Electromagnetic waves

One of the most important features in Maxwell's equations: it allows solutions of waves.

To characterize electromagnetic waves, we shall start by general characterization of waves.

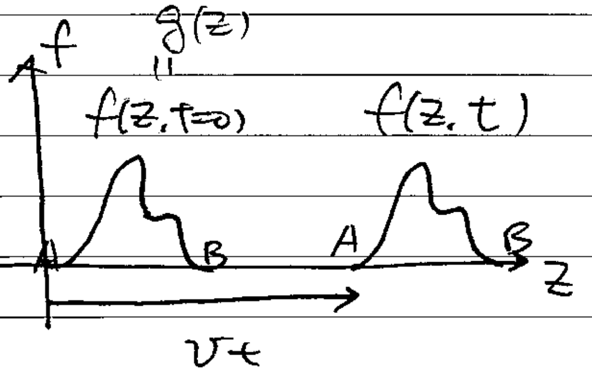
The wave equation

What is a wave?

A wave is a disturbance of configuration that can propagate in space. Usually, it is accompanied by the transfer of energy for waves discussing in physics.

In the simplest case, the configuration is the shape and the wave propagates in a medium with a fixed shape at constant speed.

In this case, there is a simple mathematical description in terms of so-called wave equations.



Consider a wave with a pulse shape at $t=0$. At t, the pulse moves to $z = vt$ with the same shape as shown in the above. v is the speed of wave.

It means $f(z,t) = f(z-vt, 0) \equiv g(z)$ ... ①

mathematics.

That is, z & t are not independant variables.

Check: A point, $t=0$ $z_A=0$ $f(0,0)=0$

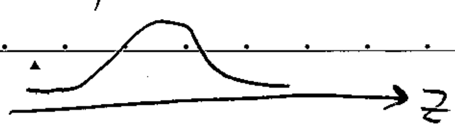
t. $z_A = vt$ $f(z_A - vt, 0) = f(0,0) = 0$

Eq. ① captures the essence of wave motion.

When ① is satisfied, $f(z,t)$ represents a wave of fixed shape ($g(z)$) travelling in the z direction at speed v.

In general, $g(z)$ can be any function such as

$f_1(z,t) = A e^{-B(z-vt)^2}$



$$f_2(z,t) = A\sin\beta(z-vt), \quad g_z(z) = A\sin\beta z$$

The relation in eq. ① implies

$$f(z,t) = g(z-vt), \quad \text{set } u = z-vt$$

$$\therefore \frac{\partial f}{\partial z} = \frac{dg}{du}\frac{\partial u}{\partial z} = \frac{dg}{du}$$

$$\frac{\partial f}{\partial t} = \frac{dg}{du}\frac{\partial u}{\partial t} = -v\frac{dg}{du}$$

$$\therefore \frac{1}{v}\frac{\partial f}{\partial t} - \frac{\partial f}{\partial z} = 0 \quad \text{--- ②}$$

i.e. the derivatives of f with respect to t & z are related, not independent from each other.

Eq. ② is only true when the wave propagates in $+z$ direction.

Clearly, for the wave $\bar{f}(z,t)$ propagating in $-z$ direction with the same speed, one has $\bar{f}(z,t) = h(z+vt)$

$$\therefore \frac{1}{v}\frac{\partial \bar{f}}{\partial t} + \frac{\partial \bar{f}}{\partial z} = 0 \quad \text{--- ③}$$

In general, for a given medium, waves can propagate (in z axis)

No.
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in +z & −z direction.

This is captured by taking the 2nd derivatives:

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{dg}{du}\right) = \frac{d^2 g}{du^2}\frac{\partial u}{\partial z} = \frac{d^2 g}{du^2}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(-v\frac{dg}{du}\right) = -v\frac{d^2 g}{du^2}\frac{\partial u}{\partial t}$$

$$= v^2\frac{\partial^2 g}{du^2}$$

$$\therefore \frac{d^2 g}{du^2} = \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \quad \text{--- (4)}$$

Eq. (4) is known as the wave equation in one dimension

It can be viewed as the combination of eqs. (2) & (3):

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right) f = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial z} - \frac{1}{v}\frac{\partial}{\partial t}\right) f = 0 \quad \text{--- (5)}$$

For $f(z,t) = g(z-vt)$, $\left(\frac{\partial}{\partial z} - \frac{1}{v}\frac{\partial}{\partial t}\right) g = 0$
$f(z,t) = h(z+vt)$, $\left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t}\right) h = 0$

thus $\left(\frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right) g \text{ (or } h) = 0$

In general, eq. (4) is linear so that

if f_1 & f_2 are solutions

$$\frac{\partial^2 f_1}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 f_1}{\partial t^2} = 0$$

$$\frac{\partial^2 f_2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 f_2}{\partial t^2} = 0$$

$$\Rightarrow \quad \frac{\partial^2}{\partial z^2}(C_1 f_1 + C_2 f_2) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}(C_1 f_1 + f_2) = 0$$

$\therefore$ $C_1 f_1 + C_2 f_2$ is also a solution.

i.e., superposition principle works for waves.

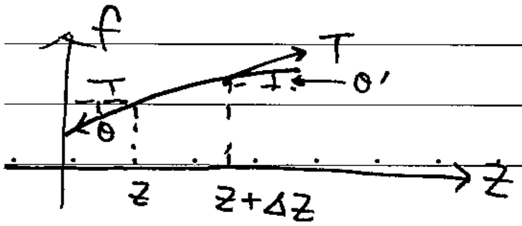
Hence, the most general solution to

the 1D wave equation of eq. (4) is

$$f(z,t) = g(z - vt) + h(z + vt) \quad \text{-- (6)}$$

1D wave on stretch string

As an example, we consider a very long string under tension T. If the string is displaced from equilibrium so that its transverse displacement at z is $f(z,t)$,



the net transverse force on

The segment between z and $z+\Delta z$ is
(as shown in the left figure)

$$\Delta F = T\sin\theta' - T\sin\theta$$

$$\simeq T\tan\theta' - T\tan\theta$$

for $\theta \ll 1$

$$= T\left(\frac{\partial f}{\partial z}\Big|_{z+\Delta z} - \frac{\partial f}{\partial z}\Big|_{z}\right)$$

$$\simeq T\frac{\partial^2 f}{\partial z^2}\Delta z \quad \cdots (7)$$

$\therefore$ If the mass per unit length is μ,

$$\Delta F = \underbrace{\mu \cdot \Delta z}_{\text{mass}} \cdot \underbrace{a}_{\text{acceleration}} = \mu\Delta z \frac{\partial^2 f}{\partial t^2} \quad \cdots (8)$$

Equating (7) and (8),

$$\therefore \frac{\partial^2 f}{\partial z^2} = \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2} \quad \cdots (9)$$

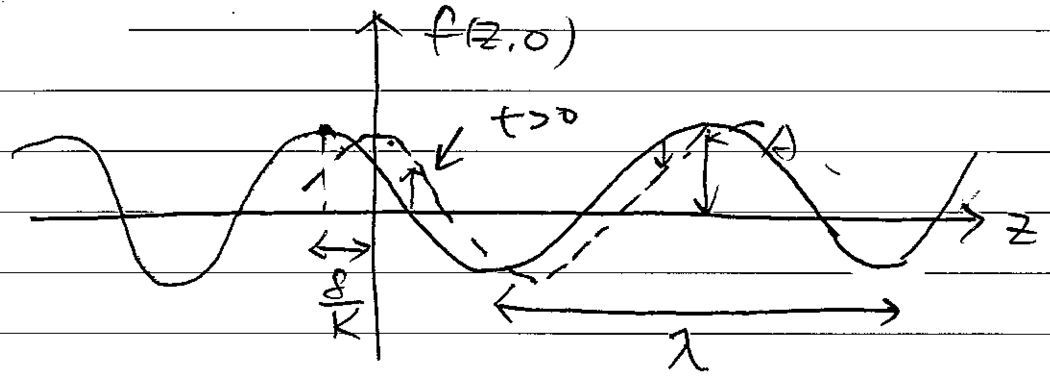
Hence the transverse displacement satisfies a wave equation with $\frac{\mu}{T} = \frac{1}{v^2}$.

$\therefore$ $v = \sqrt{\frac{T}{\mu}}$ is the speed of wave on a stretched string.

Sinusoidal waves

The simplest wave is the sinusoidal wave:

$$f(z,t) = A\cos[k(z-vt)+\delta] \quad \cdots (10)$$



$f(z,t)$ is a fundamental wave and due to the principle of superposition, one can use all possible f to construct any shape of waves via the technique of Fourier transformation.

Here at $t=0$ $\quad f(z,0) = A\cos(kz+\delta)$.

The shape oscillates in z space with A = amplitude of the wave

and f is periodic in z with period $\equiv$ wavelength λ such that

$$k\lambda = 2\pi \quad \therefore \lambda = \frac{2\pi}{k}$$

$K = \frac{2\pi}{\lambda}$ is called the wave number.

At $t=0$, $z=0$ is not at maximum of cos due to the presence of δ

$\therefore$ δ = phase constant

$K(z-vt)+\delta$ = phase (of cosine).

The speed of shape is determined by the phase. For instance, at peak,

Phase $= 2n\pi$

For constant phase, one has

$K(z-vt)+\delta = \text{Const.} \Rightarrow$ differentiating

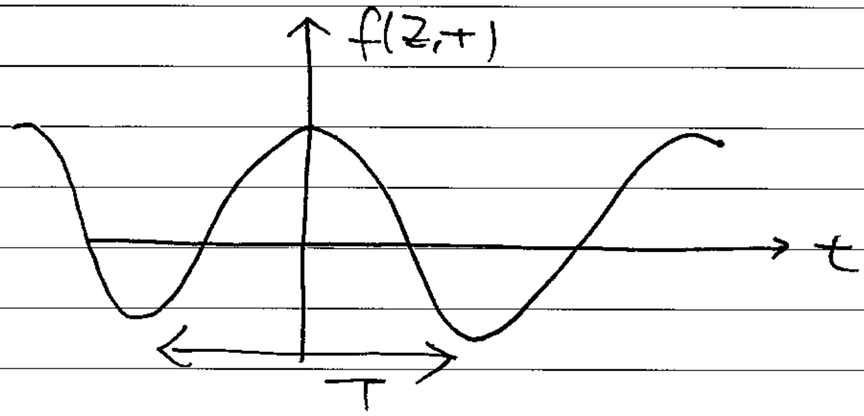
$\therefore \frac{dz}{dt} = v$ $\quad \therefore$ v = speed of shape

= speed of wave

Note that at $z = vt - \delta/K$, the phase $\Rightarrow$ is maximal!

When $t>0$ but still small, the shape of sinusoidal moves to right (dash line).

One sees that the element at fixed z starts to move up or down. The element does not move towards $\pm z$!



$\because$ For fixed z, $f = A\cos[kvt - kz - \delta]$

$\therefore$ The element oscillate in time as shown in the above. (up & down)

The period T satisfies

$$kvT = 2\pi$$

$$\therefore \quad T = \frac{2\pi}{kv}$$

$$\nu = \frac{1}{T} = \frac{kv}{2\pi} = \frac{v}{\lambda} = \text{frequency}.$$

A more convenient unit is the angular frequency $\omega = 2\pi\nu = \frac{2\pi}{T}$

(# of radians swept per unit time)

$\therefore$ $\omega = kv$. So one often writes

$$f(z,t) = A\cos(kz - \omega t + \delta) \quad \text{---} \; (11)$$

In terms of k and ω, the speed of wave is determined by setting

$$kz - \omega t + \delta = \text{constant}$$

$$\frac{d}{dt} = 0 \Rightarrow \quad \frac{dz}{dt} = \frac{\omega}{k} = v \quad \cdots (12)$$

$$\therefore \omega = vk$$

$\because v$ is independant of k. The medium is called non-dispersive medium.

In general, v may depend on k so that $\omega = \omega(k)$ is not linear in k.

The medium is dispersive and the relation $\underline{\omega = \omega(k)}$ is <u>the dispersion relation</u>

In this case, $v = \frac{\omega(k)}{k}$ is called <u>phase velocity</u>.

<u>Complex notation</u>

When the phase constant $\delta \neq 0$, The sinusoidal wave $\cos(k(z-vt)+\delta)$ is actually a combination of cos & sin waves.

$\cos(kz-\omega t+\delta)$

$=\cos(kz-\omega t)\cos\delta-\sin(kz-\omega t)\sin\delta$.

$\therefore$ $\sin(kz-\omega t)$ & $\cos(kz-\omega t)$ are two fundamental sinusoidal waves.

They can be combined using the Euler's formula : $e^{i\theta}=\cos\theta+i\sin\theta$

$$\therefore\ \cos(kz-\omega t)=\underbrace{Re}_{\text{real part}}\left[e^{i(kz-\omega t)}\right]$$

$$\sin(kz-\omega t)=Im\left[e^{i(kz-\omega t)}\right]$$

$$A\cos[kz-\omega t+\delta]=Re\ Ae^{i(kz-\omega t)+i\delta}$$
$$=Re\ \tilde{A}e^{i(kz-\omega t)}$$

with $\tilde{A}=Ae^{i\delta}$ being a complex amplitude.

In the following, $\tilde{(\cdots)}$ refers to the complex form of $(\cdots)$!

The advantage of using complex notation is that exponentials are much easier to handle.

For instance, the combination of two sinusoidal waves can be viewed as

Sum of exponentials:

$$A \cos(kz-\omega t) + A' \cos(k'z-\omega' t), \quad A, A' = \text{real}$$

$$= Re\left[A e^{i(kz-\omega t)} + A' e^{i(k'z-\omega' t)}\right]$$

$$A \cos(kz-\omega t) + A' \sin(k'z-\omega' t)$$

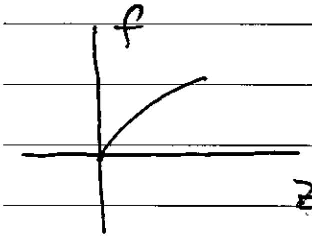
$$= Re\, A e^{i(kz-\omega t)} + Re\, A'(-i) e^{i(k'z-\omega' t)}$$

$$= Re\left[A e^{i(kz-\omega t)} + \tilde{A'} e^{i(k'z-\omega' t)}\right]$$

where $\tilde{A'} = -i A'$

Linear combination & Fourier transformation

Similiar to the Taylor's expansion:



$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

f can be expressed in terms of $\{1, z, z^2, z^3 \ldots\}$,

any wave can be expressed as a linear combination of sinusoidal waves:

express: $f(z,t) = Re\, \tilde{f}(z,t)$, $\tilde{f}$ can be then written as

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} dk\ \tilde{A}(k)\ e^{i(kz-\omega t)} \quad \cdots (13)$$

Where $\tilde{A}(k)$ is the complex amplitude for wavevector k.

For a fixed t,

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} dk\ A(k,t)\ e^{ikz} \quad \cdots (14)$$

where

$$A(k,t) = \tilde{A}(k)\ e^{-i\omega(k)t}$$

Eq. (14) is the $\underbrace{\text{Fourier transformation}}_{\text{standard form of}}$.

The inversion theorem of Fourier transform implies

$$A(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz\ e^{-ikz}\ \tilde{f}(z,t) \quad \cdots (15)$$

Hence given $\tilde{f}(z,t)$, $A(k,t)$ can be found by performing the integration of eq. (15)

Usually, $f(z,0)$ and $\frac{d}{dt}f(z,0)$ are given at $t=0$, these are known as the initial conditions of waves.

Note that given $f(z,0) = \mathrm{Re}(\tilde{f}(z,0))$, $\tilde{f}(z,0)$ is not uniquely determined.

One has to known both $f(z,0)$ & $\frac{df}{dt}(z,0)$ to determine $A(k,t)$:

$$f(z,0) = \frac{1}{2}\left(\tilde{f}(z,0) + \tilde{f}^*(z,0)\right) \quad \left(= \mathrm{Re}\,\tilde{f}(z,0)\right)$$

$$\tilde{f}(z,0) = \int_{-\infty}^{\infty} dk\ \tilde{A}(k)\, e^{ikz}$$

$$\tilde{f}^*(z,0) = \int_{-\infty}^{\infty} dk\ \tilde{A}^*(k)\, e^{-ikz}$$

$$\underset{k \to -k}{=} \int_{-\infty}^{\infty} dk\ \tilde{A}^*(-k)\, e^{ikz}$$

$$\therefore f(z,0) = \int_{-\infty}^{\infty} \frac{1}{2}\left(\tilde{A}(k) + \tilde{A}^*(-k)\right) e^{ikz} \quad \text{--- (16)}$$

$\therefore$ Knowing $f(z,0)$ only determins $\frac{1}{2}\left(\tilde{A}(k) + \tilde{A}^*(-k)\right)$

$$\frac{1}{2}\left(\tilde{A}(k) + \tilde{A}^*(-k)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz\ e^{-ikz} f(z,0) \quad \text{--- (17)}$$

Similarly, $\frac{d}{dt}f(z,0) = \frac{1}{2}\left(\frac{d\tilde{f}}{dt}(z,0) + \frac{d\tilde{f}^*}{dt}(z,0)\right)$

$$\frac{d\tilde{f}}{dt}(z,0) = \int_{-\infty}^{\infty} dk \left(-i\omega(k)\tilde{A}(k)\right) e^{ikz}$$

$$\frac{d\tilde{f}^*}{dt}(z,0) = \int_{-\infty}^{\infty} dk \left(i\omega(k)\tilde{A}^*(k)\right) e^{-ikz}$$

$$\underset{\substack{\uparrow \\ k\to -k}}{=} \int_{-\infty}^{\infty} dk \left[i\omega(-k)\tilde{A}^*(-k)\right] e^{ikz}$$

$\omega(-k) = \omega(k)$ ($\omega(k)$ is even in k : $\omega = v|k|$)

$$= \int_{-\infty}^{\infty} dk \left[+i\omega(k)\tilde{A}^*(-k)\right] e^{ikz}$$

$$\therefore \frac{df}{dt}(z,0) = \int_{-\infty}^{\infty} dk \frac{i}{2}\left(\omega\tilde{A}^*(-k) - \omega\tilde{A}(k)\right) e^{ikz} \quad \cdots (18)$$

$$\frac{i}{2}\omega\left(\tilde{A}^*(-k) - \tilde{A}(k)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz\, e^{-ikz} \frac{df}{dt}(z,0) \quad \cdots (19)$$

From eqs (17) & (18), one can determine $\tilde{A}(k)$:

$$\tilde{A}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz \left(f(z,0) + \frac{i}{\omega}\frac{df}{dt}(z,0)\right) e^{-ikz} \quad \cdots (20)$$

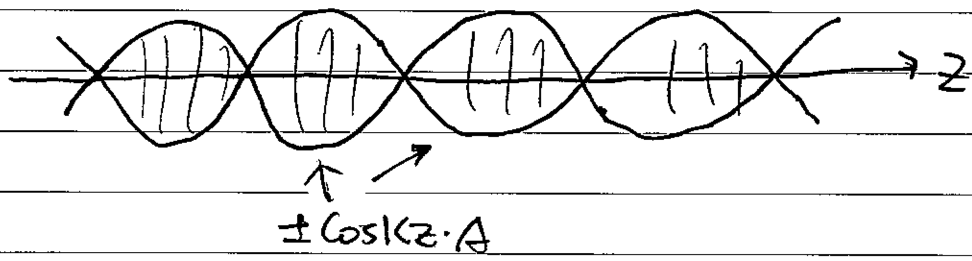
Once $\tilde{A}(k)$ is found, $f(z,t)$ can be found as

$$f(z,t) = \mathrm{Re}\int_{-\infty}^{\infty} \tilde{A}(k)\, e^{i(kz-\omega(k)t)}\, dk. \quad \cdots (21)$$

Travelling waves vs standing waves.

In general, the superposition of waves may result in disturbances that do not move. These are standing waves.

For instance, $A\cos(kz-\omega t)+A\cos(kz+\omega t)$ $=2A\cos kz\sin\omega t$ is a standing wave resulting from two sinusoidal waves travelling in opposite directions

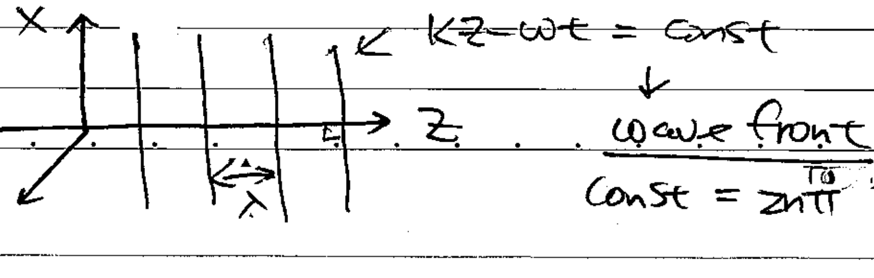


Sinusoidal waves in higher dimensions

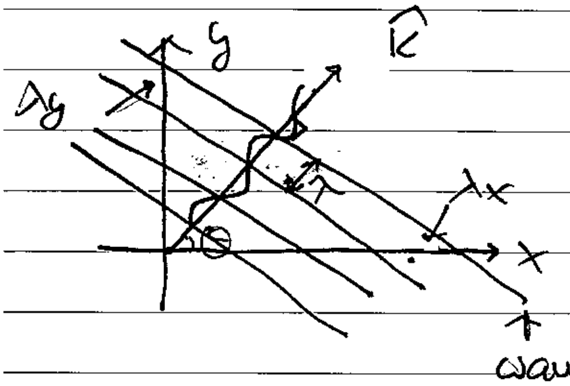
For 1D sinusoidal wave, $A\cos(kz\pm\omega t)$, the wave propagates toward $\pm z$ direction.

This wave can be viewed as part of 3D waves:

$y=A\cos(kz-\omega t)$

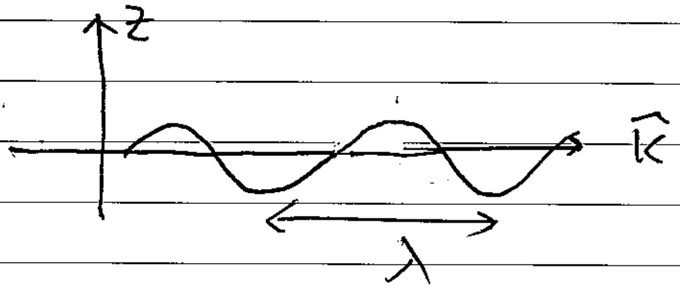


In general, the sinusoidal wave needs not to propagate along z axis only.



