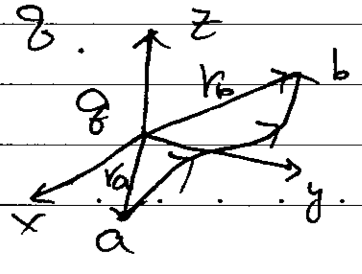
$\int_{\Gamma_1} = \int_{\Gamma_2}$ (no vortex)

$\therefore \vec{\nabla}\times\vec{E} = 0$

$\nabla\times\vec{E}$ for static charges

Due to the principle of superposition, one only needs to consider $\vec{E}$ due to a point charge q.

at O



$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$

to calculate $\int_a^b \vec{E}\cdot d\vec{l}$, one notes

that $d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

$$\therefore \vec{E}\cdot d\vec{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr$$

$$\therefore \int_a^b \vec{E}\cdot d\vec{l} = \frac{q}{4\pi\epsilon_0}\int_{r_a}^{r_b}\frac{dr}{r^2}$$

$$= \frac{-q}{4\pi\epsilon_0}\frac{1}{r}\Big|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a}-\frac{q}{r_b}\right)$$

$$\equiv -(V_b - V_a)$$

$$\therefore \quad V(r) = \frac{q}{4\pi\epsilon_0}\frac{1}{r} + \text{const}$$

Usually, by defining $V(r=\infty)=0$, one can remove the constant and gets

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \quad \text{---} \quad (32)$$

Check $-\nabla V = -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)V(r)$

$$= -\frac{q}{4\pi\epsilon_0}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{r} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} = \vec{E}$$

$$\therefore \vec{E} = -\nabla V$$

The path independence of work implies

$\vec{\nabla}\times\vec{E}=0$ which can be checked

directly: $\vec{E}=E_r\hat{r}$ using spherical coordinate

$$\vec{\nabla}\times\vec{E}=\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta E_\phi)-\frac{\partial E_\theta}{\partial\phi}\right)\hat{r}$$

$$+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\phi}-\frac{\partial}{\partial r}(rE_\phi)\right]\hat{\theta}$$

$$+\frac{1}{r}\left[\frac{\partial}{\partial r}(rE_\theta)-\frac{\partial E_r}{\partial\theta}\right]\hat{\phi}=0$$

Electric potential

As we have shown, for static charges,

$\nabla\times\vec{E}=0$. $\vec{E}=-\nabla V$

V is the electric potential.

Its unit is Newton/charge = Newton/Coulomb.
≡ Volt.

V has no absolute value as one can

add V + any constant $=V'$, V & V' will

yield the same $\vec{E}$.

Usually, one needs a reference point to
fix the constant.

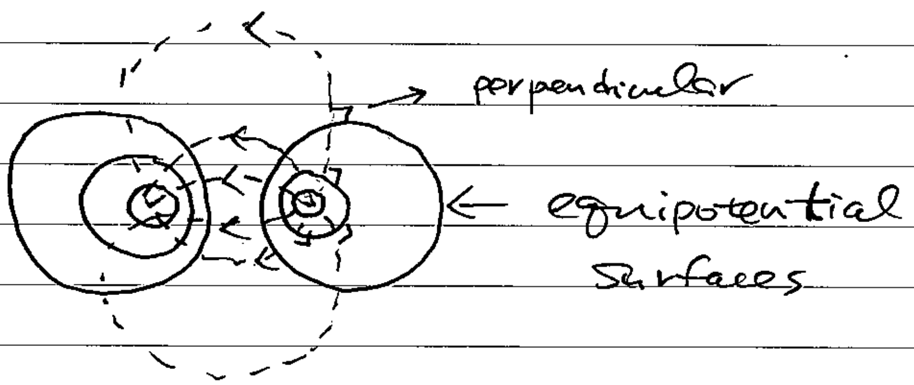
Physically, the absolute value of U has no meaning. Only the difference of U has physical meaning.

<u>Field lines</u> and <u>equipotential surfaces</u>

The relation $\vec{E} = -\nabla U$ implies that $\vec{E} \perp$ constant surfaces of U.

This results from general properties of gradient, $\vec{\nabla}$.

Since field lines are lines with tangential vectors parallel to $\vec{E}$, therefore, field lines are ~~m~~ perpendicular to equipotential surfaces.



Superposition principle

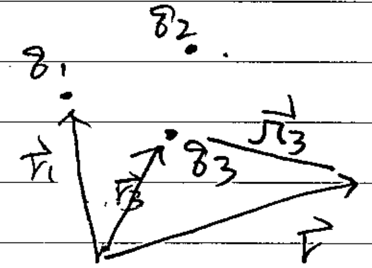
Since $\vec{E}$ obeys the superposition principle

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots,$$

the potential $\vec{E} = -\nabla V$ also obeys the superposition principle

$$V = V_1 + V_2 + V_3 \cdots$$

i.e, the potential at $\vec{r}$ = summation due to all charges. Hence eq. (32) can be generalized to



$$V(r) = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{\mathscr{R}_1} + \frac{q_2}{\mathscr{R}_2} + \cdots\right)$$

$$= \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\mathscr{R}_i} \quad \cdots (33)$$

For the continuous distribution,

$$V(r) = \frac{1}{4\pi\epsilon_0}\int \frac{dq'}{|\vec{r}-\vec{r}'|}$$

$$dq' = \rho(r)\, d\tau'$$

$$\therefore V(r) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')\, d\tau'}{|\vec{r}-\vec{r}'|} \quad \cdots (34)$$

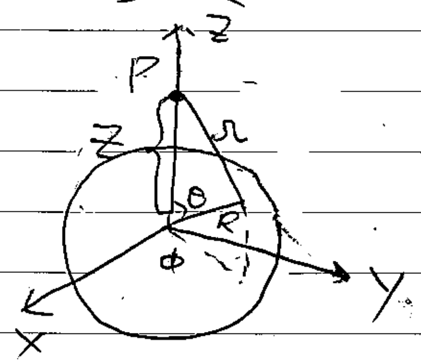
The electric field due to ρ is

$\vec{E} = -\nabla V$ which is given by eq. (7)

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')\rho(\vec{r}')\,d\tau'$$

The advantage of eq. (34) is that it is an integral over a scalar function, while one needs to consider 3-component integration in eq. (7) $\therefore$ The integration effort is often reduced.

Example: Find the potential of a uniformly charged spherical shell of radius R



an area element at (θ, ϕ)
$= R^2 \sin\theta\, d\theta\, d\phi$

$\therefore$ if σ = charge density

$$V_p = \int \frac{\sigma R^2 \sin\theta\, d\theta\, d\phi}{4\pi\epsilon_0 \mathscr{r}}$$

$\because \mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta$ independent of ϕ

$\therefore \int d\phi = 2\pi$

$$U_P = \frac{\sigma R^2}{4\pi\epsilon_0} \times 2\pi \int_0^{\pi} \frac{\sin\theta \, d\theta}{\sqrt{R^2+Z^2-2RZ\cos\theta}}$$

$$= \frac{\sigma R^2}{2\epsilon_0} \int_{-1}^{1} \frac{d\cos\theta}{\sqrt{R^2+Z^2-2RZ\cos\theta}}$$

$$= \frac{\sigma R^2}{2\epsilon_0} \frac{1}{RZ} \sqrt{R^2+Z^2-2RZ\cos\theta} \Bigg|_0^{\theta=\pi}$$

$$= \frac{\sigma R}{2\epsilon_0} \frac{1}{Z} \left[\sqrt{(R+Z)^2} - \sqrt{(R-Z)^2} \right]$$

$$\begin{cases} Z>R & = \frac{\sigma R}{2\epsilon_0} \frac{1}{Z} \left(R+Z+(R-Z) \right) = \frac{R^2 \sigma}{\epsilon_0 Z} \\ Z<R & = \frac{\sigma R}{2\epsilon_0} \frac{1}{Z} \left(R+Z+Z-R \right) = \frac{R\sigma}{2\epsilon_0} \end{cases}$$

If we set $q = 4\pi R^2 \sigma$ = total charge,

one has
$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \quad r \geq R$$
$$= \frac{1}{4\pi\epsilon_0} \frac{q}{R} \quad r \leq R$$

is consistent with the electric field calculation we did before.

<u>Poisson's equation</u> and <u>Laplace's equation</u>

— the "differential form" for V.

As we have shown, the $\vec{E}$ field for static charges obeys:

$$\left.\begin{aligned}\vec{\nabla}\times\vec{E} &= 0\\ \vec{\nabla}\cdot\vec{E} &= \frac{\rho}{\varepsilon_0}\end{aligned}\right\} \quad (35)$$

According to the Helmholtz theorem, the above two uniquely determines $\vec{E}$ in a region.

Mathematically, to solve $\vec{E}$, one starts from $\vec{\nabla}\times\vec{E}=0$ and concluds $\vec{E}=-\nabla V$.

Substituting $\vec{E}=-\nabla V$ into $\vec{\nabla}\cdot\vec{E}=\rho(\vec{r})/\varepsilon_0$, one gets

$$\nabla^2 V = -\rho/\varepsilon_0 \quad \cdots (36)$$

which is known as the Poisson's equation.

When $\rho=0$, one gets

$$\nabla^2 V = 0 \quad \cdots (37)$$

which is the Laplace equation, which will be discussed later.

Eq. (36) is the differential form of eq. (34).

Solving the differential equation is often easier than performing the integration. However, solutions to eq. 36 are only for the voltage whose absolute values have no meaning. That is, if V is a solution, V+constant is a solution. To fix the constant, one often employ the boundary condition. Eq. 36 is thus solved with the boundary conditions.

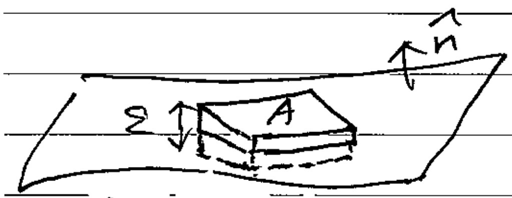
For infinite systems, we often take the boundary condition as

$$V(r) \to 0 \quad \text{at} \quad r \to \infty \quad \text{---} \; (38)$$

In general, the region of V that we are interested is finite. One needs different boundary conditions as will be discussed in below.

Boundary conditions

One of this situations for the boundary conditions is that one knows the surface charge density on a surface and needs to know how the electric-field and potential change across the surface.



As shown in the left figure, we take a small pillbox as the Gaussian surface.

The width of the pill box $\varepsilon \to 0$, hence side surfaces don't contribute

$\therefore$ If A is small enough, one gets

$$\vec{E}\cdot\vec{A}\big|_{above} + \vec{E}\cdot\vec{A}\big|_{below} = \frac{1}{\varepsilon_0}\sigma A$$

$\because \vec{A}\big|_{above} = A\hat{n}, \quad \vec{A}\big|_{below} = -A\hat{n}$

$\therefore$ The above eq. becomes $E^{\perp}_{above} - E^{\perp}_{below} = \dfrac{\sigma}{\varepsilon_0}$ --- (39)

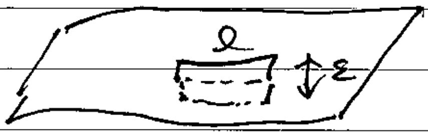
Eq.(39) states that the normal component of $\vec{E}$ across a surface is discontinuous if there is a surface charge.

On the other hand, for electrostatic field,

$\nabla \times \vec{E} = 0$ is always true.

$\therefore \oint \vec{E} \cdot d\vec{l} = 0$ for any closed surface.

Consider a small rectangular loop Γ shown in the below.



For $\varepsilon \to 0$, side edges have no contribution.

$$\therefore \oint \vec{E} \cdot d\vec{l} = \vec{E} \cdot \vec{l}\,|_{above} + \vec{E} \cdot \vec{l}\,|_{below} = 0$$

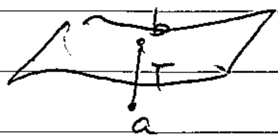
$$\vec{l}\,|_{above} = -\vec{l}\,|_{below} \quad \therefore E^{\parallel}_{above} = E^{\parallel}_{below} \quad \text{--- } (40)$$

$\parallel$ indicates components in parallel to the surface.

Combing eqs. (38) & (40), one gets $\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$ --- (41)

Eq. (41) is the boundary condition (BC) for $\vec{E}$. One needs to convert it into BC for the potential.

For the potential, $\because V_{above} - V_{below} = -\int_{a,\Gamma}^{b} \vec{E} \cdot d\vec{l}$



Taking $r_{ab} \to 0$, $\int_a^b \vec{E} \cdot d\vec{l} \to 0$

$$\therefore V_{above} = V_{below} \quad \text{--- } (42)$$

$\therefore$ The potential is continuous across the surface.