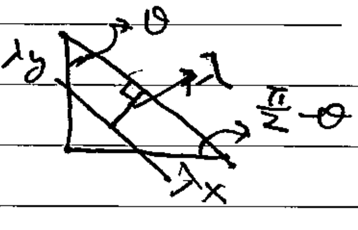
For instance, it can propagate along a direction (incline angle θ) in xy plane as shown in the left figure

The direction is denoted by $\hat{k}$. Along $\hat{k}$, one has



with λ = wavelength.

However, along x or y axis, the wavelengths appear to be enlarged:



$$\frac{\lambda}{\lambda_y} = \sin\theta \quad \text{--- (22)}$$

$$\frac{\lambda}{\lambda_x} = \cos\theta \quad \text{--- (23)}$$

$\because \lambda = \frac{2\pi}{k}$, $\therefore$ eqs (22) & (23) can be written as

$$\frac{2\pi}{\lambda_y} = k\sin\theta \quad \text{--- (24)}$$

$$\frac{2\pi}{\lambda_x} = k\cos\theta \quad \text{--- (25)}$$

Eqs (24) & (25) imply that it's more convenient to introduce the wave vector $\vec{k} = k\hat{k}$ so that

$$k\cos\theta = k_x, \quad k\sin\theta = k_y$$

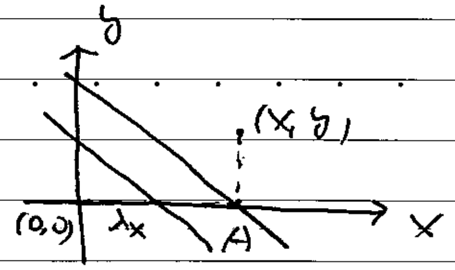
are x & y components of $\vec{k}$.

Hence $\quad k_x = \frac{2\pi}{\lambda_x}, \quad k_y = \frac{2\pi}{\lambda_y}$

$\quad = k\cos\theta \quad = k\sin\theta.$

One sees that in higher dimensions, $\vec{k}$ really becomes a vector.

<u>Phase</u> For a given point (x, y), one needs to find its phase to find the sinusoidal wave expression.



For a point (x, y), relative to $(0,0)$, the phase

$$= \underbrace{\frac{x}{\lambda_x} \times 2\pi}_{\text{relative phase between A \& 0}} + \underbrace{\frac{y}{\lambda_y} \times 2\pi}_{\text{relative phase between A \& (x, y)}}$$

$\therefore$ phase $= x \frac{2\pi}{\lambda_x} + y \frac{2\pi}{\lambda_y} + \delta$ ← phase of $(0,0)$

$$= K_x x + K_y y + \delta$$

$$= \vec{K} \cdot \vec{r} + \delta$$

Therefore, the sinusoidal waves can be generally written as

$$A \cos(\vec{K} \cdot \vec{r} - \omega t + \delta)$$

or $\cancel{A \cos(\vec{K} \cdot \vec{r} - \omega t) + B \sin(\vec{K} \cdot \vec{r} - \omega t)}$

or $\psi = Re\left[\tilde{A}\, e^{i(\vec{K} \cdot \vec{r} - \omega t)}\right]$ --- (24)

For non-dispersive medium, $\frac{\omega}{K} = v$, ψ satisfies the wave equation in 3D:

$\because \vec{\nabla} e^{i(\vec{K} \cdot \vec{r} - \omega t)} = i\vec{K}\, e^{i(\vec{K} \cdot \vec{r} - \omega t)}$

$$\left(\because \frac{\partial}{\partial x} e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \frac{\partial}{\partial x} e^{i(k_x x + \cdots - \omega t)} = ik_x e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right.$$

$$\left. \frac{\partial}{\partial y} e^{i(\vec{k}\cdot\vec{r}-\omega t)} = ik_y e^{i(\vec{k}\cdot\vec{r}-\omega t)} \cdots \right)$$

$$\nabla^2 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \vec{\nabla}\cdot\left[i\vec{k} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right]$$

$$= i\vec{k}\cdot\nabla e^{i(\vec{k}\cdot\vec{r}-\omega t)} = (i\vec{k})^2 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\therefore \nabla^2 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = -k^2 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

Similarly $\frac{\partial^2}{\partial t^2} e^{i(\vec{k}\cdot\vec{r}-\omega t)} = (-i\omega)^2 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$$= -\omega^2 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\because \omega = vk, \quad \omega^2 = v^2 k^2$$

$$\therefore \left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2} \right) e^{i(\vec{k}\cdot\vec{r}-\omega t)} = 0$$

$$\therefore \nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi \quad \cdots \quad (25)$$

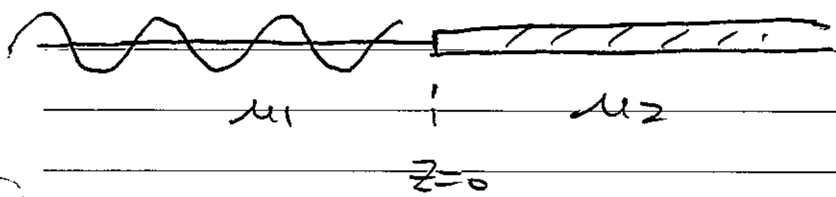
Eq. (25) is the general wave equation in 3D.

Inhomogeneous medium : reflection and transmission

In the above discussion of wave propagation, we have confined the discussion to one medium.

When there are many media, waves will be reflected with transimitted at the boundaries.

As an example, one consider two stretched strings with different mass densities μ_1 and μ_2.



For $z<0$, $v_1 = \sqrt{\frac{T}{\mu_1}}$, $z>0$, $v_2 = \sqrt{\frac{T}{\mu_2}}$

For fixed frequency ω, one has

plane waves :
$$\begin{cases} e^{\pm i(k_1 z \pm \omega t)} & z<0 \\ k_1 = \omega/v_1 & \\ e^{i\pm(k_2 z \pm \omega t)} & z>0 \\ k_2 = \omega/v_2 & \end{cases}$$

$\therefore \frac{\lambda_1}{\lambda_2} = \frac{k_2}{k_1} = \frac{v_1}{v_2}$ wavelengths are different on different media

If there is a wave incident from $z=-\infty$ and moves to $+\infty$, one can write (the string is oscillating at fixed ω at $z=-\infty$)

$$\tilde{f}(z,t) = \tilde{A}_I e^{i(k_1 z - \omega t)} \quad \text{for } z<0.$$

However, this is not sufficient as at $z=0$, the wave will get reflected.

The reflected wave moves towards $z=-\infty$ so it must be in the form

$$\tilde{A}_R e^{i(-k_1 z - \omega t)}$$

Here $e^{-i\omega t}$ is not changed as at $z=0$, the inhomogeneous part affects only on the spacial part of the wave.

To move in the opposite direction, the wave vector k_1 needs to be reversed!

Hence $\tilde{f}(z,t)$ for $z<0$ should be the superposition of $\tilde{A}_I$ & $\tilde{A}_R$

$$\tilde{f}(z,t) = \tilde{A}_I e^{i(k_1 z - \omega t)} + \tilde{A}_R e^{i(-k_1 z - \omega t)} \quad \text{.... (26)}$$

$z<0$

For $z>0$, there is only one wave moving towards $+\infty$, $\therefore$

$$\tilde{f}(z,t) = \tilde{A}_T e^{i(k_2 z - \omega t)} \quad \text{.... (27)}$$

$z>0$

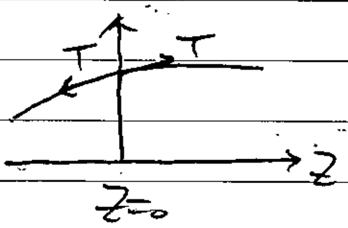
At $z=0$, $\hat{f}(z<0)$ & $\tilde{f}(z>0)$ needs to be joined together.

Since the displacement is continuous,

$$\therefore \quad f(z=0^-,t) = f(z=0^+,t) \quad \text{--- (28)}$$

For waves on strings, $\Delta F = T\frac{\partial^2 f}{\partial z^2}\Delta z$ (eq (7)), $\therefore$ $\frac{\partial f}{\partial z}$ must be continuous so that $\frac{\partial^2 f}{\partial z^2}$ can be defined!

$$\therefore \quad \frac{\partial f}{\partial z}(0^-,t) = \frac{\partial f}{\partial z}(0^+,t) \quad \text{--- (29)}$$



i.e. there is no net force acting on $z=0$

(slope is continuous across $z=0$)

$$\because \quad f(z,t) = \mathrm{Re}\,\tilde{f}(z,t) = \frac{1}{2}\left(\tilde{f}(z,t) + \tilde{f}^*(z,t)\right)$$

$$\frac{df}{dz}(z,t) = \mathrm{Re}\,\frac{d\tilde{f}}{dz}(z,t) = \frac{1}{2}\left(\frac{d\tilde{f}}{dz}(z,t) + \frac{d\tilde{f}^*}{dz}(z,t)\right)$$

L-- (29)*

Eq. (28) & (29) can be replaced by

$$\tilde{f}(z=0^-,t) = \tilde{f}(z=0^+,t) \quad \text{-- (30)}$$

$$\frac{d\tilde{f}}{dz}(z=0^-,t) = \frac{d\tilde{f}}{dz}(z=0^+,t) \quad \text{--- (31)}$$

Eqs. (30) & (31) will enforce eqs. (28) & (29) through using eq. (29)*

Combining eqs. (26), (27) & (30), (31), one gets

$$\tilde{\hat{A}}_I + \tilde{\hat{A}}_R = \tilde{\hat{A}}_T$$

$$ik_1 \tilde{\hat{A}}_I - ik_1 \tilde{\hat{A}}_R = ik_2 \tilde{\hat{A}}_T$$

$\therefore$

$$\tilde{\hat{A}}_I + \tilde{\hat{A}}_R = \tilde{\hat{A}}_T$$

$$k_1(\tilde{\hat{A}}_I - \tilde{\hat{A}}_R) = k_2 \tilde{\hat{A}}_T$$

$$\tilde{\hat{A}}_R = \frac{k_1 - k_2}{k_1 + k_2} \tilde{\hat{A}}_I, \quad \tilde{\hat{A}}_T = \frac{2k_1}{k_1 + k_2} \tilde{\hat{A}}_I \quad \text{--- (32)}$$

$\because \frac{k_1}{k_2} = \frac{v_2}{v_1}$, $\tilde{\hat{A}}_R = \underset{\substack{\uparrow \\ \text{real} > 0}}{A_R} e^{i\delta_R}$, $\tilde{\hat{A}}_T = A_T e^{i\delta_T}$

$$\tilde{\hat{A}}_I = A_I e^{i\delta_I}$$

we obtain

$$A_R e^{i\delta_R} = \frac{v_2 - v_1}{v_2 + v_1} A_I e^{i\delta_I} \quad \text{--- (33)}$$

$$A_T e^{i\delta_T} = \frac{2v_2}{v_2 + v_1} A_I e^{i\delta_I} \quad \text{--- (34)}$$

For $\mu_2 < \mu_1$ (string 2 at $z > 0$ is lighter), $v_2 > v_1$, eq (33) implies $\delta_R = \delta_I$.

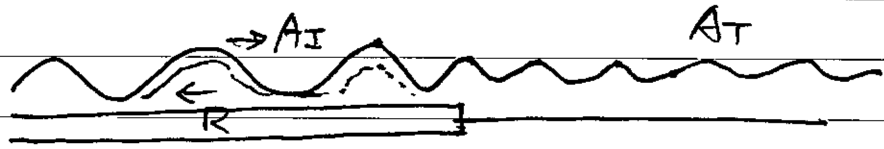
and eq (34) implies $\delta_I = \delta_T$.

$\therefore \quad \delta_R = \delta_I = \delta_T$

$$\left.\begin{aligned} A_R &= \frac{v_2 - v_1}{v_2 + v_1} A_I \\ A_T &= \frac{2v_2}{v_2 + v_1} A_I \end{aligned}\right\} \cdots (35)$$

Once A_I is given, A_R & A_T are known.

In this case, the reflected & transmitted waves don't have phase shift relative to the incident wave



For $\mu_2 > \mu_1$, $v_1 > v_2$

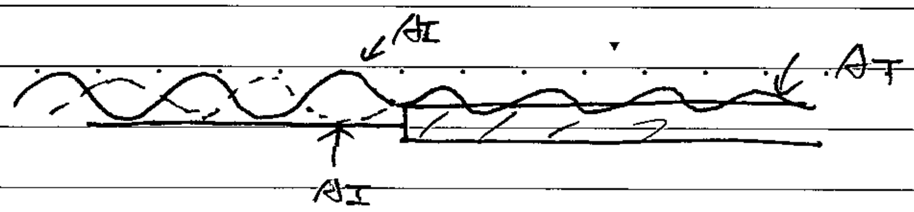
eq (33) implies $e^{i\delta_R} = \underbrace{(-1)}_{e^{-i\pi}} \times e^{i\delta_I}$

$\therefore \delta_R + \pi = \delta_I$

$\because$ eq (34) $\Rightarrow \delta_I = \delta_T$

$\therefore \quad \delta_R + \pi = \delta_T = \delta_I$

The reflected wave is shifted by π when the string at $z>0$ is heavier.



~~Here A_I, A_R & A_T still satisfy eq. (35) with $v_2 - v_1$ replaced by $v_1 - v_2$~~

If the string at $z>0$ is infinitely massive, $v_2 = \sqrt{\frac{T}{\mu_2}} = 0$,

then $$A_R = \frac{v_1 - v_2}{v_1 + v_2} A_I = A_I$$

$$A_T = \frac{2v_2}{v_1+v_2} A_I = 0$$

There is no transmitted wave, as expected.

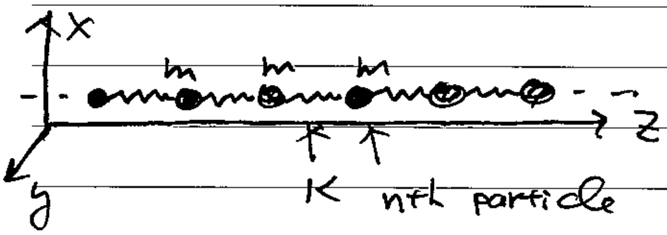
Polarization

The wave on a string is called transverse because the configuration (= the displacement of string) is perpendicular to the direction of propagation.

In general, there are different possible moving directions of elements in the medium. These different moving directions (configurations).

are called polarizations of waves.

As an example, one considers modelling the string as particles of mass m connected by springs.

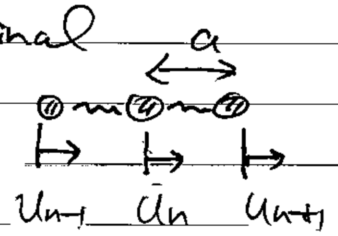


There are three possible directions for the displacement of the nth particle:

$$\underbrace{x_n, y_n}_{\text{transverse}},\ \underbrace{z_n}_{\text{longitudinal}} \equiv u_n$$

$\therefore$ polarizations:
2 transverse x/y
1 longitudinal z

(i) longitudinal



$a = $ nearby distance at equilibrium

$$m\frac{d^2u_n}{dt^2} = -K(u_n - u_{n-1}) - K(u_n - u_{n+1})$$

$$\approx -Ka\left[\frac{\partial u}{\partial z}\right]_{z=n-1/2} - Ka\left[\frac{\partial u}{\partial z}\right]_{u=n+1/2}$$

$$= Ka^2\frac{\partial^2 u}{\partial z^2}\bigg|_{u=n}$$

$\therefore \dfrac{\partial^2 u(z,t)}{\partial t^2} = \dfrac{Ka^2}{m}\dfrac{\partial^2 u(z,t)}{\partial z^2}$ describes the longitudinal wave.

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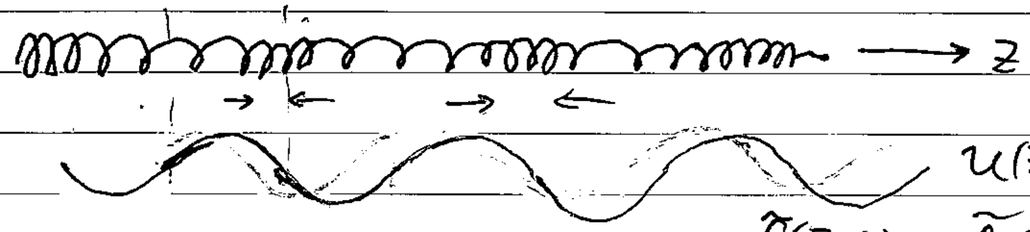
The longitudinal wave is a compression wave.

Sound wave is an example.

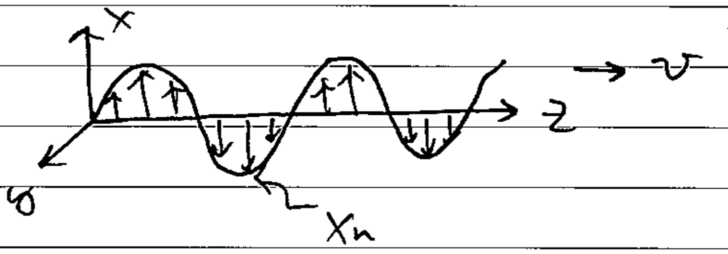
$(\rho \sim 1 - \frac{du}{dz})$



$\tilde{f}(z,t) = \tilde{A} e^{i(kz-\omega t)} \hat{z}$

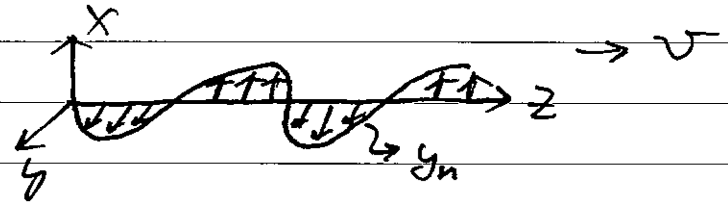
$(\hat{k})$

(ii) transverse



$\tilde{f}(z,t) = \tilde{A} e^{i(kz-\omega t)} \hat{x}$

or



$\tilde{f}(z,t) = \tilde{A} e^{i(kz-\omega t)} \hat{y}$

i.e. $\tilde{f}(z,t) = \tilde{A} e^{i(kz-\omega t)} \hat{n}$

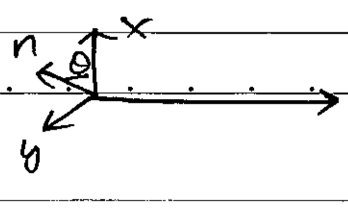
transverse: $\hat{n} \cdot \hat{z} = 0$ $\quad (\hat{n} \cdot \hat{k} = 0)$

waves with $\tilde{f} \parallel$ $\hat{x}$ or $\hat{y}$ are known as linearly polarized

In general, the wave equation is linear,

$\hat{n}$ needs not to be either $\hat{x}$ or $\hat{y}$

and can be



$\hat{n} = \cos\theta\, \hat{x} + \sin\theta\, \hat{y}$ -- (36)

θ = polarization angle.

In this case, the displacement

$$\hat{f}(z,t) = \tilde{A}\cos\theta\, e^{i(kz-\omega t)}\hat{x}$$

$$+\tilde{A}\sin\theta\, e^{i(kz-\omega t)}\hat{y}$$

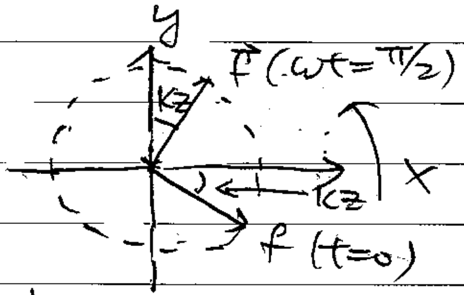
More general, one may combine $\hat{x}$ polarization & $\hat{y}$ polarization with a phase different.

For instance:

$$\vec{f}(z,t) = A\cos(kz-\omega t)\,\hat{x} + A\cos(kz-\omega t+\tfrac{\pi}{2})\,\hat{y}$$

$$= \underbrace{A\cos(kz-\omega t)}_{u_x}\hat{x} - \underbrace{A\sin(kz-\omega t)}_{u_y}\hat{y}$$

$$u_x^2+u_y^2 = A^2$$

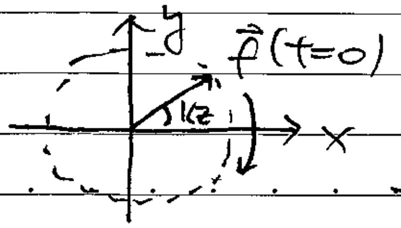


$\vec{f}$ moves counterclockwise on a circle of radius A.

This is known as left circular polarization

if $\vec{f}(z,t) = A\cos(kz-\omega t)\,\hat{x} + A\cos(kz-\omega t-\pi/2)\,\hat{y}$

$$= A\cos(kz-\omega t)\,\hat{x} + A\sin(kz-\omega t)\,\hat{y}$$



$\vec{f}$ moves clockwise on a circle of radius A

This is known as right circular polarization.

Electromagnetic waves in vacuum

As we shall show that in the vacuum, fundamental solutions to the Maxwell's equations are waves.

These waves are called electromagnetic waves.

We start from Maxwell's equations in vacuum. Here there is no charge and current

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \cdots \; (37)$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \cdots \; (38)$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \cdots \; (39)$$

$$\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \qquad \cdots \; (40)$$

taking $\vec{\nabla}\times$ on (38), one gets

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$\because\ \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{E})}_{=0\ (\text{eq. } 37)} - \nabla^2\vec{E}$$

$$\vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{B} \underset{\text{eq. } 40}{=} -\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

We arrive at

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad \cdots (41)$$

Similarly, by taking $\vec{\nabla}\times$ on (40), one gets

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \mu_0\epsilon_0 \vec{\nabla}\times\frac{\partial \vec{E}}{\partial t}$$

$$\because \quad \vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{B}}_{0}) - \nabla^2\vec{B}$$

$$\vec{\nabla}\times\frac{\partial \vec{E}}{\partial t} = \frac{\partial}{\partial t}\vec{\nabla}\times\vec{E} = -\frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\therefore \quad \nabla^2 \vec{B} = \mu_0\epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \quad \cdots (42)$$

$\therefore$ Both $\vec{E}$ & $\vec{B}$ satisfy 3D wave equation

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

With $v = \frac{1}{\sqrt{\epsilon_0\mu_0}} = 3\times10^8$ m/s

Which happens to be speed of light, c.

The coincidence leads to the identification of Electromagnetic waves as light!

Note that without the term $\epsilon_0\mu_0 \frac{\partial \vec{E}}{\partial t}$, one would not be able to get the wave equation! There would be no electromagnetic theory of light!

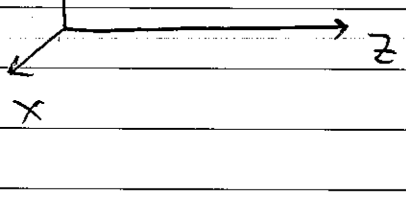
Monochromatic plane waves

It may seem that $\vec{E}$ & $\vec{B}$ are independent in eqs. (41) & (42). In fact, $\vec{E}$ & $\vec{B}$ still need to satisfy (37) – (40) so that they are not independent from each other.

The fundamental solution of EM waves in vacuum is the sinusoidal waves with fixed ω as we shall see later that those waves are generated by oscillating charges or currents.

Since fixed ω corresponds to fixed color in light, it is often referred as monochromatic plane waves.

Suppose that the plane wave travels in +z direction. The complex E & B fields are then in the form



$$\tilde{\mathbb{E}} = \tilde{\mathbb{E}}_0 \, e^{i(kz-\omega t)}$$
$$\tilde{\mathbb{B}} = \tilde{\mathbb{B}}_0 \, e^{i(kz-\omega t)} \quad \text{--- (43)}$$

$\tilde{\mathbb{E}}_0$ & $\tilde{\mathbb{B}}_0$ are complex amplitudes and $\omega = c\,k$.

Physical fields are given by

$$\vec{E}(z,t) = Re\,(\tilde{\mathbb{E}}_0 e^{i(kz-\omega t)})$$

$$\vec{B}(z,t) = Re\,(\tilde{\mathbb{B}}_0 e^{i(kz-\omega t)})$$

Eq. (43) satisfies eqs (42) & (41)

However, to satisfy (37) – (40),

one requires $\vec{\nabla}\cdot\vec{E} = 0$ & $\vec{\nabla}\cdot\vec{B} = 0$

$\because$ $\vec{\nabla}\cdot\tilde{\mathbb{E}} = \tilde{E}_{0z}\frac{\partial}{\partial z}e^{i(kz-\omega t)}$

$\therefore$ one requires $\tilde{E}_{0z} = 0$

Similarly, $\tilde{B}_{0z} = 0$

$\therefore$ $\tilde{\mathbb{E}}_0 \perp z$, $\tilde{\mathbb{B}}_0 \perp z$, $\vec{E}$ & $\vec{B}$ are

transverse --- (44)

i.e. EM waves in vacuum are transverse.

In addition, to satisfy $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$,

We require: $\vec{\nabla}\times\vec{E} = Re\,\vec{\nabla}\times(\tilde{\mathbb{E}}_0 e^{i(kz-\omega t)})$

$$= Re\left[\nabla e^{i(kz-\omega t)}\times\tilde{\mathbb{E}}_0\right]$$

$$= Re\left[ik e^{i(kz-\omega t)}\hat{z}\times\tilde{\mathbb{E}}_0\right]$$

$$-\frac{\partial \vec{B}}{\partial t} = Re\left[-\tilde{\mathbb{B}}_0 \frac{\partial}{\partial t} e^{i(kz-\omega t)}\right]$$

$$= Re\left[i\omega \tilde{\mathbb{B}}_0 e^{i(kz-\omega t)}\right]$$

$$\therefore \; ik\hat{z}\times\tilde{\mathbb{E}}_0 = i\omega\tilde{\mathbb{B}}_0$$

i.e. $\tilde{\mathbb{B}}_0 = \frac{k}{\omega}\hat{z}\times\tilde{\mathbb{E}}_0$

$$\vec{B} = \frac{k}{\omega}\hat{z}\times\vec{E} = \frac{1}{c}\hat{z}\times\vec{E} \quad \text{-- (45)}$$

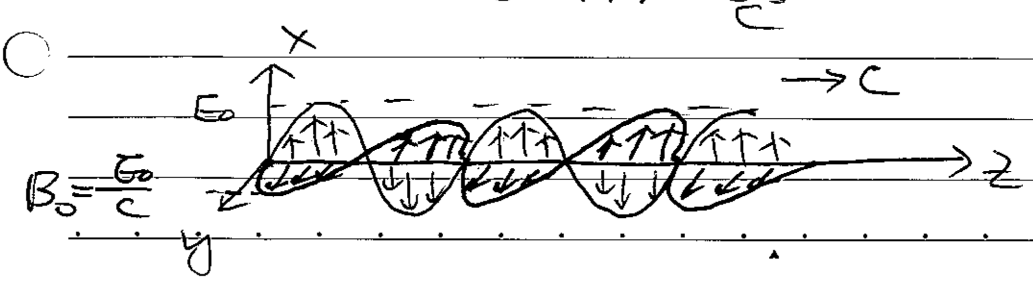
Therefore, $\vec{B}\perp\vec{E}$ and $B_0 = \frac{1}{c}E_0$ --- (46)
(Note that $\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$ does not yield a new relation.)
Combing eqs (44) & (45), for linearly polarized EM monochromatic waves, one may choose $\tilde{\mathbb{E}}_0 = e^{i\delta}E_0\hat{x}$, $\therefore \tilde{\mathbb{B}}_0 = \frac{1}{c}\hat{z}\times\hat{x}E_0 e^{i\delta}$

$E_0, B_0 = $ real $\qquad = B_0\hat{y}e^{i\delta}$

one has $\vec{E}(z,t) = Re\, E_0 e^{i(kz-\omega t+\delta)}\hat{x}$

$$= E_0\cos(kz-\omega t+\delta) \quad \text{-- (47)}$$

$$\vec{B}(z,t) = \frac{E_0}{c}\cos(kz-\omega t+\delta) \quad (48)$$



The EM wave is said to be polarized in x direction.

In general, $\vec{k}$ needs not to be in z direction. The linearly-polarized monochromotic wave is given by

$$\therefore \vec{E}(r,t) = Re\left[\tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,\hat{n}\right] \quad -(49)$$

$$\hat{n}\cdot\hat{k} = 0$$

$$\vec{B}(\vec{r},t) = Re\left[\frac{\tilde{E}_0}{c} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,\hat{k}\times\hat{n}\right]$$

$$= \frac{\hat{k}}{c}\times\vec{E}(r,t) \quad \cdots (50)$$

EM waves are classified according to ranges of frequencies as follows

f (Hz)			
10^{22}		10^{10}	microwave (10^{-1}m)
10^{21}	γ-rays	10^{9}	
10^{20}		10^{8}	TV, FM (10m)
10^{19}		10^{7}	
10^{18}	X-rays	10^{6}	AM (10^{3}m)
10^{17}		10^{5}	
10^{16}	ultraviolet (10^{-7}m)	10^{4}	RF (10^{5}m)
10^{15}	visible (10^{-6}m)	10^{3}	
10^{14}	infrared		
10^{13}			
10^{12}			
10^{11}			

Other polarizations

The linearly polarized EM waves are not the only possible monochromatic waves. For $\hat{k} = \hat{z}$, the most general electric field is given by

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[\underbrace{(\tilde{E}_{0x}\hat{x} + \tilde{E}_{0y}\hat{y})}_{\tilde{\mathbb{E}}_0} e^{i(kz-\omega t)} \right]$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}\,\hat{z}\times\vec{E}(\vec{r},t)$$

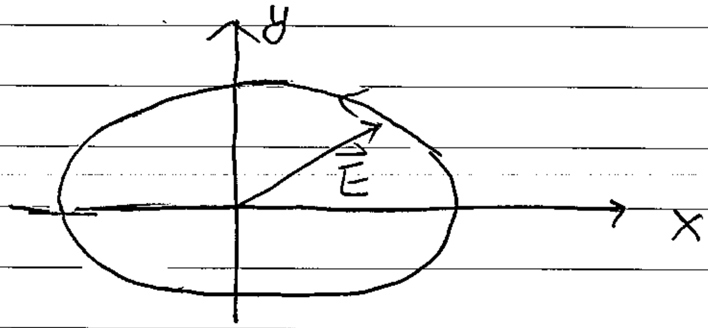
if $\tilde{E}_{0x} = E_{0x}e^{i\delta}$, $\tilde{E}_{0y} = E_{0y}\,i\,e^{i\delta}$

$$\vec{E}(\vec{r},t) = \underbrace{E_{0x}\cos(kz-\omega t+\delta)}_{E_x}\hat{x} \underbrace{- E_{0y}\sin(kz-\omega t+\delta)}_{E_y}\hat{y}$$

$$E_y = E_{0y}\sin(\omega t - kz - \delta)$$

$$\therefore \frac{E_x^2}{E_{0x}^2} + \frac{E_y^2}{E_{0y}^2} = 1$$

(E_x, E_y) moves on an ellipse



this is known as elliptically polarized.

In particular, if $E_{0y} = iE_{0x}$

$\Rightarrow$ we recovers the left circular polarization

$E_{0y} = -E_{0x} \Rightarrow$ right circular polarization

Note that the most general ellipitically polarized light can be viewed as a superposition of two linearly polarized light waves along $\hat{x}$ and $\hat{y}$.

Energy and momentum in EM waves

As we have derived, the energy density (energy per volume)

$$u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

For plane waves, $B_0^2 = \frac{E_0^2}{c^2}$
monochromatic

$$\therefore B^2 = \frac{E^2}{c^2} = \mu_0 \varepsilon_0 E^2$$

$$\therefore u = \frac{1}{2}(\varepsilon_0 E^2 + \varepsilon_0 E^2) = \varepsilon_0 E^2$$

$$= \varepsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)$$

cf. (47) — (57)

The energy flux is given by the Poynting vector

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

$$= \frac{1}{\mu_0} E_0 \hat{x} \times \frac{E_0}{c} \hat{y} \cos^2(kz - \omega t + \delta)$$

$$= c\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\,\hat{z}$$

$$= c u \hat{z}$$

$$= \text{energy density} \times \vec{c} \quad \text{--- (52)}$$

in agreement with the definition of energy flux

The magnitude of $\vec{S}$ is known as the intensity of light (I). Clearly, $I \propto E^2$. --- (53)

Momentum density

$$\vec{g} = \frac{1}{c^2} \vec{S}$$

Using eq (52), $\vec{g} = \frac{1}{c} u \hat{z} = \frac{1}{c} \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) \hat{z}$ --- (54)

∴ For monochromatic EM waves, the relation of momentum density to energy density is the same as that for a zero-mass particle in special relativity (as we shall derive later):

$$\mathcal{E} = \sqrt{p^2c^2 + m^2c^4} = pc$$

$$\Leftrightarrow \quad u = gc.$$

Average quantities

For typical light (visible), $\lambda \sim 5\times10^{-7}\,m$

$T \sim 10^{-15}$ sec

Any macroscopic measurement : $\Delta t \gg T$

The averaged quantities are what are measured.

So, one performs the averaging by taking $\langle A \rangle \equiv \frac{1}{T}\int_0^T A(t)\,dt$

$$T = \frac{2\pi}{\omega}$$

$$\because \frac{1}{T}\int_0^T \cos^2(kz - \omega t + \delta)\,dt$$

$$= \frac{1}{T}\int_0^T \frac{1}{2}\left(1 + \underbrace{\cos 2(kz - \omega t + \delta)}_{\text{equally positive and negative}}\right) dt \quad 0$$

$$= \frac{1}{2}$$

$\therefore \langle u \rangle = \frac{1}{2}\varepsilon_0 E_0^2$

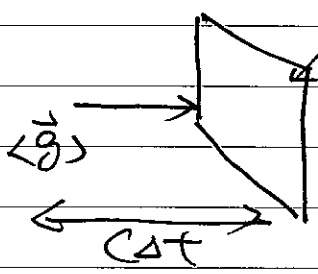
$$\langle \vec{S} \rangle = \frac{1}{2}\epsilon_0 E_0^2 c\, \hat{z}$$

$$\langle \vec{g} \rangle = \frac{1}{2c}\varepsilon_0 E_0^2 \hat{z}$$

$$I\,(\text{intensity}) \equiv \langle S \rangle = \frac{1}{2}c\,\epsilon_0 E_0^2$$

Radiation pressure

The momentum density carried by monochromatic EM wave can be exhibited as radiation pressure when light falls on a perfect absorbor:



During Δt, the total momentum absorbed for an area of A $= \langle\vec{g}\rangle \cdot (A \cdot c\Delta t) \equiv \Delta\vec{p}$

$$\therefore P\,(\text{pressure}) = \frac{\Delta p}{A\Delta t} = \langle g \rangle c$$

$$= \frac{1}{2}\varepsilon_0 E_0^2 = \frac{I}{c}$$

By measuring the radiation pressure, the momentum density can be verified.

Electromagnetic waves in matter

Linear and homogeneous media (insulator)

Inside matter, if there is no free charge or free current, the Maxwell's equations become

$$\nabla \cdot \vec{D} = 0 \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

For linear and homogeneous media,

$$\vec{D} = \epsilon \vec{E} \quad \& \quad \vec{H} = \frac{1}{\mu} \vec{B}$$

where ϵ & μ are constants.

In this case, $\vec{\nabla} \cdot \vec{D} = 0 \Rightarrow \epsilon \vec{\nabla} \cdot \vec{E} = 0 \quad \therefore \vec{\nabla} \cdot \vec{E} = 0$

$$\vec{\nabla} \times \vec{H} = \frac{1}{\mu} \vec{\nabla} \times \vec{B} = \frac{\partial \vec{D}}{\partial t} = \epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\therefore \vec{\nabla} \times \vec{B} = \mu \epsilon \frac{\partial \vec{E}}{\partial t}$$

Therefore, the Maxwell's equations are the same as those in vacuum except that ϵ_0, μ_0 are replaced by ϵ & μ.

Hence the speed of EM wave

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \sqrt{\frac{\epsilon_0\mu_0}{\epsilon\mu}}\ \underbrace{\frac{1}{\sqrt{\epsilon_0\mu_0}}}_{c}$$

$$\equiv \frac{c}{n} \quad \text{--- (56)}$$

$n \equiv \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$ is the index of refraction

for the matter. --- (57)

For most materials, $\mu \simeq \mu_0$

$$\therefore n \cong \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{\epsilon_r} \qquad \epsilon_r = \text{dielectric constant.}$$

ϵ_r is mostly > 1, $\therefore n > 1$

$$\underline{v < c}$$

i.e. light travels slowly than c in most materials (phase speed $< c$)

In addition to eq. (56), other quantities are changed accordingly:

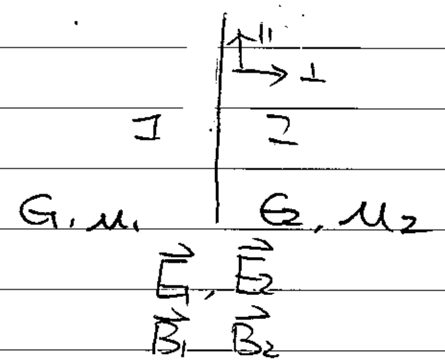
$$u = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu} B^2\right)$$

$$\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B}$$

$$I = \frac{1}{2}\epsilon E_0^2 v$$

Reflection and transmission at interface of (refraction) linear media (insulator)

Boundary conditions



In the simplest non-homogeneous situations, we have an interface formed by joining media 1 & 2 shown in the left figure

$\vec{E}$ & $\vec{B}$ at interface are connected by the boundary conditions :

(i) $\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}$ $\quad (\vec{\nabla}\cdot\vec{D} = 0)$ — (58)

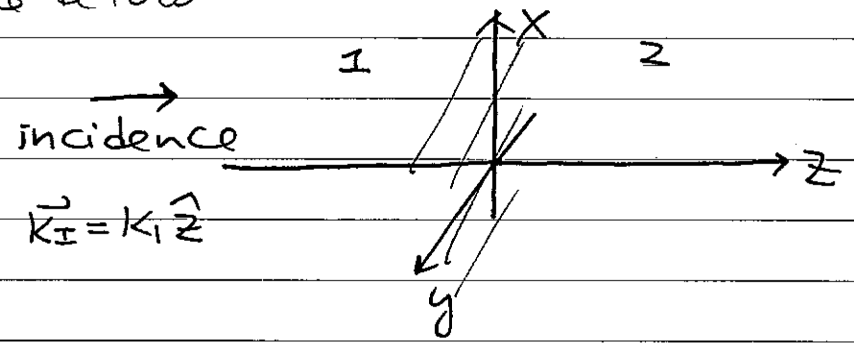
(ii) $\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$ $\quad (\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t})$ — (59)

(iii) $B_1^{\perp} = B_2^{\perp}$ $\quad (\vec{\nabla}\cdot\vec{B} = 0)$ — (60)

(iv) $\frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}$ $\quad (\vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{\partial t})$ — (61)

Normal incidence with linearly polarized light

Suppose xy plane is the interface. A plane wave of frequency ω, traveling in $+z$ direction and polarized in x direction, approaches the interface from the left ($z = -\infty$) as shown in the below



Incidence $\tilde{\mathbb{E}}_I(z,t) = \tilde{E}_{0I}\, e^{i(k_1 z - \omega t)}\, \hat{x}$ (assumed)

$$\left.\begin{array}{l} \tilde{\mathbb{E}}_I(z,t) = \tilde{E}_{0I}\, e^{i(k_1 z - \omega t)}\, \hat{x} \\ \tilde{\mathbb{B}}_I(z,t) = \underbrace{\frac{\tilde{E}_{0I}}{v_1}}_{\tilde{B}_{0I}}\, e^{i(k_1 z - \omega t)}\, \hat{y} \end{array}\right\} \quad (62)$$

$$\left(\vec{B} = \frac{1}{v_1}\, \hat{z} \times \vec{E} \right)$$

The discontinuity of ε & μ at $z = 0$ will reflect waves moving towards $z = -\infty$

$\therefore$ For the reflected wave, $k_1 \to -k_1$

$$\left.\begin{array}{l} \tilde{\mathbb{E}}_R(z,t) = \tilde{E}_{0R}\, e^{i(-k_1 z - \omega t)}\, \hat{x} \\ \vec{B} = \frac{1}{v_1}(-\hat{z}) \times \vec{E} \end{array}\right\} \quad (63)$$

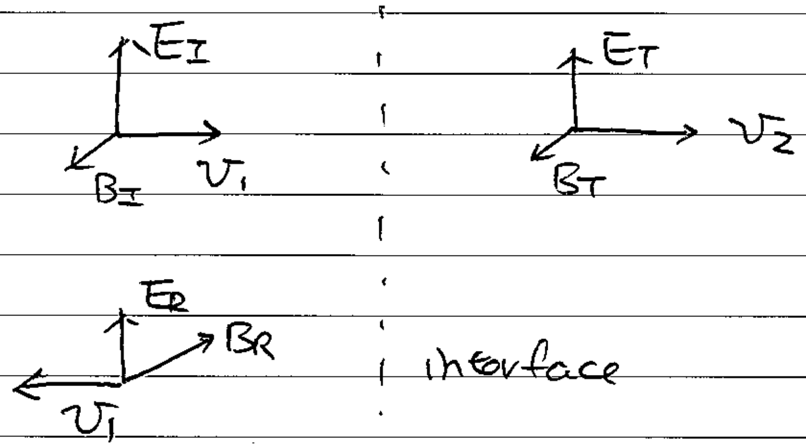
$$\therefore\ \tilde{\mathbb{B}}_R(z,t) = -\frac{1}{v_1}\, \tilde{E}_{0R}\, e^{i(-k_1 z - \omega t)}\, \hat{y}$$

There is also a transmitted wave at $z > 0$ that moves towards $z = \infty$ with $\vec{k} = k_2 \hat{z}$

$$\left.\begin{aligned} \tilde{\mathbf{E}}_T(z,t) &= \tilde{E}_{0T}\, e^{i(k_2 z-\omega t)}\,\hat{x} \\ \tilde{\mathbf{B}}_T(z,t) &= \frac{\tilde{E}_{0T}}{v_2}\, e^{i(k_2 z-\omega t)}\,\hat{y} \end{aligned}\right\} \quad (64)$$

$$\left(\vec{B} = \frac{1}{v_2}\,\hat{z}\times\vec{E} \right)$$

with the following picture:



The total fields for $z<0$ are $\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_I + \tilde{\mathbf{E}}_R$

$\tilde{\mathbf{B}} = \tilde{\mathbf{B}}_I + \tilde{\mathbf{B}}_R$,

and they have to satisfy BCs (58–61).

Eq. (59) : $\tilde{\mathbf{E}}_I + \tilde{\mathbf{E}}_R \big|_{z=0^-} = \tilde{\mathbf{E}}_T \big|_{z=0^+}$

(all $\vec{\mathbf{E}} \parallel$ interface)

$\therefore\ \tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$ --- (65)

$\because$ All $\tilde{\mathbf{E}}$ & $\tilde{\mathbf{B}}$ are parallel to interface,

$\therefore$ (i) & (iii) are satisfied ($\because \perp$ components $=0$)
(58) (60)

Eq (61) $\Rightarrow$ $\frac{1}{\mu_1}\left(\frac{1}{v_1}\tilde{E}_{0I} - \frac{1}{v_1}\tilde{E}_{0R}\right) = \frac{1}{\mu_2}\left(\frac{1}{v_2}\tilde{E}_{0T}\right)$

$$\therefore \tilde{E}_{0I} - \tilde{E}_{0R} = \underbrace{\frac{\mu_1 v_1}{\mu_2 v_2}}_{\beta} \tilde{E}_{0T} \quad \text{--- (66)}$$

Where $\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$

Eqs (65) & (66) yield

$$\tilde{E}_{0T} = \frac{2}{1+\beta}\tilde{E}_{0I}, \quad \tilde{E}_{0R} = \frac{1-\beta}{1+\beta}\tilde{E}_{0I} \quad \text{-- (67)}$$

These results are similar to those obtained for transverse waves reflected in a string: cf (35) by setting $\mu_1 = \mu_2 = \mu_0$ (vacuum value in both sides)

$$\tilde{E}_{0T} = \frac{2v_2}{v_2+v_1}\tilde{E}_{0I}, \quad \tilde{E}_{0R} = \frac{v_2-v_1}{v_2+v_1}\tilde{E}_{0I} \quad \text{-- (68)}$$

$$= \frac{2n_1}{n_1+n_2}\tilde{E}_{0I}, \quad = \frac{n_1-n_2}{n_1+n_2}\tilde{E}_{0I}$$

$\therefore$ For $v_2 > v_1$ ($n_1 > n_2$), the reflected wave is in phase with the incident wave.

While for $v_2 < v_1$ ($n_1 < n_2$), the reflected wave is out of phase with the incident wave.

Reflection and transmission coefficients

Two quantities that are most interested are

fractions of the incident energy got

reflected or transmitted.