However, derivative of V is not.

Together with $\vec{E} = -\nabla U$, eq. (41) becomes

$$\left.\begin{aligned} \nabla U_{above} - \nabla U_{below} &= -\frac{\sigma}{\varepsilon_0}\hat{n} \\ \text{or} \quad \frac{\partial U_{above}}{\partial n} - \frac{\partial U_{below}}{\partial n} &= -\frac{\sigma}{\varepsilon_0} \end{aligned}\right\} \cdots (43)$$

coordinate along $\hat{n}$ direction

i.e. $\frac{\partial U}{\partial n} = \hat{n}\cdot\nabla U$ = normal derivative of U

Eqs (42) & (43) are boundary conditions of the potential across surfaces with charges.

Work and energy in electrostatics

Work

As we have seen, if one moves a charge q in the presence of a $\vec{E}$ field from a to b, one needs to counter the electric force and performs the work

$$W = \int_a^b \vec{F}\cdot d\vec{l} = \int_a^b (-q\vec{E})\cdot d\vec{l} = q(V(b) - V(a)) \quad \cdots (44)$$

In particular, if we take the charge from ∞ to $\vec{r}$, the work needs to be done :

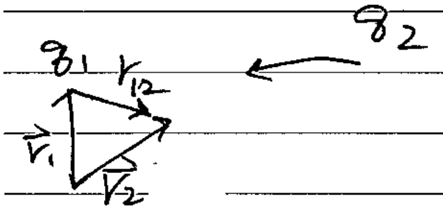
$$W = q\,(V(r) - V(\infty))$$

Setting $V(\infty) = 0$ (reference) point, we get $W = q\,V(r)$ = potential energy --- (45)

Energy of a charge distribution

Using eq. (45) (taking $r = \infty$, $V = 0$ as the reference point), one can deduce the work that needs to be done to assemble a distribution of charges.

For point charges, to assemble q_1, q_2, q_3 ..... at $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots$, one first bring in q_1 to $\vec{r}_1$, this takes no work as the space has no other charges $\therefore W_1 = 0$.



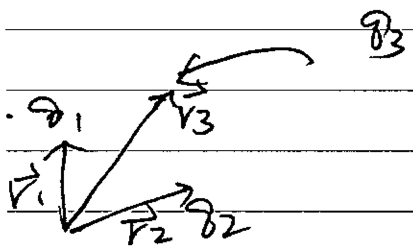
Next, we take q_2 in.

$$W_2 = q_2 V_1(\vec{r}_2)$$

$$V_1(\vec{r}_2) = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_{12}} \qquad r_{ij} \equiv |\vec{r}_i - \vec{r}_j|$$

$$\therefore W_2 = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}}$$

Now, if we bring in q_3, we need to do the work



$$W_3 = q_3 V_{1,2}(\vec{r}_3)$$

Using the principle of superposition,

$$V_{1,2}(\vec{r}_3) = V_1(\vec{r}_3) + V_2(\vec{r}_3)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_{13}} + \frac{1}{4\pi\epsilon_0}\frac{q_2}{r_{23}}$$

$$\therefore W_3 = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

Similarly, the work to bring q_4 in is

$$W_4 = \frac{1}{4\pi\epsilon_0} q_4 \left(\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}}\right)$$

and the work to bring in q_n is

$$W_n = \frac{1}{4\pi\epsilon_0} q_n \left(\frac{q_1}{r_{1n}} + \frac{q_2}{r_{2n}} + \cdots + \frac{q_{n-1}}{r_{n-1,n}} \right)$$

$\therefore$ Total work

$$W = W_1 + W_2 + W_3 + \cdots + W_n$$

$$= \frac{1}{4\pi\epsilon_0} \sum_{\substack{\text{each} \\ \text{pair} \\ (i,j)}} \frac{q_i q_j}{r_{ij}}$$

$$= \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=i+1 \\ (j>i)}}^{n} \frac{q_i q_j}{r_{ij}}$$

$j > i$ is used to avoid double-counting of (i,j) pair.

One can also include $j < i$, but due to double-counting of each (i,j) pair ((i,j) or (j,i)), we need to divide it by 2:

$$W = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \frac{q_i q_j}{r_{ij}} = \frac{1}{8\pi\epsilon_0} \sum_{i\neq j} \frac{q_i q_j}{r_{ij}}$$

$$= \frac{1}{2} \sum_{i=1}^{n} q_i \underbrace{\left(\sum_{j\neq i}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{r_{ij}} \right)}_{\text{total potential of all other charges at } r_i} \quad \text{--- (46)}$$

$$= \frac{1}{2}\sum_{i=1}^{n} q_i V(r_i) \quad \cdots (47)$$

$V(r_i)$ is the potential at r_i (excluding the potential due to q_i at r_i !)

Eq. (47) is the total work to assemble $q_1, q_2, \ldots, q_n$ at $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$.

Note that this work does not include the work that needs to assemble each q_i itself.!

For a continuous charge distribution, clearly, one replace q_i by $\rho(\vec{r})d\tau$

$$\therefore \; W = \frac{1}{2}\int \rho(\vec{r})\, V(\vec{r})\, d\tau \quad \cdots (48)$$

Now, using $\rho/\varepsilon_0 = \vec{\nabla}\cdot\vec{E}$, one gets

$$W = \frac{\varepsilon_0}{2}\int (\vec{\nabla}\cdot\vec{E})\, V\, d\tau$$

(Note that in principle V excludes the contribution of $\rho d\tau$ at $\vec{r}$, however $\because d\tau \to 0$, the contribution $\to 0$)

Using integration by parts, one has

$$(\vec{\nabla}\cdot\vec{E})\, V = \vec{\nabla}\cdot(\vec{E}V) - \vec{E}\cdot(\vec{\nabla}V) = \vec{\nabla}\cdot(\vec{E}V) + E^2$$

$$\therefore W = \frac{\epsilon_0}{2}\int E^2 d\tau + \frac{\epsilon_0}{2}\int \nabla\cdot(\vec{E}V)d\tau$$

From the divergence theorem,

$$\int_V \nabla\cdot(\vec{E}V)\,d\tau = \oint_{\partial V} V\vec{E}\cdot d\vec{a}$$

taking $V\to\infty$, $\partial V\to\infty$, $\because V\to\frac{1}{r}$, $\vec{E}\to\frac{1}{r^2}$