$\because$ The average power (unit area) transported

across $= \langle \vec{S} \rangle \cdot \hat{n}$

normal of interface

$= \langle S \rangle = I$ (intensity)

$\therefore$ fraction of reflected energy

$\equiv R = \frac{I_R}{I_I}$ = reflection coefficient --- (69)

fraction of transmitted energy

$\equiv T = \frac{I_T}{I_I}$ = transmission coefficient --- (70)

$\because I_I = I_R + I_T$, $\therefore R + T = 1$

$\because I = \frac{1}{2} \epsilon v E_0^2$

$\therefore$ For $\mu_1 = \mu_2 = \mu_0$, $v_i = \frac{c}{n_i} = \frac{1}{\sqrt{\epsilon_i \mu_0}}$

$$\therefore\quad R = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left|\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}}\right|^2 = \left(\frac{n_1-n_2}{n_1+n_2}\right)^2$$

$$T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0T}}{E_{0I}}\right)^2 = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left|\frac{\tilde{E}_{0T}}{\tilde{E}_{0I}}\right|^2$$

$$\frac{\epsilon_2 v_2}{\epsilon_1 v_1} = \frac{\epsilon_2 / \sqrt{\epsilon_2}}{\epsilon_1/\sqrt{\epsilon_1}} = \frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$

$$T = \frac{n_2}{n_1}\,\frac{4n_1^2}{(n_1+n_2)^2} = \frac{4n_1n_2}{(n_1+n_2)^2}$$

Which obeys $T+R=1$.

Real # : light goes from air ($n_1 \simeq 1$) to glass ($n_2 = 1.5$)

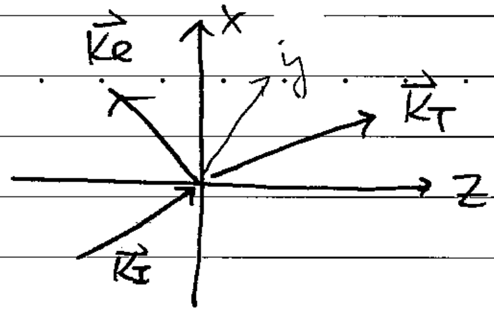
$$T = 0.96,\quad R = 0.04$$

(almost transparent.)

<u>Reflection and refraction in general incidence directions</u> (with linearly polarized light)

In the most general case, the incident $\vec{k}_I$ is not perpendicular to the interface.

$$\left.\begin{aligned} \therefore\quad \tilde{\mathbf{E}}_I(\vec{r},t) &= \tilde{\mathbf{E}}_{0I}\, e^{i(\vec{k}_I\cdot\vec{r}-\omega t)} \\ \tilde{\mathbf{B}}_I(\vec{r},t) &= \frac{1}{v_1}\,\hat{k}_I \times \tilde{\mathbf{E}}_I \end{aligned}\right\} \quad - (71)$$



The reflected wave has the same frequency and is characterized by $\vec{K_R}$ which is not $-\vec{K_I}$.

$$\left.\begin{aligned}\tilde{E}_R(\vec{r},t) &= \tilde{E}_{0R}\, e^{i(\vec{K_R}\cdot\vec{r}-\omega t)} \\ \tilde{B}_R(\vec{r},t) &= \frac{1}{v_1}\left(\hat{K}_R \times \tilde{E}_R\right)\end{aligned}\right\} \quad \cdots (72)$$

Similarly, the transmitted wave is characterized by $\vec{K_T}$.

$$\left.\begin{aligned}\tilde{E}_T(\vec{r},t) &= \tilde{E}_{0T}\, e^{i(\vec{K_T}\cdot\vec{r}-\omega t)} \\ \tilde{B}_T(\vec{r},t) &= \frac{1}{v_2}\left(\hat{K}_T \times \tilde{E}_T\right)\end{aligned}\right\} \quad \cdots (73)$$

$\because$ $\omega = vK$ is the same.

$$\therefore\ v_1 K_I = v_1 K_R = v_2 K_T$$

$$K_I = K_R = \frac{v_2}{v_1} K_T = \frac{n_1}{n_2} K_T \quad \cdots (74)$$

Now to satisfy BCs (58 – 61), we have to set, ~~for example~~,

eq (58) $\Rightarrow \left[\tilde{E}_{0I}\, e^{i(\vec{K_I}\cdot\vec{r}-\omega t)} + \tilde{E}_{0R}\, e^{i(\vec{K_R}\cdot\vec{r}-\omega t)}\right]\cdot \hat{x}\,(\text{or } \hat{y})\Big|_{z=0}$
$= \tilde{E}_{0T}\, e^{i(\vec{K_T}\cdot\vec{r}-\omega t)}\cdot\hat{x}\,(\text{or } \hat{y})\Big|_{z=0}$

Similar equations are obtained by other BCs. These equations are in the following common form:

$$\left.(\ )e^{i(\vec{k_I}\cdot\vec{r}-\omega t)}+(\ )e^{i(\vec{k_R}\cdot\vec{r}-\omega t)}\right|_{z=0}=\left.(\ )e^{i(\vec{k_T}\cdot\vec{r}-\omega t)}\right|_{z=0}$$

$e^{-i\omega t}$ gets cancelled.

∴ We have

$$\left.Ae^{i\vec{k_I}\cdot\vec{r}}\right|_{z=0}+\left.Be^{i\vec{k_R}\cdot\vec{r}}\right|_{z=0}=\left.Ce^{i\vec{k_T}\cdot\vec{r}}\right|_{z=0}$$

i.e

$$Ae^{i(xk_{Ix}+yk_{Iy})}+Be^{i(xk_{Rx}+yk_{Ry})}$$

$$=Ce^{i(xk_{Tx}+yk_{Ty})} \quad --- \quad (75)$$

Where $\vec{k_I}=(k_{Ix},k_{Iy},k_{Iz})$

$\vec{k_R}=(k_{Rx},k_{Ry},k_{Rz})$ and $\vec{k_T}=(k_{Tx},k_{Ty},k_{Tz})$

Eq. (75) has to satisfy for all possible x & y. This is possible only when all exponential terms are the same

$\therefore \; x k_{Ix} + y k_{Iy} = x k_{Rx} + y k_{Ry} = x k_{Tx} + y k_{Ty}$

i.e. ~~all phases~~ $\vec{k_I}\cdot\vec{r} = \vec{k_R}\cdot\vec{r} = \vec{k_T}\cdot\vec{r}$ are the same at $z=0$ — (76)

Consider a special case when $x=0$,

eq. (76) implies $k_{Iy} = k_{Ry} = k_{Ty}$ -- (77)

Similarly, $y=0$ leads to

$$k_{Ix} = k_{Rx} = k_{Tx} \quad --(78)$$

If we orient our axes (xyz) such that

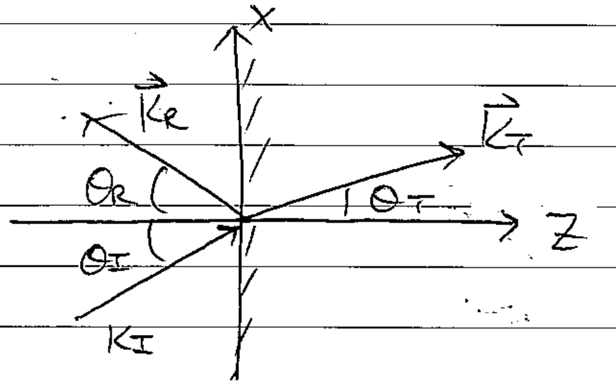
$\vec{k_I}$ lies in XZ plane, $k_{Iy}=0$.

Eq. (77) implies $k_{Ry} = k_{Ty} = 0$ as well.

Hence $\vec{k_I}$, $\vec{k_R}$ & $\vec{k_T}$ lie in the same plane. This is the first law of geometrical optics.

First law: $\vec{k_I}$, $\vec{k_R}$, $\vec{k_T}$ and normal of interface ($\hat{z}$) form a plane.
(called plane of incident)

Using the first law, one can plot $\vec{K_I}$, $\vec{K_R}$ & $\vec{K_T}$ together on XZ plane as follows



Their relative angles w.r.t. normal of interface (Z) are θ_I (angle of incident), θ_R (angle of refraction) and θ_T (angle of transmission).

Eq. (78) implies

$$K_I \sin\theta_I = K_R \sin\theta_R = K_T \sin\theta_T \quad \cdots (79)$$

Combined with eq. (74), we get

Second law : $\theta_I = \theta_R \quad \cdots (80)$

this is the law of reflection.

and the Third law, $K_I = \frac{\omega}{v_1} = n_1 \frac{\omega}{c}$, $K_T = \frac{\omega}{v_2} = n_2 \frac{\omega}{c}$

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2} \quad \cdots (81)$$

This is the law of refraction – Snell's law

Fresnel's equations

Once exponentials get cancelled, the BCs become

$$\epsilon_1(\tilde{\mathbb{E}}_{0I} + \tilde{\mathbb{E}}_{0R})_z = \epsilon_2(\tilde{\mathbb{E}}_{0T})_z \quad \text{--- (82)}$$

$$(\tilde{\mathbb{E}}_{0I} + \tilde{\mathbb{E}}_{0R})_{x,y} = (\tilde{\mathbb{E}}_{0T})_{x,y} \quad \text{--- (83)}$$

$$(\tilde{\mathbb{B}}_{0I} + \tilde{\mathbb{B}}_{0R})_z = (\tilde{\mathbb{B}}_{0T})_z \quad \text{--- (84)}$$

$$\frac{1}{\mu_1}(\tilde{\mathbb{B}}_{0I} + \tilde{\mathbb{B}}_{0R})_{x,y} = \frac{1}{\mu_2}(\tilde{\mathbb{B}}_{0T})_{x,y} \quad \text{--- (85)}$$

where $\tilde{\mathbb{B}}_0 = \frac{1}{v}\hat{k} \times \tilde{\mathbb{E}}_0$ and the interface is the xy plane.

Eqs (82) – (85) imply xy components & z component are separable. Therefore, one can divide the general cases into two cases:

(i) $(\tilde{\mathbb{E}}_{0n})_z = 0 \quad \therefore \vec{\mathbb{E}} \parallel xy$ plane (interface)
this is the TE (transverse electric field) case

(ii) $(\tilde{\mathbb{B}}_{0n})_z = 0 \quad \therefore \vec{\mathbb{B}} \parallel$ interface
this is the TM (transverse magnetic field) case.

Incident wave polarized with $\vec{E} \parallel$ plane of incident (TM case)

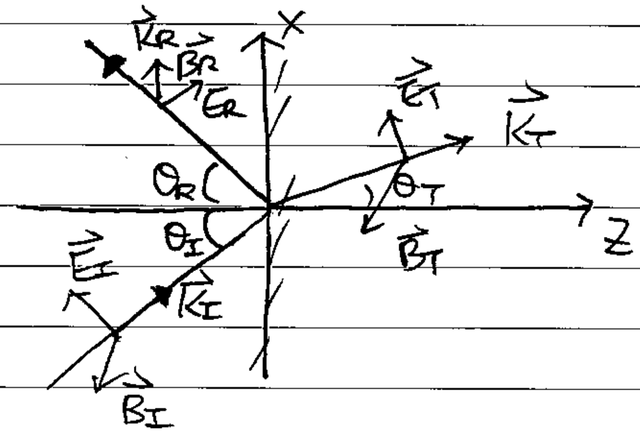
In this case $\tilde{\mathbb{B}} \parallel xy$ plane, $(\tilde{\mathbb{B}}_{0I})_z = 0$, clearly, $(\tilde{\mathbb{B}}_{0R})_z = 0$ & $(\tilde{\mathbb{B}}_{0T})_z = 0$ is a solution to eq. (84)

$\therefore$ All $\tilde{B}$ fields $\parallel$ xy plane

we can align $\tilde{B}_I \parallel y$ axis: $\tilde{B}_{0I} = \tilde{B}_{0I}\hat{y}$.

From $\tilde{B} = \frac{1}{v}\hat{k}\times\tilde{E}$, $\therefore \tilde{E}_{0I}$ is in xz plane.

The configuration is shown in below



Now if $\tilde{B}_{0R} = \tilde{B}_{0R}(\cos\theta_1\hat{x} + \sin\theta_1\hat{y})$

$\tilde{B}_{0T} = \tilde{B}_{0T}(\cos\theta_2\hat{x} + \sin\theta_2\hat{y})$,

X component of (87) implies

$$\tilde{B}_{0R}\cos\theta_1 = \frac{\mu_1}{\mu_2}\tilde{B}_{0T}\cos\theta_2 \quad \text{--- (88)}$$

On the other hand, from $\tilde{E} = v\tilde{B}\times\hat{k}$,

One has $\tilde{E}_{0R} = v_1\tilde{B}_{0R}(\cos\theta_1\hat{x} + \sin\theta_1\hat{y})\times\hat{k}_R$

$\tilde{E}_{0T} = v_2\tilde{B}_{0T}(\cos\theta_2\hat{x} + \sin\theta_2\hat{y})\times\hat{k}_T$

$\hat{x}\times\hat{k}_R = \cos\theta_R\hat{y}$, $\hat{y}\times\hat{k}_R$ lies in xz plane

$\hat{x}\times\hat{k}_T = -\cos\theta_T\hat{y}$

$\therefore$ Eq. (83)'s y component implies

$$v_1\tilde{B}_{0R}\cos\theta_1\cos\theta_R = -v_2\tilde{B}_{0T}\cos\theta_2\cos\theta_T.$$

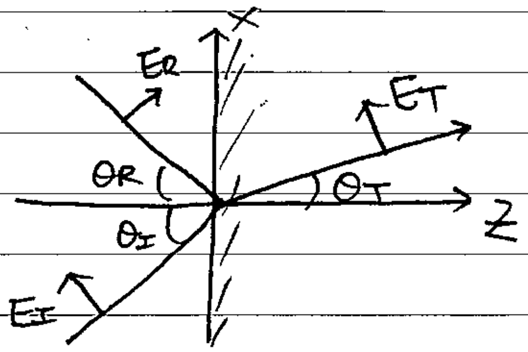
$$\therefore \tilde{B}_{0R}\cos\theta_1 = -\frac{v_2}{v_1}\frac{\cos\theta_T}{\cos\theta_R}\tilde{B}_{0T}\cos\theta_2 \quad \text{--- (89)}$$

$\because \frac{\mu_1}{\mu_2} \neq -\frac{v_2\cos\theta_T}{v_1\cos\theta_R}$, $\therefore$ Eqs (88) & (89) imply $\cos\theta_1 = \cos\theta_2 = 0$.

∴ We conclude all B fields are in y direction:

$$\tilde{\mathbf{B}}_{0I} = \tilde{B}_{0I}\hat{y}$$
$$\tilde{\mathbf{B}}_{0R} = \tilde{B}_{0R}\hat{y}$$
$$\tilde{\mathbf{B}}_{0T} = \tilde{B}_{0T}\hat{y}$$

, the directions of $\vec{E}$ are shown in the below.



∴ Eq. (82) becomes

$$\epsilon_1(-\tilde{E}_{0I}\sin\theta_I + \tilde{E}_{0R}\sin\theta_R) = \epsilon_2(-\tilde{E}_{0T}\sin\theta_T) \quad --- (88)$$

Eq. (83) becomes (x-component)

$$\tilde{E}_{0I}\cos\theta_I + \tilde{E}_{0R}\cos\theta_R = \tilde{E}_{0T}\cos\theta_T \quad -- (89)$$

Eq. (85) (y-component) becomes

$$\frac{1}{\mu_1}\frac{1}{v_1}(\tilde{E}_{0I} - \tilde{E}_{0R}) = \frac{1}{\mu_2 v_2}\tilde{E}_{0T}$$

$\left(\tilde{\mathbf{B}} = \frac{1}{v}\hat{k}\times\tilde{\mathbf{E}},\ \tilde{\mathbf{B}}_{0I} = \frac{1}{v_1}\hat{k}_I\times\tilde{\mathbf{E}}_{0I} = \frac{1}{v_1}\tilde{E}_{0I}\hat{y}\right.$

$\left. \tilde{\mathbf{B}}_{0R} = \frac{1}{v_1}\hat{k}_R\times\tilde{\mathbf{E}}_{0R} = -\frac{1}{v_1}\tilde{E}_{0R}\hat{y} \right)$

$$\therefore\ \tilde{E}_{0I} - \tilde{E}_{0R} = \beta\tilde{E}_{0T} \quad --- (90)$$

(the same as eq. (66)) ∴ $\beta = \frac{\mu_1 n_2}{\mu_2 n_1}$

$\because \theta_R = \theta_I$, $\therefore$ eq (88) becomes

$$\hat{\tilde{E}}_{0I} + \hat{\tilde{E}}_{0R} = \frac{\cos\theta_T}{\cos\theta_I}\hat{\tilde{E}}_{0T} \equiv \alpha\,\hat{\tilde{E}}_{0T} \quad \text{--- (91)}$$

$\alpha \equiv \cos\theta_T / \cos\theta_I$.

Eq. (88) becomes

$$\hat{\tilde{E}}_{0I} - \hat{\tilde{E}}_{0R} = \frac{\sin\theta_T}{\sin\theta_I}\frac{\epsilon_2}{\epsilon_1}\hat{\tilde{E}}_{0T}$$

$$= \frac{n_1}{n_2}\frac{\epsilon_2}{\epsilon_1}\hat{\tilde{E}}_{0T}$$

$$\frac{n_1}{n_2}\frac{\epsilon_2}{\epsilon_1} = \frac{\sqrt{\epsilon_1\mu_1}}{\sqrt{\epsilon_2\mu_2}}\frac{\epsilon_2}{\epsilon_1} = \frac{\mu_1}{\mu_2}\frac{\sqrt{\epsilon_2\mu_2}}{\sqrt{\epsilon_1\mu_1}} = \frac{n_2}{n_1}\frac{\mu_1}{\mu_2} = \beta$$

$$= \sqrt{\frac{\mu_1}{\epsilon_1}}\sqrt{\frac{\epsilon_2}{\mu_2}}$$

Yields the same eq. as eq. (90).

Solving eqs. (90) & (91), one finds

$$\hat{\tilde{E}}_{0R} = \frac{\alpha-\beta}{\alpha+\beta}\hat{\tilde{E}}_{0I}, \quad \hat{\tilde{E}}_{0T} = \frac{2}{\alpha+\beta}\hat{\tilde{E}}_{0I} \quad \text{--- (92)}$$

Eqs (92) are Fresnel's equations for TM case.

Note that for $\alpha < \beta$, reflected wave is 180° out of phase to the incident wave

$\alpha > \beta$, reflected wave is in the same phase as that of the incident wave.

Including α & β, eq (92) can be written as

$$\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{\frac{n_1}{\mu_1}\cos\theta_T - \frac{n_2}{\mu_2}\cos\theta_I}{\frac{n_1}{\mu_1}\cos\theta_T + \frac{n_2}{\mu_2}\cos\theta_I}$$

$$\frac{\tilde{E}_{0T}}{\tilde{E}_{0I}} = \frac{2\frac{n_1}{\mu_1}\cos\theta_T}{\frac{n_1}{\mu_1}\cos\theta_T + \frac{n_2}{\mu_2}\cos\theta_I} \quad \text{--- (93)}$$

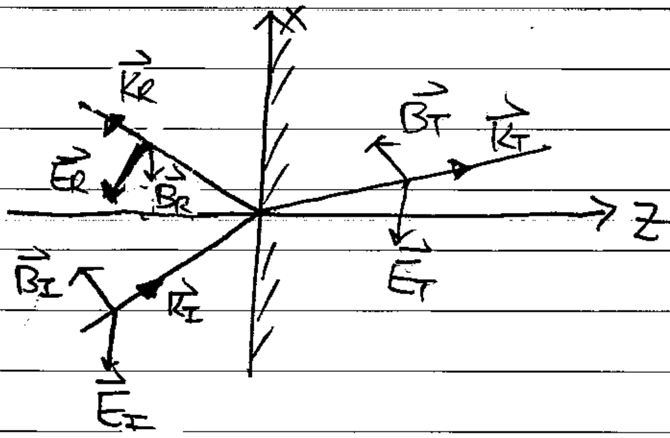
Incident wave polarized with $\vec{E} \parallel$ interface (TE case)

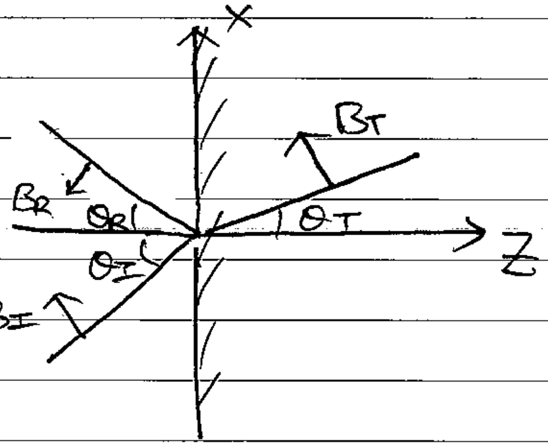
In this case $\tilde{E} \parallel xy$ plane, $\tilde{E}_{0Iz} = 0$.

Similar to the TM case, one can arrange $\tilde{E}_{0I} \parallel y$ axis and show that all $\tilde{E} \parallel y$. $\therefore$ One has $\tilde{\vec{E}}_{0I} = \tilde{E}_{0I}\hat{y}$

$\tilde{\vec{E}}_{0R} = \tilde{E}_{0R}\hat{y}$, $\tilde{\vec{E}}_{0T} = \tilde{E}_{0T}\hat{y}$.

The configuration is as shown in below.





$\therefore$ eq. (92) is satisfied.

Eq. (83) becomes $\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$ --- (94)

Eq. (84) becomes $\frac{1}{v_1}(\tilde{E}_{0I}\sin\theta_I + \tilde{E}_{0R}\sin\theta_R)$

$$= \frac{1}{v_2}\tilde{E}_{0T}\sin\theta_T \quad \text{---} \quad (95)$$

Eq. (85) (x-comp) becomes

$$\frac{1}{\mu_1 v_1}(\tilde{E}_{0I}\cos\theta_I - \tilde{E}_{0R}\cos\theta_R)$$

$$= \frac{1}{\mu_2 v_2}\tilde{E}_{0T}\cos\theta_T$$

$$\therefore \quad \tilde{E}_{0I} - \tilde{E}_{0R} \underset{\theta_I=\theta_R}{\uparrow}= \frac{\mu_1 v_1}{\mu_2 v_2}\frac{\cos\theta_T}{\cos\theta_I}\tilde{E}_{0T} = \alpha\beta\,\tilde{E}_{0T} \quad \text{-} \quad (96)$$

Eq. (95) can be rewritten as ($\because \theta_I = \theta_R$)

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \frac{v_1}{v_2}\frac{\sin\theta_T}{\sin\theta_I}\tilde{E}_{0T} = \tilde{E}_{0T}$$

which is the same as eq. (94).

$\therefore$ We need to solve eqs. (94) & (96) and got

$$\tilde{E}_{0R} = \frac{1-\alpha\beta}{1+\alpha\beta}\tilde{E}_{0I} \quad \text{---} \quad (97)$$

$$\tilde{E}_{0T} = \frac{2}{1+\alpha\beta}\tilde{E}_{0I}$$

Eqs (97) are the Fresnel's equations for the TE case which can be also put in the form

$$\frac{\tilde{\hat{E}}_{OR}}{\tilde{\hat{E}}_{OI}} = \frac{\frac{n_1}{\mu_1}\cos\theta_I - \frac{n_2}{\mu_2}\cos\theta_T}{\frac{n_1}{\mu_1}\cos\theta_I + \frac{n_2}{\mu_2}\cos\theta_T} \quad \cdots (98)$$

$$\frac{\tilde{\hat{E}}_{OT}}{\tilde{\hat{E}}_{OI}} = \frac{2\frac{n_1}{\mu_1}\cos\theta_I}{\frac{n_1}{\mu_1}\cos\theta_I + \frac{n_2}{\mu_2}\cos\theta_T}$$

Similar to the TM case, $\tilde{\hat{E}}_{OT}$ & $\tilde{\hat{E}}_{OI}$ are always in phase. For reflected waves, they are in phase with the incident wave if $\alpha\beta < 1$, otherwise the reflected wave is 180° out of phase for $\alpha\beta > 1$.

Brewster Angle (polarization angle)

$$\because \alpha = \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_I\right)^2}}{\cos\theta_I}$$

$\therefore$ As θ_I changes from 0, $\alpha = 1$, we recover the case of normal incidence, to $\theta_I = \pi/2$, $\alpha = \infty$, eqs (92) & (97) imply that the wave is totally reflected.

($\theta_I = \pi/2 \Rightarrow$ grazing incident).

One expects that there is an angle θ_B at which $E_{0R}=0$, i.e., there is no reflected wave!

To see it, we calculate reflection and transmission coefficients defined by

$$R=\frac{I_R}{I_I}, \quad T=\frac{I_T}{I_I}$$

Here $I = \langle \vec{S}\cdot\hat{z}\rangle \quad \because \hat{k}_I\cdot\hat{z}=\cos\theta_I$

$$\therefore I_I = \frac{1}{2}\epsilon_1 v_1 E_{0I}^2\cos\theta_I$$

Similarly, $I_R = \frac{1}{2}\epsilon_1 v_1 E_{0R}^2 \cos\theta_R$

$$I_T = \frac{1}{2}\epsilon_2 v_2 E_{0I}^2 \cos\theta_T$$

$$\therefore R = \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 \ (\text{TM case})$$

$$= \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2 \ (\text{TE case})$$

$$T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0R}}{E_{0I}}\right)^2\frac{\cos\theta_T}{\cos\theta_I}$$

$$= \alpha\beta\left(\frac{E_{0R}}{E_{0I}}\right)^2 = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2 \ (\text{TM case})$$

$$= \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2 \ (\text{TE case})$$

For non-magnetic materials, $\mu_1 = \mu_2 = \mu_0$.
(most materials: $\mu_1 \simeq \mu_2$)

$$\beta = \frac{n_2}{n_1} = \frac{\sin\theta_I}{\sin\theta_T}, \quad \alpha = \frac{\cos\theta_T}{\cos\theta_I}$$

$\therefore$ TM case:
$$\frac{\alpha-\beta}{\alpha+\beta} = \frac{\sin\theta_T\cos\theta_T - \sin\theta_I\cos\theta_I}{\sin\theta_T\cos\theta_T + \sin\theta_I\cos\theta_I}$$

$$= \frac{\sin 2\theta_T - \sin 2\theta_I}{\sin 2\theta_T + \sin 2\theta_I} = \frac{\cos(\theta_T+\theta_I)\sin(\theta_T-\theta_I)}{\sin(\theta_T+\theta_I)\cos(\theta_T-\theta_I)}$$

$$\therefore R_{TM} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 = \frac{\tan^2(\theta_I-\theta_T)}{\tan^2(\theta_I+\theta_T)} \quad \text{--- (99)}$$

TE case:
$$\frac{1-\alpha\beta}{1+\alpha\beta} = \frac{\sin\theta_T\cos\theta_I - \sin\theta_I\cos\theta_T}{\sin\theta_T\cos\theta_I + \sin\theta_I\cos\theta_T}$$

$$= -\frac{\sin(\theta_I-\theta_T)}{\sin(\theta_I+\theta_T)}$$

$$\therefore R_{TE} = \frac{\sin^2(\theta_I-\theta_T)}{\sin^2(\theta_I+\theta_T)} \quad \text{--- (100)}$$

From eq^s (99) & (100), one sees that only
R_{TM} can go to 0 if $\tan^2(\theta_I+\theta_T) \to \infty$
while R_{TE} is always finite.

This happens when $\theta_I + \theta_T = \pi/2$, $\theta_I = \theta_B$
is called the Brewster Angle.

$\because\ n_1 \sin\theta_I = n_2 \sin\theta_T$

$\theta_I + \theta_T = \pi/2$ implies that $n_1 \sin\theta_B = n_2 \cos\theta_B$

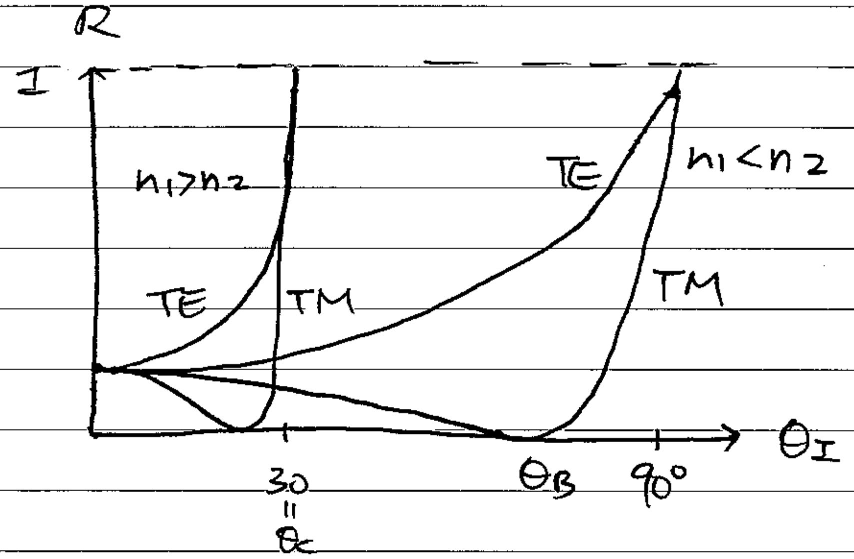
$$\therefore\ \tan\theta_B = \frac{n_2}{n_1}$$

$$\theta_B = \tan^{-1}\frac{n_2}{n_1} \quad \cdots (101)$$

$\because$ At the Brewster angle, only R_{TM} goes to zero. $\therefore$ If the incident wave's field is in general direction (superposition of TE and TM), only TE wave is reflected. Hence light reflected at Brewster angle incidence is perfected polarized! $\vec{E}_R \parallel \hat{y}$!

In practice, the property of the Brewster angle is used to eliminate unwanted reflection by using Polaroid glasses with the transmission axis vertical to y, or to generate linearly polarized light.

By solving $\theta_I = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_I\right)$, one obtains T & R as functions of θ_I:



Clearly, one sees that $R=0$ happens only for TM case.

Total internal reflection

From $R(\theta_I)$, one sees that for $n_1>n_2$, there is an angle θ_c at which R reaches I. This is the situation when the incident wave is totally reflected.

From $n_1 \sin\theta_I = n_2 \sin\theta_T$, one sees that this is possible only when $n_1>n_2$, at $\theta_I=\theta_c$, $\theta_T=\pi/2$, $\sin\theta_T$ reaches maximum

$$\therefore \theta_c = \sin^{-1}\frac{n_2}{n_1} \quad \cdots (102)$$

What happens to the transmitted wave when $\theta > \theta_c$?

To see it, we recall the relation $n_1 \sin\theta_I = n_2 \sin\theta_T$ originates from eq. (79) $k_I \sin\theta_I = k_T \sin\theta_T$ and eq. (78)

$$k_{Ix} = k_{Tx}$$

$$\because k_I = \frac{\omega}{c} n_1, \quad k_T = \frac{\omega}{c} n_2 = \sqrt{k_{Tx}^2 + k_{Tz}^2} \quad \text{---} (103)$$

$$\therefore k_{Ix} = k_I \sin\theta_I = \frac{\omega}{c} n_1 \sin\theta_I$$

For $\theta_I > \theta_c$, $n_1 \sin\theta_I > n_2$

$$\therefore k_{Tx} = k_{Ix} = \underbrace{\frac{\omega}{c} n_1}_{k_I} \sin\theta_I > \frac{\omega}{c} n_2 = k_T = \frac{\omega}{c} n_1 \sin\theta_c$$

$\therefore k_{Tx} > k_T$ together with eq. (103)

$\therefore$ It implies that $\theta_I > \theta_c$

$$k_{Tz}^2 = k_T^2 - k_{Tx}^2 < 0$$

$\therefore k_{Tz}$ is purely imaginary.

$\therefore$ We set $k_{Tz} = ik$, $\therefore k = \sqrt{k_{Tx}^2 - k_T^2} = (\pm)\frac{\omega}{c} n_1 \sqrt{\sin^2\theta_I - \sin^2\theta_c}$ --- (104)

$$\vec{k}_T = (k_{Tx}, 0, ik) \quad \text{---} (105)$$

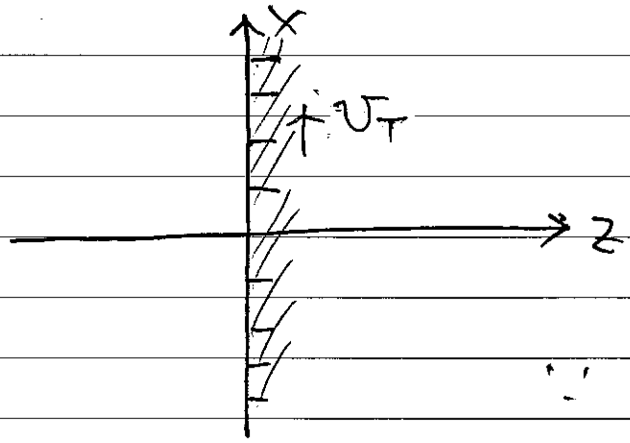
$$\therefore \tilde{E}_T(x,z,t) = \tilde{E}_{0T}\, e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}$$

$$= \tilde{E}_{0T}\, e^{-\kappa z}\, e^{i(k_{Tx}x-\omega t)}$$

$$= \tilde{E}_{0T}\, e^{-\kappa z}\, e^{i(k_I x \sin\theta_I-\omega t)} \quad \leftarrow \text{-- (106)}$$

$\therefore \theta_I > \theta_c$, $\tilde{E}_T$ decays exponentially in z direction (take + in (104)) with κ^{-1} being the decaying length.

Only within $0 \le z < \kappa^{-1}$, $\tilde{E}_T$ has appreciable values. The transmitted wave is thus an interface wave (or evanescent wave) and propagates in x direction :



$$v_T = \frac{\omega}{k_{Tx}}$$

$$= \frac{\omega}{k_I \sin\theta_I} = \frac{c}{n_1\sin\theta_I} > \frac{c}{n_1} = v_1$$

$\because n_1\sin\theta_I \geqq n_1 \sin\theta_c = n_2$

$$\therefore v_T \le \frac{c}{n_2} = v_2$$

$$\therefore v_1 \le v_T \le v_2$$

The field amplitudes of transmitted wave for $\theta_I > \theta_c$ can be obtained by replacing $\vec{k}_T$ by eq. (105) and expressing all boundary conditions by components of $\vec{k}$. $\left(\frac{\omega}{v_2}\cos\theta_T = k_{Tz} = ik\right)$

$\therefore$ For TE case,

Eq. (96) $\Rightarrow \frac{1}{\mu_1} k_{Iz} (\tilde{E}_{0I} - \tilde{E}_{0R}) = \frac{1}{\mu_2} k_{Tz} \tilde{E}_{0T}$

$\frac{\omega}{v_1} = k_I$

$k_I \cos\theta_I = k_{Iz}$

$$\therefore \tilde{E}_{0I} - \tilde{E}_{0R} = \frac{\mu_2}{\mu_1} \frac{k_{Tz}}{k_{Iz}} \tilde{E}_{0T}$$

$\alpha\beta$ becomes $\frac{\mu_2}{\mu_1} \frac{k_{Tz}}{k_{Iz}} = \frac{\mu_2}{\mu_1} \frac{ik}{k_{Iz}}$

$\therefore \frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{1 - \frac{\mu_2}{\mu_1}\frac{ik}{k_{Iz}}}{1 + \frac{\mu_2}{\mu_1}\frac{ik}{k_{Iz}}}$ which has the form of $\frac{z}{z^*}$ (z = complex #)

$\therefore R_{TE} = \left| \frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} \right|^2 = 1$

For non-magnetic materials, $\mu_2 = \mu_1 = \mu_0$

$\therefore \frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{k_{Iz} - ik}{k_{Iz} + ik} = \frac{\cancel{k_1}\cos\theta_I - i\cancel{k_1}\sqrt{\sin^2\theta_I - \sin^2\theta_c}}{\cancel{k_1}\cos\theta_I + i\cancel{k_1}\sqrt{\sin^2\theta_I - \sin^2\theta_c}}$

$$\frac{\tilde{\hat{E}}_{OT}}{\tilde{\hat{E}}_{OI}} = \frac{2K_{IZ}}{K_{IZ}+iK} = \frac{2\cos\theta_I}{\cos\theta_I + i\sqrt{\sin^2\theta_I - \sin^2\theta_c}}$$

Which can be also obtained by setting $\mu_1 = \mu_2$

$\frac{\omega}{c} n_1 = K_I$, $\frac{\omega}{c} n_2 = K_T$, and $K_T \cos\theta_T = iK$

in eq. (98).

For TM case, however, one needs to

go back to eqs. (88) & (89).

$\because \vec{K}_T \cdot \tilde{\hat{E}}_T = 0 \quad \therefore K_{TX}\tilde{E}_{OTX} + K_{TZ}\tilde{E}_{OTZ} = 0$ --- (107)

Eq. (88) $\Rightarrow \quad \frac{\varepsilon_1}{\varepsilon_2}\left(-\tilde{\hat{E}}_{OI}\sin\theta_I + \tilde{\hat{E}}_{OR}\sin\theta_R\right) = \tilde{E}_{OTZ}$ --- (108)

Eq. (89) $\Rightarrow \quad \tilde{\hat{E}}_{OI}\cos\theta_I + \tilde{\hat{E}}_{OR}\cos\theta_R = \tilde{E}_{OTX}$ --- (109)

(108) $\times K_{TX}$ + (109) $\times K_{TZ}$ & using eq (107), one

obtains

$$\frac{\tilde{\hat{E}}_{OR}}{\tilde{\hat{E}}_{OI}} = \frac{-\frac{\varepsilon_1}{\varepsilon_2}\sin\theta_I K_{TZ} + \cos\theta_I K_{TX}}{\frac{\varepsilon_1}{\varepsilon_2}\sin\theta_R K_{TZ} + \cos\theta_R K_{TX}}$$

$\theta_I = \theta_R \quad K_{TZ} = iK \quad \therefore \frac{\tilde{\hat{E}}_{OR}}{\tilde{\hat{E}}_{OI}}$ is in the form $\frac{z}{z^*}$.

$$\therefore R_{TM} = \left|\frac{\tilde{\hat{E}}_{OR}}{\tilde{\hat{E}}_{OI}}\right|^2 = 1.$$

$\because K_{Tx} = K_{Ix} = K_I \sin\theta_I = \frac{\omega}{c} n_1 \sin\theta_I$

$K_{Tz} = iK = i\frac{\omega}{c} n_1 \sqrt{\sin^2\theta_I - \sin^2\theta_C}$

$\mu_1 = \mu_2 \quad , \quad \left(\frac{\varepsilon_1}{\varepsilon_2}\right) = \left(\frac{n_1}{n_2}\right)^2$

$$\therefore \frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{-\left(\frac{n_1}{n_2}\right)^2 \sin\theta_I \, i n_1 \sqrt{\sin^2\theta_I - \sin^2\theta_C} + n_1 \sin\theta_I \cos\theta_I}{\left(\frac{n_1}{n_2}\right)^2 \sin\theta_I \, i n_1 \sqrt{\sin^2\theta_I - \sin^2\theta_C} + n_1 \sin\theta_I \cos\theta_I}$$

$$= \frac{-i n_1^2 \sqrt{\sin^2\theta_I - \sin^2\theta_C} + n_2^2 \cos\theta_I}{i n_1^2 \sqrt{\sin^2\theta_I - \sin^2\theta_C} + n_2^2 \cos\theta_I}$$

with $R_{TM} = \left| \frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} \right| = 1$

Electromagnetic waves in conductors — dissipation

In vacuum and linear media, EM waves do not dissipate and can propagate without being dissipated away.

This is no longer true of EM waves enter into conductors.

The primary reason is that inside conductors, there are free charges that can move freely and can be accelerated by E & B to absorb energy of EM wave. Hence the energy of EM wave will be dissipated.

∴ The dissipation mainly lies in the presence of ρ_f & $\vec{J}_f$.

For conductors, the Maxwell's equations become

(i) $\vec{\nabla}\cdot\vec{E} = \rho_f/\varepsilon$

(ii) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

(iii) $\vec{\nabla}\cdot\vec{B} = 0$

(iv) $\vec{\nabla}\times\vec{B} = \mu\left(\vec{J}_f + \epsilon\frac{\partial\vec{E}}{\partial t}\right)$

Here $\vec{J}_f = \sigma\vec{E}$

The continuity equation

$$\frac{\partial\rho_f}{\partial t} + \vec{\nabla}\cdot\vec{J}_f = 0$$

becomes $\frac{\partial\rho_f}{\partial t} + \sigma\vec{\nabla}\cdot\vec{E} = 0$

Using (i) ∴ $\frac{\partial\rho_f}{\partial t} = -\frac{\sigma}{\varepsilon}\rho_f$

$$\rho_f(t) = e^{-\sigma/\varepsilon\, t}\,\rho_f(0) \quad \text{--- (110)}$$

Hence any charges inside conductors dissipate in a time scale $\tau = \varepsilon/\sigma$.

For good & neutral conductors, $\sigma = \infty$, $\tau = 0$. ∴ We may set $\rho_f = 0$ (if the conductor is neutral,

(f. on surface = 0 as well.)

For ordinary conductors, we consider frequency ω such that $\tau \ll 1/\omega$

$$\therefore \quad \vec{\nabla}\cdot\vec{E} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu\epsilon\frac{\partial \vec{E}}{\partial t} + \mu\sigma\vec{E}$$

taking $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E})$ & $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})$, we obtain

$$\nabla^2\vec{E} = \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} + \mu\sigma\frac{\partial\vec{E}}{\partial t} \quad \text{— (111)}$$

and $$\nabla^2\vec{B} = \mu\epsilon\frac{\partial^2\vec{B}}{\partial t^2} + \mu\sigma\frac{\partial\vec{B}}{\partial t} \quad \text{— (112)}$$

Eqs. (111) & (112) no longer take the standard form of the wave equation.

The additional terms $\mu\sigma\frac{\partial\vec{E}}{\partial t}$ & $\mu\sigma\frac{\partial\vec{B}}{\partial t}$ cause dissipation.

They still allow plane-wave solutions

$$\tilde{\mathbb{E}}(z,t) = \tilde{\mathbb{E}}_0\, e^{i(\tilde{k}z-\omega t)}$$

$$\tilde{\mathbb{B}}(z,t) = \tilde{\mathbb{B}}_0\, e^{i(\tilde{k}z-\omega t)}$$

Upon substitution into eqs (11) & (12), one gets:

$$\because \frac{d^2}{dz^2} e^{i(\hat{k}z-\omega t)} = (i\hat{k})^2 e^{i(\hat{k}z-\omega t)}$$

$$\therefore \hat{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \quad \text{---} \; (13)$$

From eq. (13), one sees that $\sigma \neq 0$, $\hat{k} \neq \sqrt{\epsilon\mu}\,\omega$.

Furthermore, $\hat{k}$ is not real.

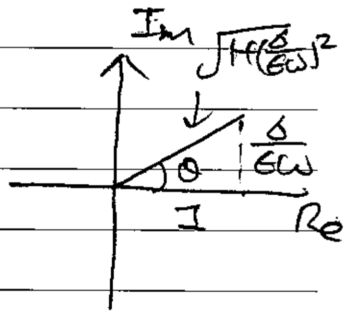
To solve $\hat{k}$ in terms of ω, one needs to try complex $\hat{k}$:

$$\hat{k} = k + i\kappa$$

We first write $\mu\epsilon\omega^2 + i\mu\sigma\omega$

$$= \mu\epsilon\omega^2\left[1 + \frac{i\sigma}{\epsilon\omega}\right]$$

$$= \mu\epsilon\omega^2\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}\, e^{i\theta}$$



$$\therefore \hat{k} = \sqrt{\mu\epsilon\omega^2\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}}\, e^{i\theta/2}$$

$$\cos\theta/2 = \sqrt{\frac{\cos\theta+1}{2}} = \sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}}\right)}$$

$$\sin\theta/2 = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1}{2}\left(1-\frac{1}{\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}}\right)}$$

$$\therefore \tilde{k} = \omega\sqrt{\frac{\epsilon\mu}{2}}\left\{\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{\frac{1}{2}} + i\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{\frac{1}{2}}\right\}$$

$$\therefore k = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{\frac{1}{2}} \quad \text{--- (114)}$$

$$\kappa = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}$$

The imaginary part of $\tilde{k}$ results in the attenuation of wave in $+z$ direction:

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}$$
$$\tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{-\kappa z} e^{i(kz-\omega t)} \quad \text{--- (115)}$$

The distance it takes $\vec{E}$ & $\vec{B}$ to decay to the original amplitude by the factor e^{-1} is called the <u>skin depth</u> δ, determined by κ.

$$\therefore \delta = 1/\kappa \quad \text{--- (116)}$$

(Typical: Silver, $\delta \leq 100\,\text{Å}$ in optical frequencies)

The real part of $\tilde{k}$ still determines the wavelength & the propagation speed. (Similar to the transmission wave in the case total internal reflection)

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\mathrm{Re}\,\tilde{k}}, \quad v = \frac{\omega}{k} = \frac{\omega}{\mathrm{Re}\,\tilde{k}}$$

$$n = \frac{ck}{\omega} \quad \text{--- (117)}$$

The relation between $\vec{E}$ & $\vec{B}$ also changes due to $\tilde{k}$ being complex.