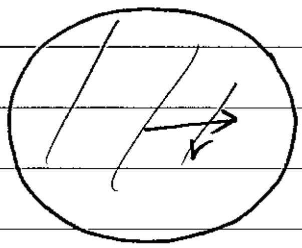
$$\therefore \oint_{\partial V} V\vec{E}\cdot d\vec{a} \longrightarrow 0 \quad (\text{as } d\vec{a}\to r^2)$$

$\therefore$ One obtains $\quad W = \frac{\varepsilon_0}{2}\int E^2 d\tau \quad \ldots$ (49)

i.e. work is stored in field with energy density per volume in the space $= \frac{1}{2}\varepsilon_0 E^2 \quad \ldots$ (50)

Note that when q_i is replaced by $\rho(r)d\tau$ in eq. (48), eq. (48) also includes work that needs to assemble q_i itself! (self-energy) (as $V(r)$ is the full potential used in (48)!) That is why eq. (50) is always positive while eq. (47) may become negative.

example: Find the energy of a uniformly charged sphere with total charge q and radius R.



Using Gauss's law, E at r satisfies

$$E \cdot 4\pi r^2 = \frac{1}{\varepsilon_0} Q_{enclosed}$$

$$= \frac{1}{\varepsilon_0}\left(\frac{4\pi}{3}r^3\right) \cdot \rho$$

$$\therefore E = \frac{\rho}{3\varepsilon_0} r \qquad \text{for } 0 \le r \le R$$

For $r > R$, $E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$ $\left(\rho = \frac{q}{\frac{4\pi}{3}R^3}\right)$

For the potential, one has

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \qquad r \ge R$$

$$V = -\int^r E\,dr = \frac{-\rho}{6\varepsilon_0}r^2 + \text{const} \qquad 0 \le r \le R$$

$$= \frac{-q}{8\pi\varepsilon_0 R^3}r^2 + C$$

$\because$ V has to be continuous at $r = R$,

$$\therefore \frac{1}{4\pi\varepsilon_0}\frac{q}{R} = \frac{-q}{8\pi\varepsilon_0 R^3}R^2 + C$$

$$\therefore C = \frac{+3}{8\pi\epsilon_0}\frac{q}{R}$$

$$\therefore V = \frac{-q}{8\pi\epsilon_0 R^3} r^2 + \frac{3}{8\pi\epsilon_0}\frac{q}{R} \quad \text{for } 0 \le r \le R$$

$$\therefore W = \frac{1}{2}\int \rho V \, d\tau$$

$$= \frac{\rho}{2}\int_0^R 4\pi r^2 dr \left(\frac{-q}{8\pi\epsilon_0 R^3} r^2 + \frac{3}{8\pi\epsilon_0}\frac{q}{R} \right)$$

$$= \frac{\rho}{2}\cdot \frac{3q}{8\pi\epsilon_0}\frac{1}{R}\cdot \frac{4\pi}{3}R^3$$

$$+ \frac{\rho}{2}\frac{(-q)}{8\pi\epsilon_0 R^3}\cdot 4\pi \int_0^R r^4 dr$$

$$= \frac{1}{8\pi\epsilon_0 R}\frac{3q^2}{2} - \underbrace{\frac{\rho}{2}\frac{q}{8\pi\epsilon_0 R^3} 4\pi \frac{R^5}{5}}_{= \frac{3}{5}\frac{q^2}{16\pi\epsilon_0 R}}$$

$$= \frac{3}{5}\frac{q^2}{4\pi\epsilon_0 R}$$

One can also use eq. (44) to calculate and find the same result.

Clearly, as $R \to 0$, $W \to \infty$. Therefore, the classical point self-energy of $= \infty$.

The divergence of self-energy can be directly revealed by considering the $\vec{E}$ field of a point charge q

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

$$\therefore W = \frac{\epsilon_0}{2}\int E^2 d\tau$$

$$= \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \int \frac{1}{r^4} r^2 \underbrace{\sin\theta d\theta d\phi}_{4\pi} dr$$

$$= \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \cdot 4\pi \int_0^\infty \frac{dr}{r^2} = \infty$$

The infinite energy of a point charge is a recurring source of difficulty in E&M theory and has plagued physicists even in quantum theory of E&M.

We shall discuss it later in this course.

Note that because W depends on E quadratically, it does not obey a superposition principle:

if $\vec{E} = \vec{E}_1 + \vec{E}_2$

$$W_{total} = \frac{\epsilon_0}{2}\int E^2 d\tau = \frac{\epsilon_0}{2}\int (\vec{E}_1 + \vec{E}_2)^2 d\tau$$

$$= \frac{\epsilon_0}{2}\int (E_1^2 + E_2^2 + 2\vec{E}_1\cdot\vec{E}_2) d\tau = W_1 + W_2 + \epsilon_0\int \vec{E}_1\cdot\vec{E}_2 d\tau$$

$$\neq W_1 + W_2$$

Electric properties of conductors

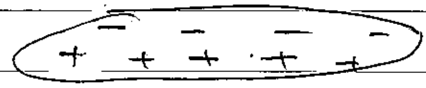
Insulator, conductor and perfect conductor

Materials are often classified according to their transport properties which as we shall discuss later, are the abilities to carry currents.

They are insulators, conductors & semi-conductors

The ability to carry currents depends on the availability of electrons

that are not bound to atoms and can move freely. (called free electrons)



tightly bound

For insulators, there is no free electron and all electrons can not move in long distance and are attached to atoms.

On the other hand, in a conductor, which is often metallic, there are one or more electrons per atom that can move freely.

When there are unlimited free charges, those are perfect conductors.

Perfect conductors are idealized conductors.

In real materials, there is no perfect conductor but metals are close for most purposes.

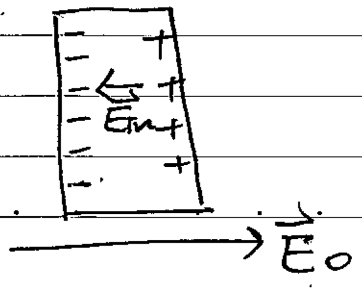
We shall assume conductors in this course are perfect.

Basic electrostatic properties of ideal conductors (perfect)

(i) $\vec{E}=0$ inside.