Choose the axes so that $\vec{E}$ is polarized in x direction.

$$\tilde{\mathbf{E}}(z,t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)} \hat{x} \quad \text{--- (118)}$$

$$\vec{\nabla} \times \tilde{\mathbf{E}} = \frac{\partial}{\partial z} \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)} \hat{y}$$

$$= i\underbrace{(k+i\kappa)}_{\tilde{k}} \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)} \hat{y}$$

$\therefore$ To satisfy $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, we obtain

$$\tilde{\mathbf{B}}(z,t) = \frac{\tilde{k}}{\omega} \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)} \hat{y} \quad \text{--- (119)}$$

Eqs. (118) & (119) imply $\vec{E} \perp \vec{B}$ is still correct.

Futhermore, both $\vec{E}$ & $\vec{B}$ are perpendicular to the propagation direction: $\underline{\mathrm{Re}\,\tilde{\mathbf{k}}}$ $\quad \tilde{\mathbf{k}} = (0, 0, k+i\kappa)$

However, the amplitudes of $\vec{E}$ & $\vec{B}$ are not in the same phase:

If we write $\tilde{k} = \bar{k} e^{i\theta}$, we

have $\bar{k} = \omega\sqrt{\epsilon\mu}\left(1+\left(\frac{\sigma}{\epsilon\omega}\right)^2\right)^{\frac{1}{4}}$, $\theta = \tan^{-1}\frac{\kappa}{k}$

Hence if $\tilde{E}_0 = E_0 e^{i\theta_E}$, eq. (19) implies
$\tilde{B}_0 = B_0 e^{i\theta_B}$

$$B_0 e^{i\theta_B} = \frac{\bar{k}}{\omega} E_0 e^{i(\theta_E+\theta)}$$

$$\therefore \frac{B_0}{E_0} = \frac{\bar{k}}{\omega} = \sqrt{\epsilon\mu}\left[1+\left(\frac{\sigma}{\epsilon\omega}\right)^2\right]^{\frac{1}{4}} = \frac{c}{n}\left[1+\left(\frac{\sigma}{\epsilon\omega}\right)^2\right]^{\frac{1}{4}}$$

$$\theta_B - \theta_E = \theta$$

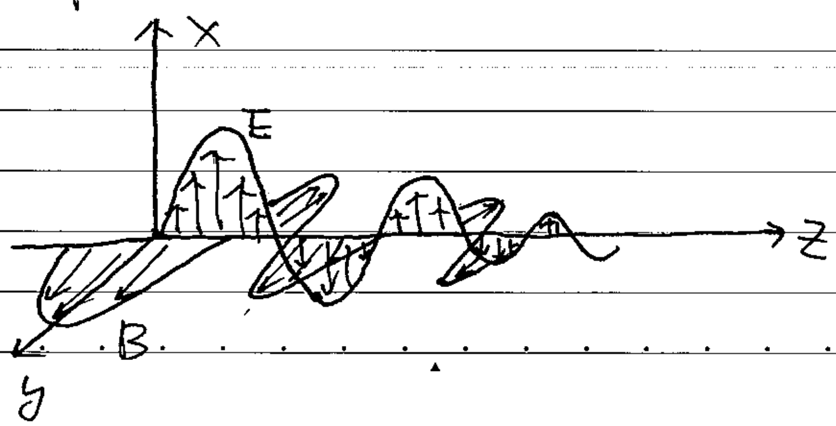
The magnetic field is behind the electric field in phase by θ.

$\therefore$ For real fields, one has

$$\vec{E}(z,t) = E_0 e^{-\kappa z}\cos(kz-\omega t+\theta_E)\hat{x}$$

$$\vec{B}(z,t) = B_0 e^{-\kappa z}\cos(kz-\omega t+\theta_E+\theta)\hat{y}$$

with the picture:



~~Note that~~

For good conductors, $\sigma \to \infty$

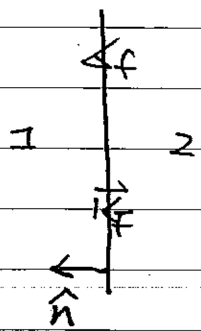
$$\frac{\varkappa}{k} \overset{\uparrow}{\underset{(1k_p)}{=}} \frac{\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{\frac{1}{2}}}{\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{\frac{1}{2}}} \to 1$$

$$\therefore \ \theta = \tan^{-1}\frac{\varkappa}{k} = \tan^{-1}(1) = \pi/4$$

The magnetic field lags the electric field by $\pi/4$

<u>Reflection at the surface of a conductor</u>

Since the surface of conductor can host surface charge σ_f and surface current $\vec{K}_f$, the boundary conditions have to be modified :



(i) $\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f$

(ii) $\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$

(iii) $B_1^{\perp} = B_2^{\perp}$

(iv) $\frac{1}{\mu_1}\vec{B}_1^{\parallel} - \frac{1}{\mu_2}\vec{B}_2^{\parallel} = \vec{K}_f \times \hat{n}$

For ohmic conductors, $\vec{J}_f = \sigma \vec{E}$. The surface current $\vec{K}_f$ is $\vec{J}_f \delta(z)$ if $\hat{n} = \hat{z}$. Which implies $\vec{E} \propto \delta(z) = \infty$ at $z=0$.

Therefore, this is not physical. In other words, no matter how narrow $\vec{J}_f$ is distributed, we shall treat it as a volume current density. Hence one sets $\vec{K}_f = 0$

If $\sigma_f = 0$ (the conductor is neutral), BCs (i)-(iv) are the same as BCs for two insulators, except that k_T for the conductor is complex: $\tilde{k}_T = k_2 + i\kappa$

Hence for normal incidence, eqs (67), are still correct with

$$\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{1-\tilde{\beta}}{1+\tilde{\beta}}$$

$$\frac{\tilde{E}_{0T}}{\tilde{E}_{0I}} = \frac{2}{1+\tilde{\beta}} \qquad \text{--- (120)}$$

$\tilde{\beta} = \frac{\mu_1}{\mu_2}\frac{v_1}{\tilde{v}_2}$ where $\tilde{v}_2 = \frac{\omega}{\tilde{k}_T}$, with $\tilde{k}_T = k_2 + i\kappa$

For good conductors, $\sigma \to \infty$

$$\therefore k_2 = \omega\sqrt{\frac{\epsilon_2\mu_2}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon_2\omega}\right)^2}+1\right]^{\frac{1}{2}} \to \infty$$

$\tilde{\beta} \to \infty \quad \therefore \tilde{E}_{0T} = 0, \quad \tilde{E}_{0R} = -\tilde{E}_{0I} \qquad$ --- (121)

The EM wave is totally reflected with a 180° phase shift. That's why good conductors can be used for mirrors.

For general incidences, we simply replace n_2 by a complex $\tilde{n}_2$ in eqs (93) (TM) & eqs (98) (TM)

$$k_T \equiv \hat{n}_2 \frac{\omega}{c}$$

$$\because \hat{n}_2 = n_2' + i n_2''$$

$$n_2' = c\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]^{\frac{1}{2}}$$

$$n_2'' = c\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right]^{\frac{1}{2}}$$

Clearly, for good conductors, $\sigma \to \infty$

$\hat{E}_{0T} = 0$ for both TE & TM cases

and $\tilde{E}_{0R} = -\tilde{E}_{0I}$

The reflect wave is also shifted by a 180° phase!

Frequency dependence of permitivity $\epsilon(\omega)$

In the above discussions so far, we have not discussed the frequency dependence of μ & ε.

As we have mentioned, in real materials, ε & μ may depend on wavelength (λ) (or frequency), i.e., the medium is dispersive.

The dependence of ε & μ on A

is equivalent to the dependence

of ε & μ on ω ($\because \omega = vk$)

Hence the refraction index

$n = \frac{1}{\sqrt{\mu\epsilon}}$ also depends on ω or k.

The dependence of ε or μ on ω

implies a nonlocal response in time.

For instance, $\because \vec{p} = \epsilon_0 \chi_e \vec{E}$

$$\vec{D} = \vec{E} + \epsilon_0 \vec{P} = \underbrace{\epsilon_0(1+\chi_e)}_{\varepsilon} \vec{E}$$

$\therefore$ $\vec{D}(\omega) = \epsilon(\omega)\vec{E}(\omega)$ implies that in

t-space, $\vec{D}(t) = \int_{-\infty}^{\infty} \epsilon(t-t')\, \vec{E}(t')\, dt'$ --- (122)-1

i.e $\vec{D}(t) = \epsilon_0 \int_{-\infty}^{t} \chi_e(t-t')\, \vec{E}(t')\, dt'$ --- (122)-2

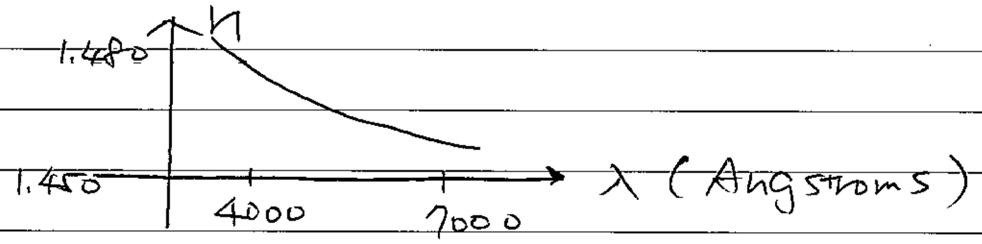
$$\epsilon(t-t') = \epsilon_0 \delta(t-t') + \epsilon_0 \chi_e(t-t')\, \Theta(t-t')$$

Only the "cause" (E field) from $t' < t$ can induce $\vec{p}$ at t.
($\because t' < t$, $\chi_e \neq 0$). $\vec{E}$ from the future $t' > t$
Can not affect $\vec{p}$ at t.

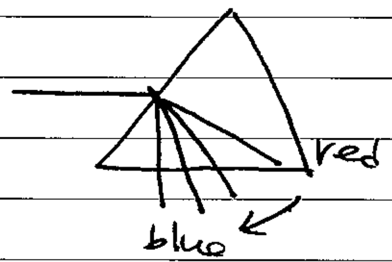
The value of $\vec{D}(t)$ depends on $\vec{E}$ at $t' \neq t$ is
a nonlocal relation in time.

For a typical glass, n behaves

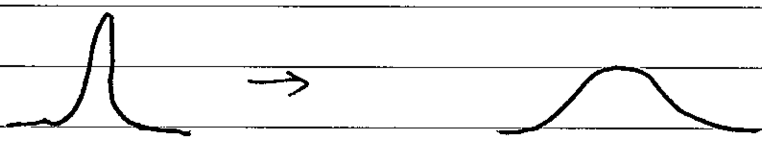
as



Therefore, waves of different wavelengths travel at different speed $v = \omega/k$. This is the origin of the phenomenon that a prism or a raindrop bends light of different colors differently.



Since waves of different wavelengths travel at different speeds, a sharp wave packet formed by different wavelengths will become broad and broad as it moves



The speed for each wavelength is determined by the phase velocity

$$v_p = \omega/k \quad \text{---} \quad (122)\text{-}3$$

While the is for a sharp wavepacket is determined by the group velocity. v_g.

There are also other velocities one can define, such as speed that energy is transferred & speed that signal is transferred, or the front speed. (the speed for the front when the wave increases from zero to an appreciable value.)

These velocities are also different from the phase velocity. Some of them can even be greater than c. In special relativity, as we shall see later, the causal signal velocity is less or equal to c.

The group velocity

The effect of dispersion relation on the velocity can be illustrated by considering two waves:

$$E_1(z,t) = E_0 \cos(\omega_1 t - k_1 z), \quad \omega_1 \sim \omega_2$$

$$E_2(z,t) = E_0 \cos(\omega_2 t - k_2 z) \quad k_1 \sim k_2$$

the total field

$$E(z,t) = E_1(z,t) + E_2(z,t)$$

$$= E_0\left[\cos(\omega_1 t - k_1 z) + \cos(\omega_2 t - k_2 z)\right]$$

$$= 2E_0 \cos\left[\tfrac{1}{2}(\omega_1 - \omega_2)t - \tfrac{1}{2}(k_1 - k_2)x\right]$$

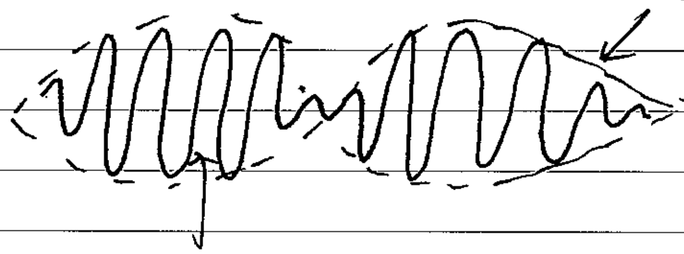
$$\times \cos\left[\tfrac{1}{2}(\omega_1 + \omega_2)t - \tfrac{1}{2}(k_1 + k_2)x\right]$$

$\because \omega_1 + \omega_2 > |\omega_1 - \omega_2|, \quad k_1 + k_2 > |k_1 - k_2|$

$\therefore \cos\left[\tfrac{1}{2}(\omega_1 + \omega_2)t - \tfrac{1}{2}(k_1 + k_2)x\right]$ is fast oscillating which $\cos\left[\tfrac{1}{2}(\omega_1 - \omega_2)t - \tfrac{1}{2}(k_1 - k_2)x\right]$ oscillates slowly.

One has

$\cos\left[\tfrac{1}{2}(\omega_1 - \omega_2)t - \tfrac{1}{2}(k_1 - k_2)x\right]$



$\cos\left[\tfrac{1}{2}(\omega_1 + \omega_2)t - \tfrac{1}{2}(k_1 + k_2)x\right]$

The envelope exhibits "beat" phenomenon.

and its speed is determined by

$$V_g = \frac{\omega_1 - \omega_2}{k_1 - k_2}$$

It is conceivable when $k_2 \to k_1$, $\omega_2 \to \omega_1$,

one gets
$$v_g = \frac{d\omega}{dk} \quad \cdots \quad (123)$$

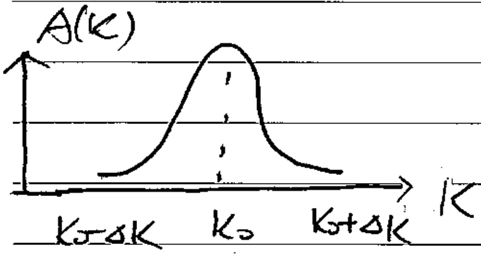
Indeed, in the general wavepacket

$$\tilde{E}(z,t) = \int A(k)\, e^{i(\omega t - kz)}\, dk$$

One assumes $A(k) \neq 0$ around k_0

$$A(k) \neq 0 \qquad k_0 - \Delta k < k < k_0 + \Delta k$$

$$\Delta k \ll k_0$$



$\because \Delta k \ll k_0$, $\therefore\ \omega(k) \cong \underbrace{\omega(k_0)}_{\omega_0} + \underbrace{\left.\frac{d\omega}{dk}\right|_{k_0}}_{\omega_0'} (k - k_0)$

$$\omega t - kz \simeq \omega_0 t - k_0 z$$
$$+ \omega_0' (k - k_0) t - (k - k_0) z$$

$$\tilde{E}(z,t) = \underbrace{\left\{ \int dk\, A(k)\, e^{i(k-k_0)[\omega_0' t - z]} \right\}}_{\text{envelope}} \underbrace{e^{i(\omega_0 t - k_0 x)}}_{\text{fast oscillation}}$$

$$= f(z - v_g t)$$

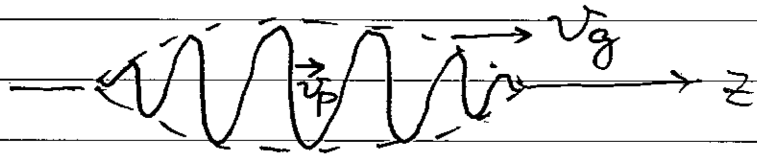
$$v_g = \left.\frac{d\omega}{dk}\right|_{k=k_0}$$

Therefore, the envelope is the
speed of the

group velocity $\underline{v_g = \frac{d\omega}{dk}}$, while the speed for fast oscillation is the phase velocity

$$\underline{v_p = \frac{\omega_0}{k_0}}.$$

One has the following picture:



the fast oscillation is the rapid oscillation inside the wavepacket.

The envelope moves with speed v_g.
(dash line)

In the case with dissipation, only the real part of k matters.

$$\therefore \ v_g = Re\left[\frac{d\omega}{dk}\right] \quad \text{--- (124)}$$

Consider $k = k(\omega)$

$$\therefore \ v_g = Re\left[\frac{1}{\frac{dk(\omega)}{d\omega}}\right] \equiv \frac{c}{n_g}$$

The frequency dependence can be summarized in the refraction index $n(\omega)$

by $\frac{\omega}{K} = v_p = \frac{c}{n(\omega)}$

$\therefore K(\omega) = \frac{\omega}{c} n(\omega)$

$$\frac{dK}{d\omega} = \frac{1}{c}\left[n(\omega) + \omega \frac{dn}{d\omega}\right]$$

$$\therefore v_g = Re\left[\frac{c}{n(\omega) + \omega \frac{dn}{d\omega}}\right] \quad --- (125)$$

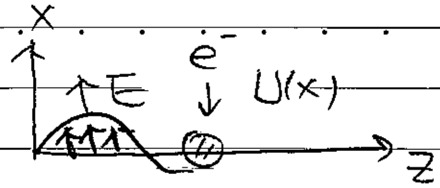
For normal dispersion, $\frac{dn}{d\omega} > 0 \therefore v_g < \frac{c}{n} < v_p$

Anomalous dispersion, $\frac{dn}{d\omega} < 0$, $v_g > \frac{c}{n} > v_p$.

Frequency dependence of ε

For most materials that are non-magnetic, $\mu \simeq \mu_0$, $n \simeq \sqrt{\varepsilon}$. The frequency dependence is determined by $\varepsilon(\omega)$.

A simplified model (known as the Lorentz model) to find $\varepsilon(\omega)$ for insulators is to assume that the electron is bound to specific molecules (see below). This is an approximation, however, it captures important features of $\varepsilon(\omega)$.



The binding potential $U(x)$ is generally complicated.

However, if $x=0$ is the equilibrium position of the electron, $U'(0)=0$.

One may expand $U(x)$ as

$$U(x) = U(0) + U'(0)x + \frac{x^2}{2}U''(0) + \dots$$

$$\approx U(0) + \frac{1}{2}U''_{(0)}x^2 \quad \text{for small } x.$$

Shifting energy's zero point so that $U(0)=0$, one has

$$U(x) = \frac{1}{2}Kx^2, \quad \text{with } K = U''(0).$$

For stable equilibrium, $U''(0)>0$, $\therefore K>0$

Therefore, the binding force of the electron can be written as $F_{binding} = -Kx$

In addition to the binding force, there will be damping force $F_{damping} = -m\gamma\frac{dx}{dt}$

which is due to radiation of the electron (we shall come back to its origin later) and other dissipation.

Together with the force due to the electric field in EM wave. ($B = \frac{E}{v} \sim \frac{E}{c} \ll E$, F_B can be neglected):

$$F_E = qE = qE_0 \cos\omega t$$

We obtain the equation of motion

$$m\frac{d^2x}{dt^2} = -m\gamma\frac{dx}{dt} + Kx + qE_0\cos\omega t$$

$$\therefore \quad \frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{q}{m}E_0\cos\omega t \quad \text{--- (126)}$$

Where $\omega_0^2 = K/m$. Eq (126) describes an electron bound to the molecule in the presence an E field.

To solve eq. (126), one can view x as the real part of some complex number

$$x = \mathrm{Re}\,\tilde{x}. \quad \text{-- (127)}$$

$$\because \quad E_0\cos\omega t = \mathrm{Re}\, E_0 e^{-i\omega t}$$

$\therefore$ Eq (126) becomes

$$\mathrm{Re}\left[\frac{d^2\tilde{x}}{dt^2} + \gamma\frac{d\tilde{x}}{dt} + \omega_0^2\tilde{x}\right] = \mathrm{Re}\left[\frac{q}{m}E_0 e^{-i\omega t}\right]$$

Hence it's suffice to solve

$$\frac{d^2\tilde{x}}{dt^2} + \gamma\frac{d\tilde{x}}{dt} + \omega_0^2\tilde{x} = \frac{q}{m}E_0 e^{-i\omega t} \quad \text{--- (128)}$$

In the steady state, we know that the electron oscillates with the same frequency ω.

$$\therefore \tilde{x} = \tilde{x}_0 e^{-i\omega t}$$

Inserting it into eq. (128), we obtain

$$(-\omega^2+\omega_0^2-i\gamma\omega)\,\tilde{x}_0 = \frac{q}{m}E_0$$

$$\therefore \tilde{x}_0 = \frac{q/m}{\omega_0^2-\omega^2-i\gamma\omega}\,E_0$$

$\therefore$ The dipole moment

$$\tilde{p}(t) = q\,\tilde{x}(t)$$

$$= \frac{q^2/m}{\omega_0^2-\omega^2-i\gamma\omega}\,E_0 e^{-i\omega t} \quad \text{-- } (129)$$

One can see that there is a phase lag between p & E: phase lag $= \tan^{-1}\left(\frac{\gamma\omega}{\omega_0^2-\omega^2}\right)$

In general, there are many electrons bound to a molecules. Different electron is in different orbit, hence their natural frequency ω_0 and damping are different.

Suppose that there are f_j electrons with natural frequency ω_j and damping γ_j in a

There are N_0 molecules in a unit volume

a molecule, and $\therefore$ Eq. (129) is generalized to

$$\tilde{\mathbb{P}} = \frac{N_0 q^2}{m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega} \tilde{\mathbb{E}}$$

--- (130)

Which, in comparison to the relation

$\vec{P} = \epsilon_0 \chi_e \vec{E}$, indicates that we

can generalize χ_e to complex susceptibility $\tilde{\chi}_e$

so that $\quad \tilde{\mathbb{P}} = \epsilon_0 \tilde{\chi}_e \tilde{\mathbb{E}}$.

As a result, $\tilde{\mathbb{D}} = \varepsilon_0 \tilde{\mathbb{E}} + \tilde{\mathbb{P}}$

$$= \underbrace{\epsilon_0 (1 + \tilde{\chi}_e)}_{} \tilde{\mathbb{E}}.$$

$\tilde{\epsilon}$ = complex permittivity.

$\therefore$ One defines the complex permittivity

$$\tilde{\varepsilon} = \varepsilon_0 (1 + \tilde{\chi}_e)$$

and the complex dielectric constant

$$\tilde{\varepsilon}_r = \frac{\tilde{\varepsilon}}{\varepsilon_0} = 1 + \tilde{\chi}_e$$

$$= 1 + \frac{N_0 q^2}{m \varepsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega} \quad \text{---} (131)$$

Note that in the original linear relation between $\vec{P}$ & $\vec{E}$: $\vec{P} = \epsilon_0 \chi_e \vec{E}$, it is not possible to have a phase difference between $\vec{P}$ & $\vec{E}$.

For instance, $\vec{P} = \epsilon_0 \chi_e A \cos(\omega t + \pi/3)\,\hat{x}$

$$\vec{E} = A \cos(\omega t)\,\hat{x}$$

can't be fit into the relation $\vec{P} = \epsilon_0 \chi_e \vec{E}$.

In general, $\vec{P}$ may have a phase difference (lag) to $\vec{E}$.

Using the complex number allows one to generalize the linear relation:

$$\vec{P} = Re\,\tilde{\mathbb{P}}, \quad \vec{E} = Re\,\tilde{\mathbb{E}}$$

$$\tilde{\mathbb{P}} = \epsilon_0 \tilde{\chi}_e \tilde{\mathbb{E}}$$

In the previous example: $\vec{P} = Re\, \underbrace{\epsilon_0 \chi_e A e^{-i(\omega t + \pi/3)}}_{\tilde{\mathbb{P}}} \hat{x}$

$$\vec{E} = Re\, \underbrace{A e^{-i\omega t}}_{\tilde{\mathbb{E}}} \hat{x}$$

$$\therefore \tilde{\mathbb{P}} = \epsilon_0 \chi_e e^{-i\pi/3}\, \tilde{\mathbb{E}}, \quad \tilde{\chi}_e = \chi_e e^{-i\pi/3}.$$

In the complex notations, just as we go from eq. (126) to eq. (128), $\tilde{\mathbb{B}}, \tilde{\mathbb{E}}, \tilde{\mathbb{P}}, \tilde{\mathbb{D}}, \tilde{\mathbb{H}}, \tilde{\mathbb{M}}$ replaces $\vec{B}, \vec{E}, \vec{P}, \vec{D}, \vec{H}, \vec{M}$ and $\tilde{\epsilon}$ replaces ϵ. Hence for non-magnetic materials, we obtain

$$\nabla^2 \tilde{\mathbb{E}} = \mu_0 \tilde{\epsilon} \frac{\partial^2 \tilde{\mathbb{E}}}{\partial t^2} \quad \text{--- (132)}$$

It allows the solution:

$$\tilde{E}(z,t) = \hat{\tilde{E}}_0 e^{i(\tilde{k}z-\omega t)}$$

With complex wave number $\tilde{k}$ satisfying

$$\tilde{k}^2 = \mu_0 \tilde{\epsilon}\omega^2$$

$$\therefore \tilde{k} = \sqrt{\tilde{\epsilon}\mu_0}\,\omega \quad \cdots \quad (133)$$

$\because$ $\tilde{\epsilon}$ is complex, $\tilde{k}$ is also complex.

Writing $\tilde{k} = k + i\kappa$

The electric field $\tilde{E}(z,t) = \hat{\tilde{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}$

is attenuated due to damping.

$\because$ Intensity $\propto |\hat{\tilde{E}}_0 e^{-\kappa z}|^2$ $\therefore$ I decays

exponentially: $I = I_0 e^{-2\kappa z}$

$\therefore$ One defines $\alpha = 2\kappa$ = absorption coefficient.

Note that the traveling speed of $\tilde{E}$ is

controlled by $e^{i(kz-\omega t)}$, $\therefore$ the wave speed $= \frac{\omega}{k}$

$$n = \frac{c}{\omega}k = Re\,\frac{c\tilde{k}}{\omega} \quad \cdots \quad (134)$$

If the insulator is a gas, N is small.

The 2nd term in eq. (131) is small, one may expand $\sqrt{\varepsilon_r}$ as

$$\sqrt{\tilde{\varepsilon}_r} \approx 1 + \frac{1}{2}\frac{N_0 q^2}{m\varepsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}.$$

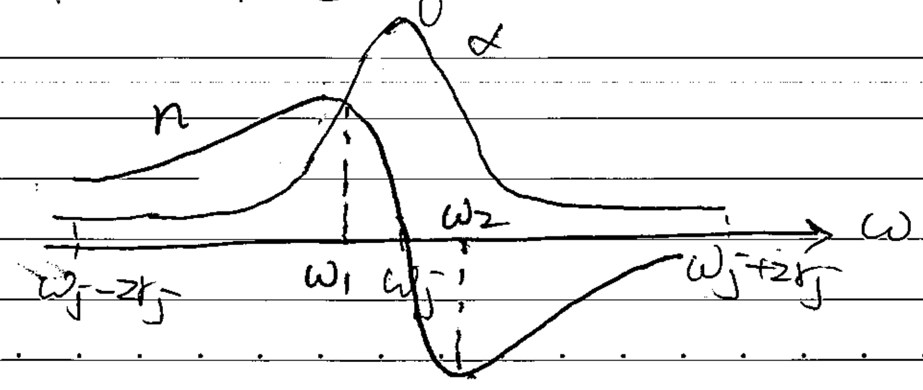
$$\therefore \tilde{k} = \frac{\omega}{c}\sqrt{\tilde{\varepsilon}_r} \simeq \frac{\omega}{c}\left[1 + \frac{N_0 q^2}{2m\varepsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right]$$

$$\therefore n = \mathrm{Re}\,\frac{c\tilde{k}}{\omega} \simeq 1 + \frac{N_0 q^2}{2m\varepsilon_0}\sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}$$

--- (135)

$$\alpha = 2\kappa \simeq \frac{N_0 q^2 \omega^2}{m\varepsilon_0 c}\sum_j \frac{f_j \gamma_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2} \quad \text{--- (136)}$$

Resonance : α shows a resonance structure in eq. (136): Near $\omega = \omega_j$, $\gamma_j\omega$ dominates, for $\omega \ll \omega_j$, $(\omega_j^2 - \omega^2)^2 \sim \omega_j^4$ in the denominator dominates. $\therefore$ Plots of α & n near ω_j are



The resonance at $\omega = \omega_j$ is clear:

the amplitude of oscillation for the electron is largest and hence energy being absorbed is largest.

On the other hand, near $\omega = \omega_j$, n starts to drop and becomes less than one near $\omega = \omega_j$.

For $\omega_1 \le \omega \le \omega_2$, $\frac{dn}{d\omega} < 0$, the dispersion

$$k = \frac{\omega}{c} n(\omega)$$ is anomalous dispersion.

Close to $\omega = \omega_j$, eventually, $n < 1$, phase velocity $v_p = \frac{\omega}{k} = \frac{c}{n} > c$. In fact, at $\omega = \omega_j$, $n = 0$, $v_p = \infty$. The speed $v_p > c$ may sound to be a problem as we shall learn that in special relativity, no speed can exceed c. However, more precisely, it is experimentally found that the signal velocity can not exceed c. Other speeds such as phase velocity & group velocity are not signal velocity. Hence they can exceed c.

Away from the resonance; the damping term in the denominator can be neglected.

$$\therefore\ n = 1 + \frac{Ne^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2} \quad \text{---} \ (137)$$

For most materials, ω_j's are scattered all over the spectrum. However, for transparent materials, they are transparent in the visible light, the closest resonance to the visible light is in the ultraviolet.

(optical region)

Hence for the visible light, $\omega < \omega_j$.

$$\therefore\ \frac{1}{\omega_j^2 - \omega^2} = \frac{1}{\omega_j^2}\left(1 - \frac{\omega^2}{\omega_j^2}\right)^{-1} \simeq \frac{1}{\omega_j^2}\left(1 + \frac{\omega^2}{\omega_j^2}\right)$$

$$\therefore\ n \cong 1 + \frac{Ne^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2} + \omega^2\left(\frac{Ne^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^4}\right)$$

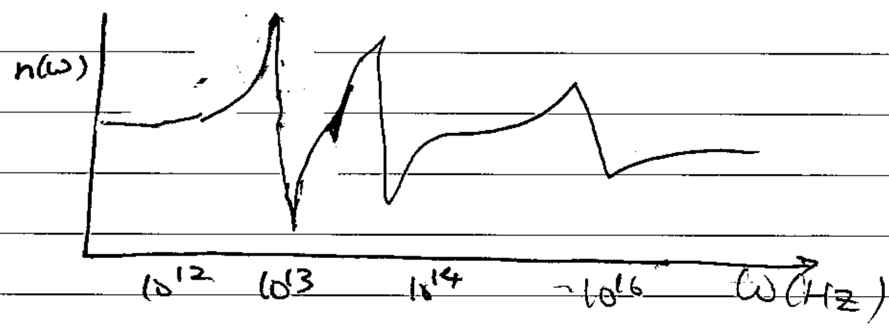
In terms of λ in vacuum, $\lambda = \frac{2\pi c}{\omega}$, the refraction index is in the form

$$n = 1 + A\left(1 + \frac{B}{\lambda^2}\right) \quad \text{---} \ (138)$$

which is known as the Cauchy's formula. With A being the coefficient of refraction and B being the coefficient of dispersion.

We see that far from the resonance, n decreases monotonically as λ increases, which is the typical behavior of n in the optical region.

The Lorentz model is a simplified model. However, for most condensed matter systems, the dielectric constant ε_r can be modelled in the same as that of the Lorentz model as sum of oscillator contributions. The following exptal data for SiO_2 is an example



The exact shape of $n(\omega)$ may not be captured by the Lorentz model but the resonance picture is the same.

Propagation of EM waves in plasma

For insulating dielectrics, if the frequence ω of EM waves exceed all natural frequency ω_j, $\omega \gg \omega_j$

$$\tilde{\varepsilon}_r = 1 + \frac{N_0 e^2}{m\varepsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega} \quad (\text{cf. } 131)$$

$$\rightarrow 1 - \frac{1}{\omega^2} \frac{N_0 e^2}{m} \sum_j f_j \quad \text{for } \omega \gg \omega_j$$

$$\because N_0 \sum_j f_j = \text{total \# of electrons} = N$$

$$\therefore \tilde{\varepsilon}_r = 1 - \frac{\omega_p^2}{\omega^2}, \quad \omega_p^2 \equiv \frac{Ne^2}{m\varepsilon_0} \text{ is}$$

$\llcorner$ -- (139)

the so-called plasma frequency.

In the limit $\omega \gg \omega_j$ for all j, the electron does not feel binding force from the molecule it binds to. The state is similar to ionized gas in which atoms/molecules are ionized so that positive charges (ions) & negative charges (electrons) do not bind together individually, i.e., electrons do not belong a particular molecule any more. They are free to move.

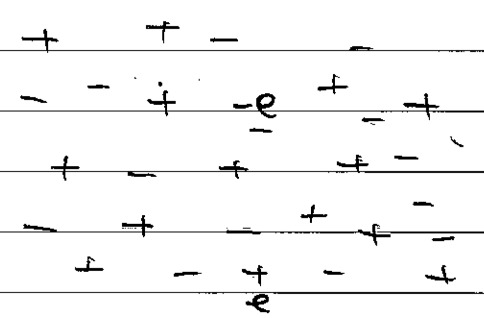
Furthermore, collision between electrons & ions and dissipation can be neglected.

Such state is called plasma.

Eq. (129) indicates that in $\omega \gg \omega_j$, the dielectrics behaves as a plasma.

Natural frequency of a plasma

In the unperturbed state, the plasma is neutral with equal # of positive charges & negative charges, distributed uniformly as shown in below.

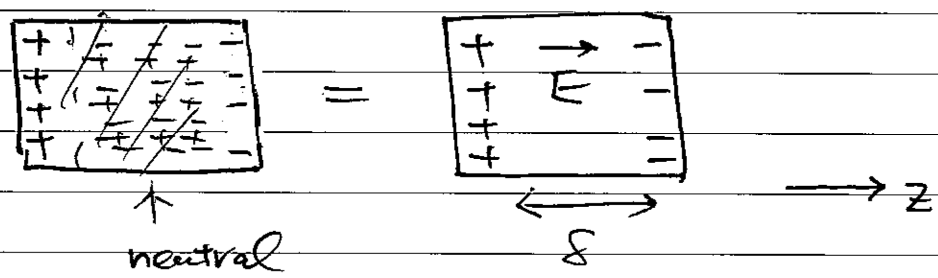


The center of positive charges coincides
= the center of negative charges

Under a perturbation, if there is separation δ between positive charge center & negative charge center, one can view it as a relative displacement δ between positive charges &

negative charges locally.

$\therefore$ Locally, one can view the local plasma as



in the center, the plasma is still neutral.

The displacement induces extra separated charges on edges.

Clearly, this can be viewed as a small parallel plate capacitor with surface charge density $\sigma = e n \delta$, n = electron density

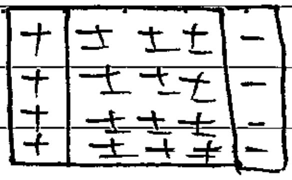
└ - - (140)

The charge density yields an electric field $E = \frac{\sigma}{\varepsilon_0} = \frac{ne\delta}{\varepsilon_0}$.

Since the positive charges (ion) are heavy, one ~~can treat them as fixed charges.~~

The electric field E acts on the center of negative charges with $f = -NeE = -\frac{Nne^2}{\varepsilon_0}\delta$

f: force per volume; N: # of e per volume



$\because f = Nm \underbrace{\frac{d^2\delta}{dt^2}}_{\text{acceleration of the center for negative charges}}$

$$\therefore \; Nm\frac{d^2\delta}{dt^2} = -N\frac{ne^2}{\varepsilon_0}\delta$$

$$\frac{d^2\delta}{dt^2} = -\frac{ne^2}{m\varepsilon_0}\delta = -\omega_p^2\delta \quad \cdots \; (141)$$

$\therefore$ ω_p is the natural frequency that tries to restore the charge neutrality of plasma.

<u>Propagation EM wave</u> (Longitudinal wave)

In the presence of EM wave, the electron experiences the external force

$$\vec{f} = (-e)(\vec{E}_{ext} + \vec{v}\times\vec{B})$$

$\because \frac{B}{E} \doteq \frac{1}{c} \ll 1 \quad \therefore \; \vec{f} = (-e)\vec{E}$

For $\vec{E}_{ext} = \vec{E}_0 \cos\omega t$, we have $\tilde{\mathbb{E}}_{ext} = \tilde{E}_0 e^{i\omega t}$.

The external force acting on electrons per volume $= N(-e)\tilde{\mathbb{E}}_{ext}$

$\therefore$ Eq. (141) becomes $Nm\frac{d^2\tilde{\delta}}{dt^2} = -N\frac{ne^2}{\varepsilon_0}\tilde{\delta} + N(-e)\tilde{\mathbb{E}}_{ext}$ $\quad \cdots \; (142)$

$$\therefore \frac{d^2\tilde{\delta}}{dt^2} = -\omega_p^2\tilde{\delta} - \frac{e\tilde{E}_0}{m}e^{i\omega t} \quad \text{--- (143)}$$

This is a forced simple harmonic oscillation.

$\tilde{\delta} = \tilde{\delta}_0 e^{i\omega t}$ in the steady state

with $\tilde{\delta}_0 = \dfrac{e\tilde{E}_0/m}{\omega^2 - \omega_p^2}$

Therefore, $\tilde{\delta}(t) = \dfrac{e/m}{\omega^2 - \omega_p^2}\tilde{\mathbb{E}}_{ext}(t)$. The wave is longitudinal in when $\tilde{\delta} \parallel \tilde{E}_{ext}$.

Using $\tilde{\delta}(t)$, eq. (142) implies that

The total force acting on electrons per volume

$$= -N\frac{ne^2}{\varepsilon_0}\tilde{\delta} - Ne\tilde{\mathbb{E}}_{ext}$$

$$= -N\left[\frac{ne^2}{\varepsilon_0}\frac{e/m}{\omega^2 - \omega_p^2} + e\right]\tilde{\mathbb{E}}_{ext}$$

$$= -Ne\tilde{\mathbb{E}}_{total}$$

$$\therefore \tilde{\mathbb{E}}_{total} = \left[1 + \frac{ne^2/m\varepsilon_0}{\omega^2 - \omega_p^2}\right]\tilde{\mathbb{E}}_{ext}$$

$$= \left(1 + \frac{\omega_p^2}{\omega^2 - \omega_p^2}\right)\tilde{\mathbb{E}}_{ext} = \frac{\tilde{\mathbb{E}}_{ext}}{1 - \frac{\omega_p^2}{\omega^2}} \quad \text{--- (144)}$$

The total field $\tilde{\mathbb{E}}_{total}$ is screened electric field $\tilde{\mathbb{E}}$ (macroscopic field). Its relation to

$\tilde{E}_{ext}$ is. $\tilde{E}_{tot} = \frac{\tilde{E}_{ext}}{\varepsilon_r}$

Hence $\varepsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$

$$\varepsilon(\omega) = \varepsilon_0 \varepsilon_r = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right) \quad \text{-- (145)}$$

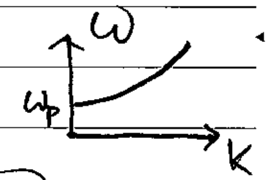
The refraction index

$$n(\omega) = \sqrt{\varepsilon_r} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

and the speed of EM wave

$$v_p = \frac{1}{\sqrt{\mu_0 \varepsilon}} = \frac{c}{\sqrt{1 - \omega_p^2/\omega^2}} > c \quad \text{--- (146)}$$

$\because v_p = \omega/k \quad \therefore \frac{\omega}{k} = \frac{c}{\sqrt{1 - \omega_p^2/\omega^2}}$



We get

$$\omega^2 = \omega_p^2 + c^2 k^2 \quad \text{--- (147)}$$

which is the dispersion relation of EM wave in plasma.

The group velocity

$$v_g = \frac{d\omega}{dk} = c\sqrt{1 - \omega_p^2/\omega^2} < c \quad \text{-- (148)}$$

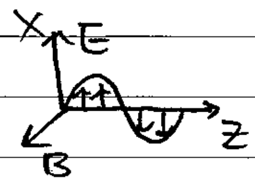
$\underline{\omega < \omega_p}$

Clearly, for $\omega < \omega_p$, n becomes purely imaginary. Hence $k = i\kappa$ is purely imaginary for a given ω.

Therefore, $\tilde{E} \propto e^{-\kappa z}$ decays exponentially.

$\therefore$ Longitudinal EM waves with $\omega < \omega_p$ can not propagate in plasma.

Transverse wave



When $\vec{E}_{ext} \perp Z$ (propagation direction), electrons move up & down in X directions and inducing local change of charge density that depends on Z.

Hence the restoring force is in Z direction.

In X direction, the force on the electron is $\vec{f} = (-e)(\vec{E}_{ext} + \vec{v} \times \vec{B}) \approx (-e)\vec{E}_{ext}$ $\left(\because \frac{B}{E} = \frac{1}{c} \ll 1\right)$

$$\therefore \quad \frac{d^2x}{dt^2} = -\frac{e}{m}E \quad (\because \text{internal field} \parallel Z, \therefore E = E_{ext})$$

For $\tilde{E} = \tilde{E}_0 e^{i\omega t}$, $\tilde{x} = \frac{e\tilde{E}_0}{m\omega^2} e^{i\omega t}$

i.e, the electron is a free particle under the action of E field.

Hence. $\frac{d\tilde{x}}{dt} = i\omega \frac{e\tilde{E}_0}{m\omega^2} e^{i\omega t}$

The current density

$$\tilde{J}_f = N(-e)\frac{d\tilde{x}}{dt}$$

$$= -i\frac{Ne^2}{m\omega}\tilde{E}_0 e^{i\omega t}$$

$$\equiv \sigma \tilde{E}$$

With the conductivity $\sigma(\omega) = -i\frac{Ne^2}{m\omega}$ --- (149)

From the equation

$$\nabla^2 \vec{E} = \mu_0\epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} + \mu_0 \sigma \frac{\partial \vec{E}}{\partial t},$$

one gets $K^2 = \omega^2\epsilon_0\mu_0 - i\omega\mu_0\sigma(\omega)$

$$= \omega^2\epsilon_0\mu_0\left(1 - \frac{i\sigma}{\omega\epsilon_0}\right)$$

$$= \omega^2\mu_0\epsilon_0\left(1 - \frac{\frac{Ne^2}{m\epsilon_0}}{\omega^2}\right)$$

$$= \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right) \equiv \frac{(n\omega)^2}{c^2} = \left(\frac{\omega}{v_p}\right)^2$$

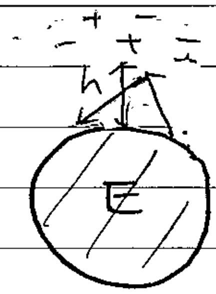
Hence $n(\omega) = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$, $v_p = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}$

Which are the same as those for the longitudinal waves

Therefore, both transverse and longitudinal waves

With $\omega < \omega_p$ can't propagate in plasma.

Example: Reflection of radio waves by the ionosphere



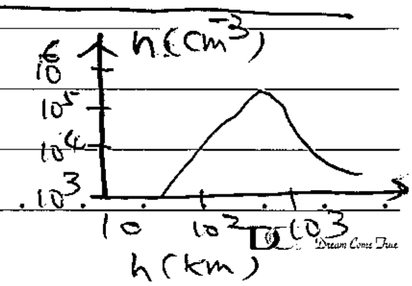
In 1901, G. Marconi demonstrated that the radio wave could be transimitted from one side of the Atlantic ocean to the other.

What is the mechanism for reflection?

O. Heaviside & A. Kennely proposed that there is a conducting layer located above 100 Km above the surface of the Earth, which reflects the radio waves

Direct experimental evidence of the conducting layer (known as the ionosphere) was reported by E. V. Appleton in 1942.

The ionosphere is a plasma which is an ionized gas mostly due to ultraviolet radiation of the Sun with electron density as shown in the right figure.