If there is any field $\vec{E} \neq 0$, due to the unlimited free charges, those free charges will move and become current. This would not be electrostatic any more.

More precisely, for a given ideal conductor shown on below, in the presence of external $\vec{E}_0$, charges will move in response to $\vec{E}_0$. However, due to the boundary of the



conductor, charges will accumulate at surfaces of conductors

These surface charges, in turn, give rise to an internal field $\vec{E}_{in}$ to cancel $\vec{E}_0$! $\vec{E} = \vec{E}_0 + \vec{E}_{in} = 0$

The process occurs in very shot of time, practically instantaneous!

Note that the above is true only when there are unlimited free charges. For conductors with finite charges, the cancellation of $\vec{E}$ will not be complete if $\vec{E}$ is too strong!

(ii) $\rho = 0$ inside

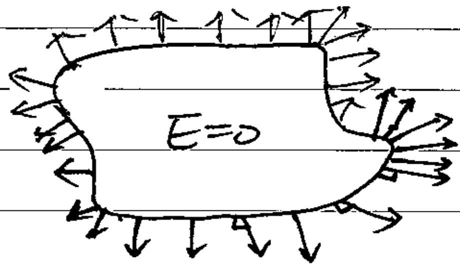
From (i), $\because \vec{E} = 0$, from $\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0$

$\therefore \rho = 0$

(iii) Any net charge resides on the surface. (consequence of (ii))

(iv) A conductor is an equipotential.

$\because \vec{E}=0 \quad \therefore$ for any two points a & b inside a conductor the difference of their potential



$$U_a - U_b = -\int_b^a \vec{E}\cdot d\vec{\ell} = 0$$

$\therefore \quad V_b = V_a$

Hence inside the conductor, $U=$ const.

Outside the conductor, $\vec{E} \neq 0$

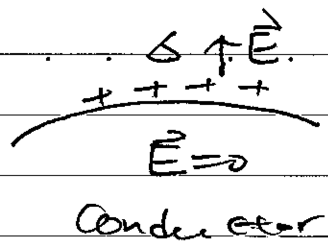
but $\because \vec{E} \perp$ equipotential surface,

$\therefore \quad \vec{E} \perp$ surface of a conductor.

Induced Charges & force on a Conductor

As explained in the above, in the presence of external $\vec{E_0}$, charges on a conductor can only reside on surfaces.

As a result, there are surface charges characterized by surface charge density $\sigma(\vec{r})$ on a conductor.



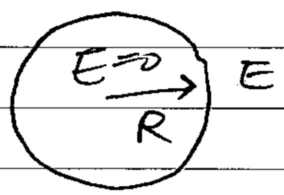
Since $\vec{E}=0$ inside the conductor, according to the boundary condition we derived, the E field outside must be

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n} \quad --- \quad (51)$$

i.e. $\sigma = -\varepsilon_0 \frac{\partial U}{\partial n}$ $\quad \hat{n}$ = unit normal vector

Eq. (51) is not sufficient to solve $\vec{E}$ or σ on a conductor. It, however, gives $\vec{E}$ once σ is known.

Physically, σ is related to the curvature of the surface. For a conducting sphere of radius R, one has $E = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}$



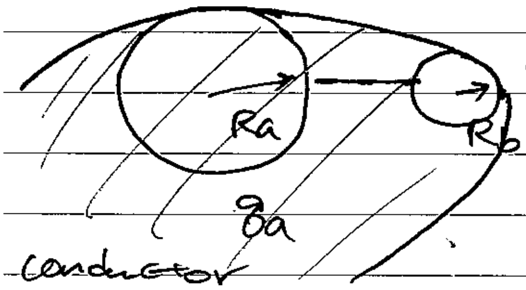
$$\therefore \sigma = \frac{q}{4\pi R^2}$$

Therefore, for a fixed q, σ increases, if R reduces

i.e. $R\downarrow$ ($\underbrace{\text{curvature}\uparrow}_{1/R}$), $E\uparrow$

Since the surface of a conductor consists of parts with different curvatures, q is not fixed, one needs to consider the equilibrium between two spheres

with R_a, R_b : potentials are the same



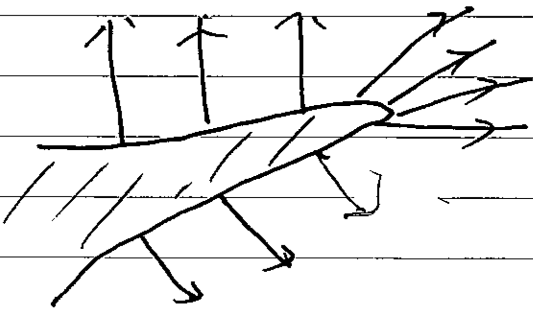
$$\frac{q_a}{4\pi\epsilon_0 R_a} = \frac{q_b}{4\pi\epsilon_0 R_b}$$

$$\therefore \frac{q_a}{q_b} = \frac{R_a}{R_b}$$

$$\therefore \frac{E_a}{E_b} = \frac{q_a}{R_a^2} \Big/ \frac{q_b}{R_b^2} = \frac{q_a}{q_b}\frac{R_b^2}{R_a^2} = \frac{R_b}{R_a}$$

Therefore, $E \propto \frac{1}{R}$ (= curvature)

$$\sigma \propto \frac{1}{R} \quad \text{--- (1-2)}$$



Hence density of E lines near a conductor is larger when the curvature is larger (see the left figure)

This is often summarized as the statement: the electric field near a sharp area of a conductor is enhanced!

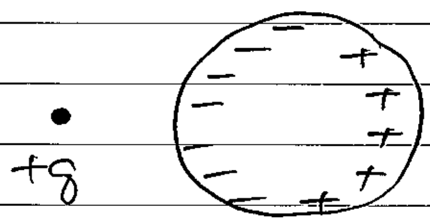
If the electric field is large enough so that air molecules are accelerated and collide with each other that leads to breakdown of air, it may lead to discharge phenomena. (尖端放电)

Induced charges

The integration of σ over the surface of the conductor gives the net charge induced on the conductor.

These are known as induced charges.