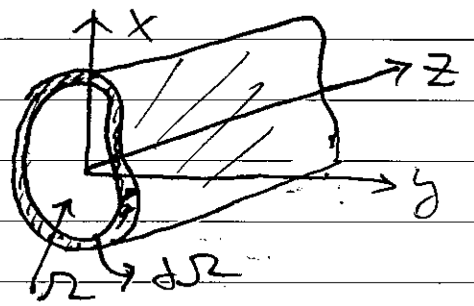
For $n \sim 10^5\,\mathrm{cm}^{-3}$, $f_p = \frac{\omega_p}{2\pi} \sim 3\,\mathrm{MHz}$.

Guided waves

Wave guids — hollow conducting tubes

In real application of EM waves, it is useful if one can transmit EM waves through a device to other locations.

Such a device is usually amplimented by a hollow conducting tube to guide the EM wave from one location to the other, shown in below: $\because$ cross sections at different z are the same.



We shall assume that the conducting tube is a perfect conductor so that $\sigma = \infty$

$$\because \vec{J}_f = \sigma \vec{E}, \therefore \vec{E} = 0$$

From $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, $\therefore \frac{\partial \vec{B}}{\partial t} = 0$. If the magnetic field $\vec{B}$ started at $\vec{B}(0) = 0$, $\vec{B} = 0$

Applying the BCs for metal surface, one has the BCs: (i) $E^{\parallel}|_{\partial\Omega} = 0$ $(E_1^{\parallel} = E_2^{\parallel} = 0)$

(ii) $B^{\perp}|_{\partial\Omega} = 0$ $(B_1^{\perp} = B_2^{\perp} = 0)$

We shall be interested in the monochromatic waves that can propagate along the tube.

$\vec{E}$ & $\vec{B}$ have the following forms in complex notations:

$$\tilde{\mathbb{E}}(x,y,z,t) = \tilde{\mathbb{E}}_0(x,y)\, e^{i(kz-\omega t)} \quad \text{--- (151)}$$

$$\tilde{\mathbb{B}}(x,y,z,t) = \tilde{\mathbb{B}}_0(x,y)\, e^{i(kz-\omega t)} \quad \text{--- (152)}$$

Let $\tilde{\mathbb{E}}_0(x,y) = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$ (E_i is complex)

$\tilde{\mathbb{B}}_0(x,y) = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$

$(\vec{\nabla}\times\vec{E}) = -\frac{\partial \vec{B}}{\partial t}$ implies

(i) $\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z$ (z-component)

(ii) $\frac{\partial E_z}{\partial y} - \underbrace{ikE_y}_{\frac{\partial E_y}{\partial z}} = i\omega B_x$ (x-component)

(iii) $\underbrace{ikE_x}_{\frac{\partial E_x}{\partial z}} - \frac{\partial E_z}{\partial x} = i\omega B_y$ (y-component)

— (152) — 1

$\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$ implies

(iv) $\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \frac{-i\omega}{c^2}E_z$ (z-component)

(v) $\frac{\partial B_z}{\partial y} - ikB_y = \frac{-i\omega}{c^2}E_x$ (x-component)

(vi) $ikB_x - \frac{\partial B_z}{\partial x} = \frac{-i\omega}{c^2}E_y$ (y-component)

Eqs (ii), (iii), (v) & (vi) allow one to solve

E_x, E_y, B_x, B_y in terms of $\frac{\partial E_z}{\partial x}, \frac{\partial E_z}{\partial y}, \frac{\partial B_z}{\partial x}$ & $\frac{\partial B_z}{\partial y}$.

For instance,

$$E_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$$

$$E_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial y}\right) \quad \text{--- (153)}$$

& $\begin{cases} E_x \to B_y \\ E_y \to B_x \end{cases}$ $\left(B_z \to \frac{E_z}{c^2},\ E_z \to B_z\right)$

Therefore, once E_z & B_z are known

all $\vec{E}$ & $\vec{B}$ are known.

Another implication of eq. (153) is that E_z & B_z can't both vanish! If they do,

$\vec{E}$ & $\vec{B} \equiv 0$, there is no EM wave.

More precisely, if $E_z = 0$, we call the EM waves as TE (transverse electric) waves, while if $B_z = 0$, we call the waves as TM (transverse magnetic) waves.

If both $E_z = 0$ and $B_z = 0$, we call them TEM waves.

Both E_z & B_z still satisfy the wave equation with $\frac{\partial^2}{\partial z^2}$ replaced by $-k^2$ and $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ replaced by $-\left(\frac{\omega}{c}\right)^2$:

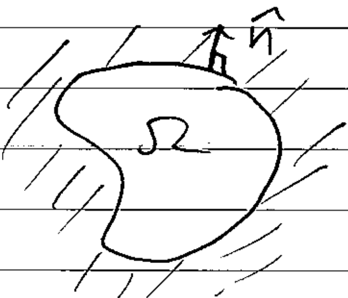
$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k^2\right]E_z=0 \quad \cdots (154)$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k^2\right]B_z=0 \quad \cdots (155)$$

Theorem: TEM waves can not occur in a hollow wave guide

Pf: For TEM waves, $B_z=0$, $E_z=0$

$$\left.\begin{aligned} \therefore\ \vec{\nabla}\cdot\vec{E} &= \frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}=0 \\ (\vec{\nabla}\times\vec{E})_z &= -\frac{\partial B_z}{\partial t}=0.\quad \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=0 \end{aligned}\right\}\ \text{in } \Omega \text{ (cross section)}$$



$\therefore \vec{E}$ satisfies electrostatic eq. in 2D.

One can write $\vec{E}=(E_x,E_y)=\left(-\frac{\partial\phi}{\partial x},-\frac{\partial\phi}{\partial y}\right)$ with ϕ being the electrostatic potential.

To satisfy $\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}=0$ $\therefore$ ϕ satisfies 2D Laplace equation $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$

with BC: $\vec{E}^{\parallel}\big|_{\partial\Omega}=0$

$\therefore \vec{E} \parallel \hat{n}$.

i.e. $\partial\Omega$ is an equipotential $\therefore$ curve of ϕ

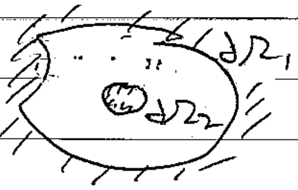
Since the Laplace's equation does not allow local maxima or minima in Ω (can appear only in $\partial\Omega$), it implies ϕ = constant in Ω. $\therefore \vec{E} \equiv 0$ in Ω.

i.e. $E_x = E_y = 0$ in Ω $\because E_z = 0$, $\therefore \vec{E} \equiv 0$ inside tube

Once $\vec{E} \equiv 0$, eqs. (152)-1 imply $\vec{B} = 0$.

$\therefore$ There is no wave in a hollow tube.

Note that this conclusion is only true for a hollow tube. As we shall see, if there is a separate conducting line inside the tube, there are two equipotential curves

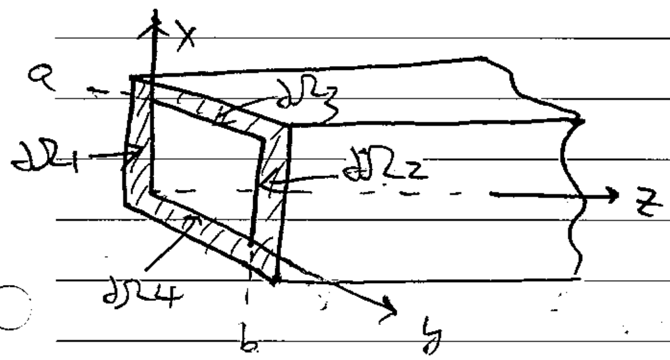


$\partial\Omega_1$ & $\partial\Omega_2$. Their potentials need not to be the same.

Hence ϕ needs not to be a constant in Ω! In this case, TEM waves can propagate in the tube!

TE waves in a rectangular wave guide

We shall illustrate the guided waves by TE waves in a rectangular wave guide. The TM waves are analyzed similarly and those are homework ~~as an exercise (prob. 9.31)~~



$E_z = 0$

B_z satisfies eq. (155) with ~~$\vec{B}\cdot\hat{n}=0$ at the~~ boundaries.

The geometry & BCs are separable.

$\therefore$ We try the solution

$$B_z(x,y) = X(x)\,Y(y)$$

Eq. (155) becomes

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right] X\,Y = 0$$

$$\therefore \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \left(\frac{\omega}{c}\right)^2 - k^2 = 0 \quad \cdots (156)$$

We can set $\frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2$, $\frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2$ (157)

with $k_x^2 + k_y^2 + k^2 = \left(\frac{\omega}{c}\right)^2$ --- (158)

$$\therefore \begin{cases} \cancel{Y(y) = C\sin k_y y + D\cos k_y y} \\ X(x) = A\sin k_x x + B\cos k_x x \end{cases}$$

Now, B_x & B_y are given by eqs. (153)

$$B_x = \frac{i}{(\frac{\omega}{c})^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$$

$$= \frac{ik}{(\frac{\omega}{c})^2 - k^2}\frac{\partial B_z}{\partial x} \quad \cdots \quad (159)$$

$$B_y = \frac{i}{(\frac{\omega}{c})^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$$

$$= \frac{ik}{(\frac{\omega}{c})^2 - k^2}\frac{\partial B_z}{\partial y} \quad \cdots \quad (160)$$

On $\partial\Omega_1$ & $\partial\Omega_2$, one requires $\hat{y}\cdot\vec{B} = 0$

i.e. $B_y(y=0) = B_y(y=b) = 0$

Eq. (160) implies $\left.\frac{dY}{dy}\right|_{y=0} = \left.\frac{dY}{dy}\right|_{y=b} = 0$

$\therefore C = 0$, $\sin k_y b = 0$, $k_y = \frac{n\pi}{b}$, $n = 0, 1, 2, 3 \ldots$

On $\partial\Omega_3$ & $\partial\Omega_4$, one requires $\hat{x}\cdot\vec{B} = 0$

$\therefore B_x(x=0) = B_x(x=a) = 0$

Eq. (159) implies $\left.\frac{dX}{dx}\right|_{x=0} = \left.\frac{dX}{dx}\right|_{x=a} = 0$

$\therefore A = 0$, $k_x a = m\pi$, $k_x = \frac{m\pi}{a}$, $m = 0, 1, 2, \ldots$

Hence. $B_z = B_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$.

The solution is called TEmn mode

with at least one index being non-zero.

From eq. (158), one can find

$$K = \sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]} \quad \text{--- (161)}$$

Together with eqs. (158) & (160), one

can compute B_x & B_y.

Using eqs. (152)-1, E_x, E_y can be computed

as well.

Eq. (161) imply that for a given (m, n),

$$\omega < c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \equiv \omega_{mn},$$

the $\underbrace{\text{TEmn}}_{\text{wave number}}$ is purely imaginary. It's

exponentially attenuated.

ω_{mn} is called the cutoff frequency for

TEmn mode.

The lowest cutoff frequency is the cutoff

frequency of TE_{01} mode : $\omega_{10} = c\pi/a$, $(a>b)$

$\therefore$ For $\omega < \frac{c\pi}{a}$, there is no TE mode

that can propagate in the wave guide.

For general TE_{mn} mode; the wave number

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2} \quad (\text{eg. } 161)$$

∴ The phase velocity

$$v_p = \frac{\omega}{k} = \frac{c}{\sqrt{1-(\frac{\omega_{mn}}{\omega})^2}} > c \quad \cdots (162)$$

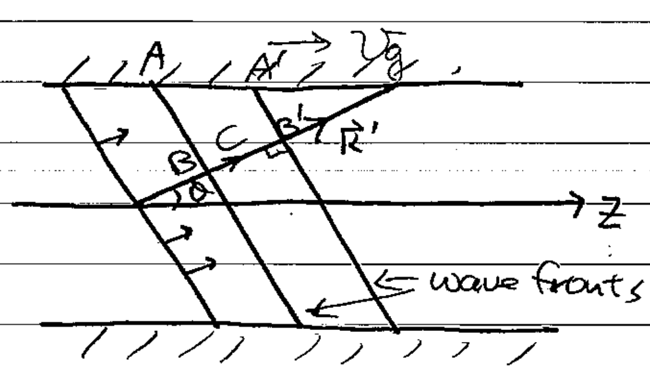
However, the energy carried by the wave travels at the group velocity

$$v_g = \frac{d\omega}{dk} = \frac{1}{dk/d\omega}$$

$$= c\sqrt{1-(\frac{\omega_{mn}}{\omega})^2} < c \quad \cdots (163)$$

Internal reflection

The propagation of waves (e.g. TE mode) in the wave guide can be also understood via the view of internal reflections.



$B_z = B_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \times e^{i(kz-\omega t)}$

consits travelling waves of $\vec{k}' = (\frac{m\pi}{a}, \frac{n\pi}{b}, k)$

$(\because \cos\frac{m\pi x}{a} = \frac{1}{2}(e^{i\frac{\pi x m}{a}} + e^{-i\frac{m\pi}{a}x}))$

and $\left(-\frac{m\pi}{a}, \frac{n\pi}{b}, k\right)$, $\left(\frac{m\pi}{a}, -\frac{n\pi}{b}, k\right)$, $\left(-\frac{m\pi}{a}, -\frac{n\pi}{b}, k\right)$.

We shall focus on the propagation of $\vec{k}'$ as shown in the above.

The analysis for other travelling waves is similar.

$$\because \omega = c|\vec{k}'| = c\sqrt{k^2 + \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]}$$

$\therefore$ The angle of $\vec{k}' \wedge z \equiv \theta$

$$\cos\theta = \frac{k}{|\vec{k}'|} = \frac{k}{\omega/c} = \sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2}$$

Propagation along $\hat{k}'$: speed $= c$.

$\therefore$ The effect velocity along the wave guide

$$v_g = c\cos\theta = c\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2}$$

in agree with eq. (163)

On the other hand, the phase velocity along the wave guide can be found by examing points of constant phase, say, point A (phase $= 0$, maximal value of B_z).

During one wavelength, $B \to B'$, A moves to A'

$\because B \to B'$, speed $= c$.

$\therefore A \to A' \Rightarrow$ phase velocity $= \dfrac{c}{\cos\theta} = \dfrac{c}{\sqrt{1-\left(\frac{\omega_{mn}}{\omega}\right)^2}}$

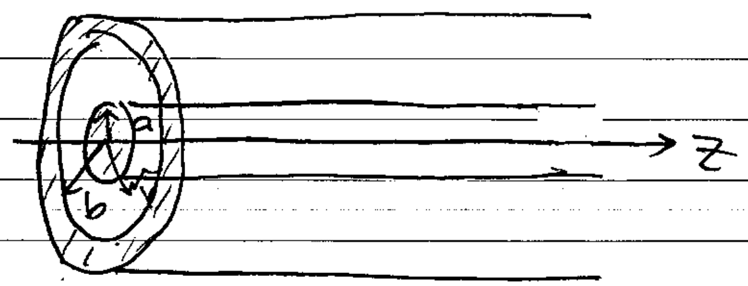
in agreement with eq. (162).

The coaxial transmission line

As mentioned, for a hollow wave guide, there is no TEM wave that can propagate inside it.

This can be overcome by introducing another metal surface inside the guide.

A coaxial transmission line is such a design with a long straight wire of radius a surrounded by a (coaxial) cylindrical conducting sheath of radius b. (as shown in below)



In this case, TEM waves can be supported.

Using eqs. (153)-1, one obtains

$$\left.\begin{aligned}&(\text{ii}')\ -kE_y = \omega B_x \\ &(\text{vi})\ -kB_x = \frac{\omega}{c^2}E_y\end{aligned}\right\} \Rightarrow k^2 = \frac{\omega^2}{c^2},\quad k = \omega/c$$

$$\therefore cB_x = -E_y$$

Similarly,

$$\left.\begin{aligned}&(\text{iii}')\quad kE_x = \omega B_y \\ &(\text{v})\quad kB_y = \frac{\omega}{c^2}E_x\end{aligned}\right\} \Rightarrow \omega = ck$$

$$\therefore cB_y = E_x$$

$$\therefore (E_x, E_y)\cdot(B_x, B_y) = 0 \quad \vec{E}\perp\vec{B},\quad \vec{B} = \frac{1}{c}\hat{z}\times\vec{E} \quad \text{--- (164)}$$

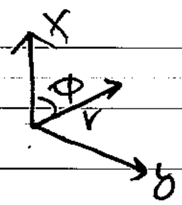
Now, $\because \vec{\nabla}\cdot\vec{E} = 0 \quad (\vec{\nabla}\times\vec{E})\cdot\hat{z} = -\frac{\partial B_z}{\partial t} = 0$,

$$\therefore \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0, \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0$$

These are electrostatic equations for $\vec{E}$ in 2D.

$\because \vec{E}_{\parallel}\big|_{r=a} = 0,\ \vec{E}_{\parallel}\big|_{r=b} = 0 \quad \therefore \vec{E}\parallel\hat{r}\big|_{r=a \text{ or } r=b}$.

$r=a$ & $r=b$ are equipotentials. The solution to $\vec{E}$ is the same as that an infinite line charge located at $r=0$.



$$\therefore \vec{E} = \hat{E}_0\, e^{i(kz-\omega t)}$$

$$\hat{E}_0(r,\phi) = \frac{A}{r}\hat{r}, \quad A = \text{some constant}$$

Using eq. (164), $\hat{B}_0(r,\phi) = \frac{A}{cr}\hat{\phi}$

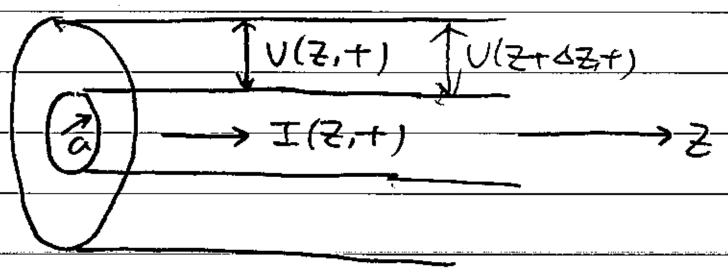
Taking the real part of $\hat{E}$, we obtain the general solution for TEM mode in the coaxial line:

$$\vec{E}(r,\phi,z,t) = \frac{A\cos(kz-\omega t)}{r}\hat{r}$$

$$\vec{B}(r,\phi,z,t) = \frac{A\cos(kz-\omega t)}{cr}\hat{\phi} \quad \text{--- (165)}$$

Telegrapher's equation

In practice, it's often convenient to describe the coaxial transmission line in terms of the propagation of voltage $V(z,t)$ (potential difference $V_b - V_a$) and current $I(z,t)$ ($r=a$).



Treating the coaxial line as a circuit, it has L_0 (inductance per length along z) and C_0 (capacitance " " " z).
Neglecting the resistance (perfect conductor), one gets

$\because \Delta V = -L\frac{dI}{dt}$ across a circuit element.

$$\therefore V(z+\Delta z,t) - V(z,t) = -L_0\Delta z\frac{\partial I(z,t)}{\partial t}$$

$$\therefore \frac{\Delta V}{\Delta z} = -L_0\frac{\partial I}{\partial t}$$

$$\text{i.e.}\quad \frac{\partial V(z,t)}{\partial z} = -L_0\frac{\partial I(z,t)}{\partial t} \quad \text{--- (166)}$$

Similarly,

$$I(z+\Delta z,t) - I(z,t) = -\frac{\partial Q}{\partial t}$$

$$= -C_0\Delta z\frac{\partial V(z,t)}{\partial t}$$

$$\therefore \frac{\partial I(z,t)}{\partial z} = -C_0\frac{\partial V}{\partial t} \quad \text{--- (167)}$$

Eqs. (166) & (167) are a pair of telegrapher's equations. Eliminating I or V, one gets

$$\frac{\partial^2 V(z,t)}{\partial z^2} = L_0C_0\frac{\partial^2 V(z,t)}{\partial t^2}$$

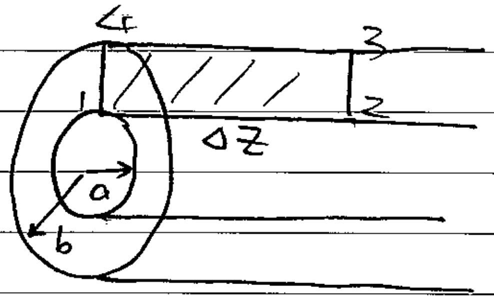
$$\frac{\partial^2 I(z,t)}{\partial z^2} = L_0C_0\frac{\partial^2 I(z,t)}{\partial t^2} \quad \text{--- (168)}$$

$\therefore$ Both V & I satisfy wave equations with wave velocity $v = \frac{1}{\sqrt{L_0C_0}}$ --- (169).

For the coaxial line with I on $r=a$,

$$\vec{B} = \frac{\mu_0}{2\pi r}I(z,t)\,\hat{\phi}$$

The inductance per length (L_0) can be determined by $\Delta V = -\frac{d\Phi_B}{dt}$.



Φ_B = flux passing

$$\phi = \phi_0 \left({}^{4}_{1}\boxed{\;\;}^{3}_{2} \right)$$

$$= \left(\int_a^b \frac{\mu_0}{2\pi r} I \, dr\right) \Delta z$$

$$= \left(\frac{\mu_0}{2\pi} I(z,t) \ln b/a \right) \Delta z$$

$$\therefore \Delta V = -\Delta z \frac{\mu_0}{2\pi} \ln b/a \frac{\partial I}{\partial t} = -L_0 \Delta z \frac{\partial I}{\partial t}$$

$$\therefore L_0 = \frac{\mu_0}{2\pi} \ln b/a \quad \text{---} \quad (170)$$

The capacitance per length (C_0) can be determined by setting $\lambda(z,t)$ (charge density per length) on $r=a$ and $-\lambda$ on $r=b$.

The electric field $\vec{E}(z,t) = \frac{\lambda(z,t)}{2\pi\epsilon_0 r}\hat{r}$

$$\therefore U(z,t) = \int_a^b \vec{E} \cdot d\vec{r} = \frac{\lambda(z,t)}{2\pi\epsilon_0} \int_a^b \frac{dr}{r}$$

$$= \frac{\lambda(z,t)}{2\pi\epsilon_0} \ln b/a$$

For a segment Δz, $Q(z,t) = \lambda \Delta z$

$$\therefore U(z,t) = \frac{Q(z,t)}{\Delta z \, 2\pi\epsilon_0 / \ln b/a} = \frac{Q}{\Delta C} \quad \therefore \frac{\Delta C}{\Delta z} = C_0$$

$$= \frac{2\pi\epsilon_0}{\ln b/a} \quad \text{---} \quad (171)$$

Combing eqs. (169), (170) & (171), we find

$$v = \frac{1}{\sqrt{L_0 C_0}} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$$

is the speed of light, in agreement with the propagation speed we obtained for travelling TEM modes.