For instance, in the presence of a positive charge, the induced charge of a sphere near +q is negative,

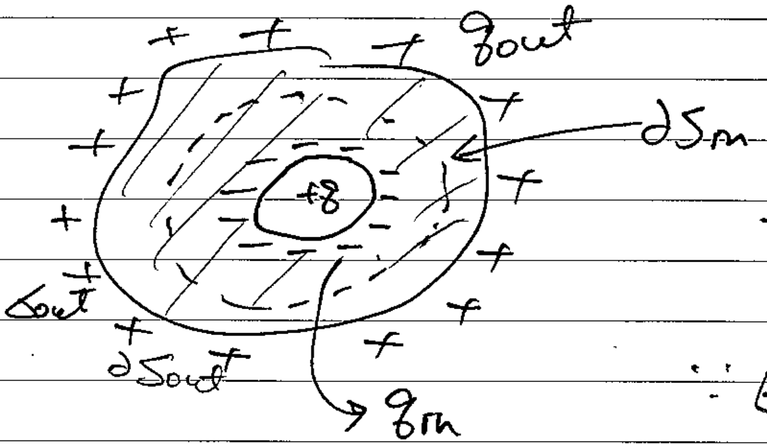


while the induced charge away from +q is positive.

As a result, the net induced charge on the sphere is zero:

$$\oint \sigma \, da = 0$$

The net induced charge on a closed surface is not necessary zero. For instance, if there is a cavity inside the conductor as shown in below, the induced charge on the inner surface $\sigma_{in} \neq 0$:



Consider a Gaussian surface ∂S_{in} that is buried inside the conductor.

$\because \vec{E} = 0$ on ∂S_{in}

$$\therefore \oint_{\partial S_{in}} \vec{E} \cdot d\vec{a} = 0 = \frac{1}{\varepsilon_0} q_{total}$$

$$\therefore q_{total} = q + q_{in} = 0$$

$$q_{in} = -q \, !$$

Note that if the conductor is neutral before $+q$ is introduced, we must have

$$q_{in} + q_{out} = 0$$

$$\therefore \; q_{out} = -q_{in} = +q \; !$$

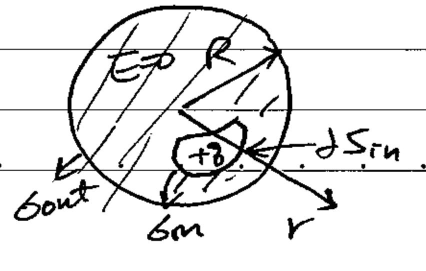
Effect of shielding

As we have seen, when there is a cavity in a neutral conductor, the net charge on the outside surface is q.

(i) With charges inside cavity

In fact, as we shall argue in below, σ_{out} is independent of the shape of cavity and is solely determined by geometry of outside surface.

Example: ∂S_{out} = Sphere of radius R

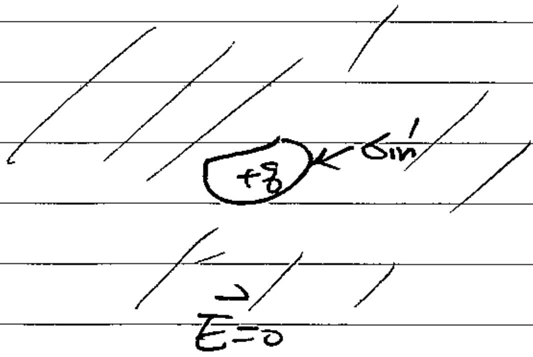


We shall argue that

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \quad \cdots (53)$$

for $r > R$

Consider a different case when $R \to \infty$



Let σ_{in}' be the surface charge in this case.

Clearly,

$$\vec{E}_q + \vec{E}_{in} = 0 \text{ for}$$

$$\vec{r} \in \text{conductor}$$

Now for the problem of finite R, we shall set $\sigma_{in} = \sigma_{in}'$, therefore

$$\vec{E}_q + \vec{E}_{in} = 0 \text{ for all points}$$

(including $r > R$ region) outside the cavity.

Therefore, outside the cavity, $\vec{E}$ is entirely determined by σ_{out}.

From the above construction, one has

$$\vec{E}(\vec{r}) = \vec{E}_q(\vec{r}) + \vec{E}_{\sigma_{in}}(\vec{r}) + \vec{E}_{\sigma_{out}}(\vec{r})$$

which satisfies (i) $\vec{E} = 0$ inside conductor

(ii) $\vec{E} \to 0$, $r \to \infty$

(iii) $\vec{\nabla} \cdot \vec{E} = \frac{q}{\varepsilon_0} \delta(\vec{r})$ inside the cavity.

Due to uniqueness of solutions for E field as we shall prove later, the constructed solution (via principle of superposition) is the solution!

Hence, $\vec{E}$ is entirely determined by $\vec{E}_{\sigma out}$, i.e., by σ_{out}.

For spherical shape that σ_{out} is distributed, clearly by symmetry, σ_{out} is a constant

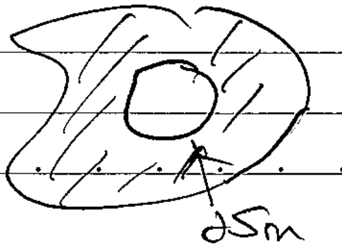
$$\therefore \quad \sigma_{out} = \frac{q}{4\pi R^2}$$

$$\therefore \quad \vec{E}_{\sigma out} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \quad \text{for } r > R$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \quad \text{for } r > R$$

(ii) No charge inside cavity

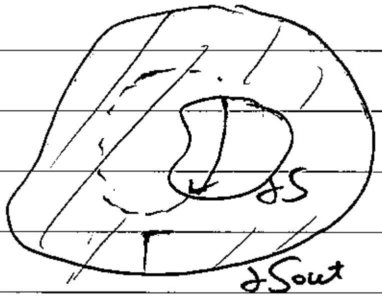
When there is not any charge inside the cavity as shown on the left figure, there is no charge on ∂S_{in}.



∂S_{in}

No.
Date. 2-13.

Clearly, if this is not true, there must be positive & negative charges on ∂S_{in} and the E field inside the cavity must start from some positive charge and end at some negative charge on ∂S_{in} as shown in below.



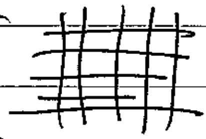
This is, however, not possible because one can form a closed curve Γ, the integral $\oint \vec{E} \cdot d\vec{\ell} \neq 0$

which violates $\oint \vec{E} \cdot d\vec{\ell} = 0$ for E fields of electrostatics.

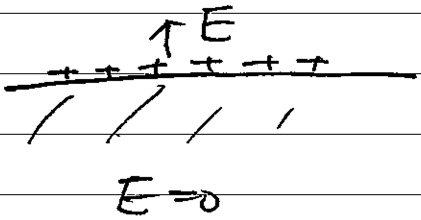
Hence one concludes that there is no charge on ∂S!

That is, no matter what is arranged on ∂S_{out} & outside, there is no charge on ∂S. The cavity appears to be protected against charges outside (electrostatics).

This is the ~~effect~~ of shielding: a conductor can shield electrostatic perturbations from perturbing objects in cavity inside the conductor.

The shielding ~~effect~~ is usually implemented using the Faraday cage made by metal. In practice, the enclosure does not have to be solid conductor, and chicken wire is suffice. ( ← chicken wire as a net.)

Force acting on a conductor



As we have seen, the E field on the surface of a conductor is given by

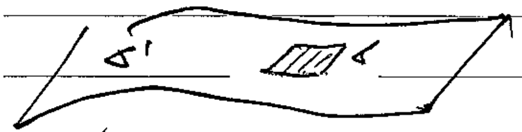
$$\vec{E} = \frac{\sigma}{\varepsilon_0} \hat{n}.$$

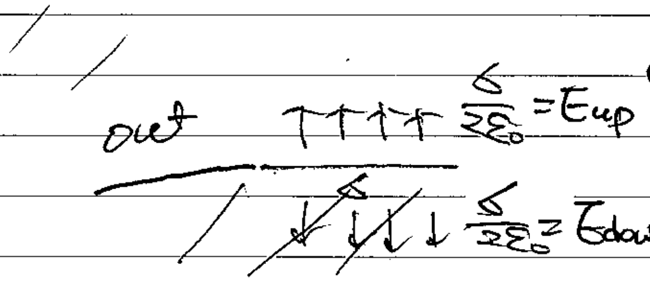
However, this field comes from two contributions: external field (other than σ) & field produced by σ.

Only the external field yields a force on σ.

While the field produced by σ itself does not act on σ itself!

For this purpose, we consider a small parallelogram on the surface shown in the left figure.





Close to the parallelogram, the E field due to σ is $\frac{\pm\sigma}{2\varepsilon_0}$ as shown.

The total E is the sum of $\pm\frac{\sigma}{2\varepsilon_0}$ and E' due to other surface charges σ' & external charges

$$\because \quad \vec{E}_{down} + \vec{E}' = 0 \quad \text{(inside conductor)}$$

$$\therefore \quad \vec{E}' = \frac{\sigma}{2\varepsilon_0}\hat{n} \quad \text{--- (54)}$$

$$\therefore \quad \vec{E}_{out} = \vec{E}' + \vec{E}_{up} = \frac{\sigma}{2\varepsilon_0}\hat{n} + \frac{\sigma}{2\varepsilon_0}\hat{n} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

in agreement with the solution $\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$.

The force is due to E'.

Therefore, for a small area a around $\vec{r}$,

the charge on a : $q = a\sigma$

$\therefore$ the force that acts on a

$$= q \cdot \vec{E'} = \frac{\sigma^2}{2\epsilon_0} a \, \hat{n}$$

$\therefore$ Force per area at $\vec{r}$

$$\vec{f} = \frac{1}{2\epsilon_0} \sigma^2(\vec{r}) \, \hat{n} \quad \text{--- (55)}$$

This is essentially the pressure (P) that acts on $\sigma(\vec{r})$.

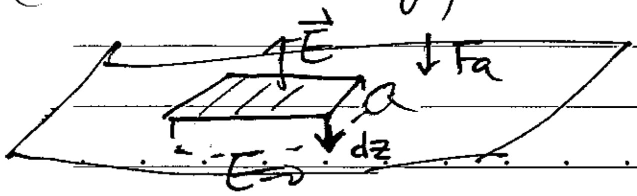
$$P = \frac{1}{2\epsilon_0} \sigma^2 \underset{\uparrow}{=} \frac{1}{2} \epsilon_0 E^2 \quad \text{--- (56)}$$

$$E = \frac{\sigma}{\epsilon_0}$$

$\frac{1}{2}\epsilon_0 E^2$ is the energy density at $\vec{r}$.

Eq. (56) states that the electrostatic pressure at $\vec{r}$

= energy density. This can be understood by imaging the area a moving into the conductor. dz



via the applied for F_a

$\because$ F_a has to counter P

$\therefore$ $F_a/a = P$

The work done by F_a = $F_a \, dz$ -- (5.7)

When a is moving down dz, the energy of the system increases by $\underline{\frac{\epsilon_0}{2} E^2 a \, dz}$ -- (5.8)

because before the movement, $E=0$ in $a dz$, E is now non-vanishing in $a dz$

Hence the work must go to $\frac{\epsilon_0}{2} E^2 a dz$

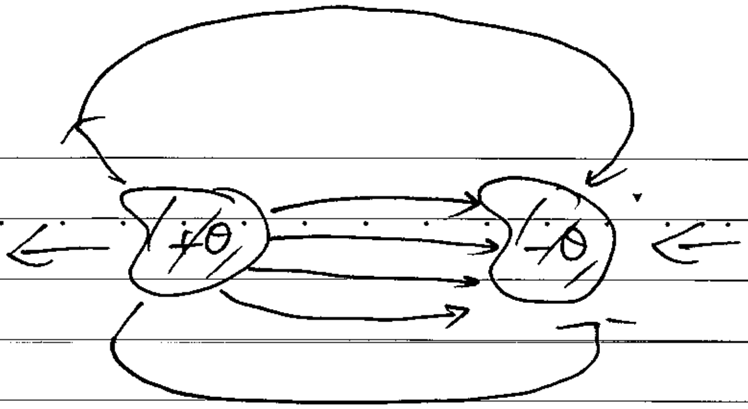
Equating (5.7) & (5.8) yields

$$F_a = \frac{\epsilon_0}{2} E^2 a = a p$$

$$\therefore \; P = \frac{1}{2}\epsilon_0 E^2$$

Capacitors

A capacitor is a special arrangement of two conductors <u>with $+Q$ on one</u> and <u>$-Q$ on the other</u> as shown in below.



Clearly, the charges on two conductors re-arrange electric fields such that more E lines are confined between ±Q conductors.

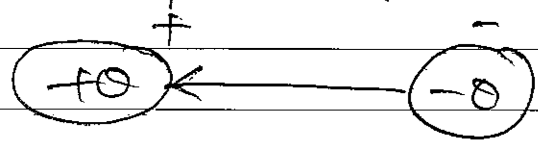
∵ Energy density $= \frac{1}{2}\varepsilon_0 E^2$, one can see that energy is now stored between two conductors. ∴ The capacitor can be used to store electric energy.

The total energy stored $= \int \frac{1}{2}\varepsilon_0 E^2 dz$ is the work done to separate ±Q charges.

To find the total work, one needs to know the potential difference (dV) between two conductors so that moving dq from one conductor to another is known.

$$dW = dq \cdot dV \quad \text{--- (59)}$$

For a given charge Q, the potential difference



$$V \equiv V_+ - V_- = -\int_-^+ \vec{E}\cdot d\vec{l} \quad \text{--- (60)}$$

Now, $$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')\,d\tau'$$

From the principle of superposition, doubling Q doubles $\vec{E}$ if ρ is doubled.

This is clearly a solution. From the uniqueness of $\vec{E}$ (electrostatic), one concludes that $\vec{E} \propto Q$. --- (61)

Hence eqs. (60) implies $V \propto Q$.

We write $$\frac{Q}{V} \equiv C \quad \text{--- (62)}$$

With C being defined as capacitance.

Clearly, C is a purely geometrical quantity determined by sizes, shapes & separation of two conductors. Its unit is Coulomb/volt ≡ farads (F)

F is a large unit. In practice,
one uses micro farad (10^{-6}F) and
pico farad (10^{-12}F) more often.

In practice, sometimes, one discusses
the capacitance for a single conductor.
In that case, the second conductor
is set at $r = \infty$ with $V = 0$.

$\therefore$ $C = Q/V$ Q & V are
charge and voltage of the conductor
— (63)

This definition of capacitance is the
most general definition of capacitance.

In the case with many conductors with
Q_i, V_i for each conductor, one has

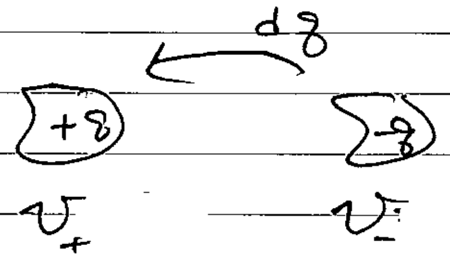
Q_1 V_1 Q_2 V_2 Q_3 V_3 - .. ..

$$Q_i = \sum_{j=1}^{n} C_{ij} V_j$$

with $C_{ii} \equiv$ capacitance
C_{ij} $(i \neq j) \equiv$ coefficients of induction.

We shall stick to the case with one or two conductors, and the definition of capacitance using eqs. (62) & (63).

To find the work for separating the charges into $\pm Q$, we start from an intermediate step when two conductors carry $\pm q$ charges.



To move dq from $-q$ to $+q$, one needs to do the work

$$dW = V_+ dq + V_-(-dq)$$
$$= (V_+ - V_-)\,dq \underset{\text{eq. (62)}}{=} \frac{q}{C}\,dq$$

$$\therefore\ W = \int dW = \int_0^Q \frac{q}{C}\,dq = \frac{1}{2C}Q^2$$
$$\underset{Q=CV}{=} \frac{1}{2}CV^2 \quad \text{--- (63)}$$

with V being the final potential charged!

Note that eq. (63) can be derived by using eq. (48) directly

$$W = \frac{1}{2}\int \rho V d\tau$$

$$= \frac{1}{2}\left(\oint_{+} \sigma da\right) V_{+} + \frac{1}{2}\left(\oint_{-} \sigma da\right) V_{-}$$

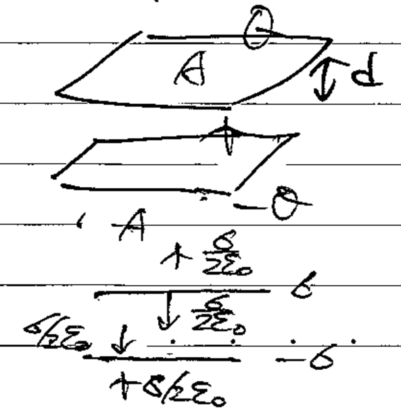
$$= \frac{1}{2} Q V_{+} - \frac{1}{2} Q V_{-}$$

$$= \frac{1}{2} Q V = \frac{1}{2} C V^2$$

From the derivation, it is clearly the W includes not only the work separating $+Q$ and $-Q$ but also the work between charges on Q ($-Q$) themself.

$\therefore$ W includes self energy of $+Q$ & $-Q$.

Example: Parallel-plate capacitor



$$\sigma = Q/A$$

$$E = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

$$V = Ed = \frac{\sigma}{\varepsilon_0} d = \frac{Q}{A\varepsilon_0} d = \frac{Q}{C}$$

$$\therefore\ C = \frac{A\varepsilon_0}{d}$$

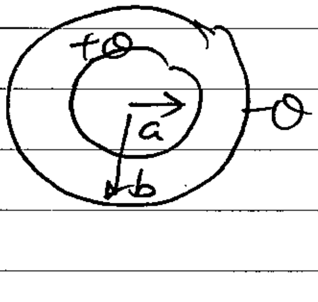
$$W = \frac{1}{2} C U^2 = \frac{1}{2} \frac{A\varepsilon_0}{d} U^2$$

$$= \frac{1}{2} \frac{A\varepsilon_0}{d} (Ed)^2$$

$$= \frac{1}{2} \varepsilon_0 E^2 \cdot \underbrace{(Ad)}_{\text{volume between two-plates}}$$

$\therefore$ Clearly, both $\frac{1}{2}CU^2$ and $\int \frac{1}{2}\varepsilon_0 E^2$ are the work for creating a capacitor with voltage U.

Example : Capatiance of two concentric spherical metal shells with radii a and b $(b>a)$



$\therefore$ for $a<r<b$, $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$

$$\therefore\ U = -\int_b^a \vec{E}\cdot d\vec{r} = -\frac{Q}{4\pi\varepsilon_0}\int_b^a \frac{dr}{r^2}$$

$$= \frac{Q}{4\pi\varepsilon_0} \frac{1}{r}\Big|_b^a = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{Q}{C}$$

$$\therefore\ C = 4\pi\varepsilon_0 \frac{a\cdot b}{b-a}$$