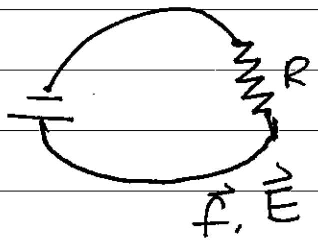
Electrodynamics

Electromotive force



As we discussed, charges are pushed through a resistor by $\vec{E}$.

In steady state, $\vec{\nabla} \times \vec{E} = 0$

$\therefore$ The force by $\vec{E}$ per unit charge

$$\vec{f}_E = \vec{E}$$

and $\oint \vec{f}_E \cdot d\vec{l} = 0$

There is no net work done by $\vec{E}$!

Then, what keep charges move around the circuit ?

Usually, this is provided by some driving sources such as battery.

In this case, charges ... experience :

$\vec{f}_s$ (usually in some portion of circuit)

$$\therefore \vec{f} = \vec{f}_s + \vec{f}_E = \vec{f}_s + \vec{E}$$

is the total driving force.

Note that f_s may not be a real force.
For instance, for a battery, $\vec{f_s}$ results from chemical energy.

Therefore, what is more important is the quantity

$$\mathcal{E} = \oint \vec{f} \cdot d\vec{\ell}$$

Which is the work done on the charge by all driving sources. $\mathcal{E}$ is called electromotive force, or emf.

$$\because \oint \vec{f_E} \cdot d\vec{\ell} = 0$$

$$\therefore \mathcal{E} = \oint \vec{f_s} \cdot d\vec{\ell} = \text{work done per unit charge by source}$$

a —|⊢ b   Inside an ideal source of emf ($f_s \neq 0$) (from a to b), $f = 0 = \frac{J}{\sigma}$ (i.e. $\sigma = \infty$) resistanless battery

$$\vec{f_s} = -\vec{E}$$

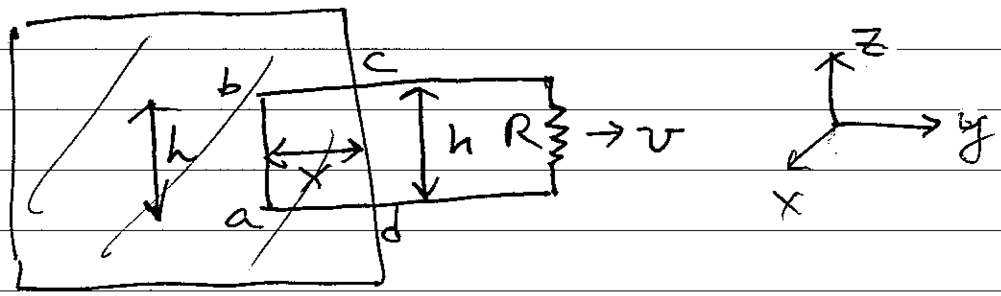
$$\therefore \mathcal{E} = \oint \vec{f_s} \cdot d\vec{\ell} = \int_a^b \vec{f_s} \cdot d\vec{\ell} = -\int_a^b \vec{E} \cdot d\vec{\ell}$$

$$= V_b - V_a$$

$\therefore$ The battery maintains a voltage difference across the battery, that drives currents flow outside the battery.

Motional emf

In addition to the battery, generators are another way to provide emf.
In this case, one exploits emf due to motions of conducting wires.



Consider a primitive model for a generator, in which a loop with resistance is pulled out from a region with uniform magnetic field, as shown on the above figure.
If the velocity of the loop is v, then charges in segment ab experience a magnetic force whose perpendicular component is $F_z = qvB$

Hence

$$f^{z}_{mag} = \frac{F_z}{q} = vB$$

$$\mathcal{E} = \oint \vec{f}_{mag} \cdot d\vec{l} = \int_a^b f^{z}_{mag}\, dl$$

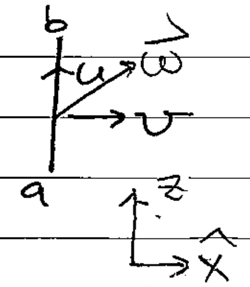
$$= vBh \quad --- \ (1)$$

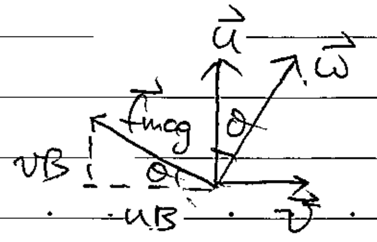
$\therefore$ Lorentz force generates an emf $\mathcal{E} = vBh$ in this case, It is called motional emf.

This seems to stem from the magnetic field but the magnetic force does not do work. Instead, the work is done by the force that pulls the loop!

Similar to the case we mentioned at the beginning of Chapter 5. Charges in ab segment have velocity

$$\vec{w} = u\hat{z} + v\hat{x} \quad -- \ (2)$$





$$\therefore \vec{f}_{mag} = \vec{w} \times (B\hat{y})$$

$$= -uB\hat{x} + Bv\hat{z}$$

$\vec{f}_{mag}$ is still in perpendicular to $\vec{w}$

$\therefore$ It does not do work on charges.

However, it yields a force $-uB\hat{x}$.

To counter it, one has to apply a

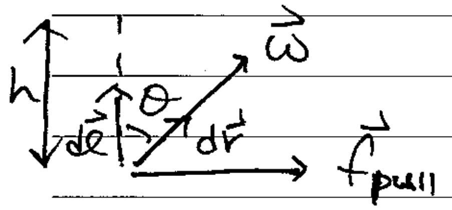
force $\vec{f}_{pull} = uB\hat{x}$ --- (3)

As a result of $\vec{f}_{pull}$, the total force acting

on unit charge is $\underline{\vec{f}_{mag} + \vec{f}_{pull} = Bv\hat{z}}$

does work on charges.

The work done on a unit charge is



$$W = \int (\vec{f}_{mag} + \vec{f}_{pull}) \cdot d\vec{r} \quad \text{--- (4)}$$

$$= \int \vec{f}_{pull} \cdot d\vec{r}$$

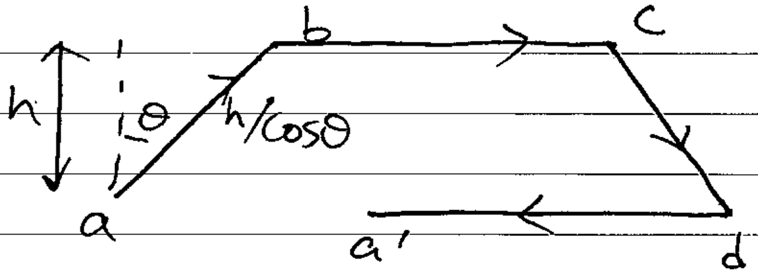
$\vec{f}_{mag} \perp d\vec{r}$

$$= \int f_{pull}\, dr \sin\theta$$

$$= f_{pull} \sin\theta \frac{h}{\cos\theta} = Bhu\tan\theta$$

$$= vBh = \mathcal{E}_m \; (1).$$

Note that the path of a charge moves during a period is



$\because \ \vec{f}_{mag} + \vec{f}_{pull} = f_{mag}^{z}\,\hat{z}$

$\therefore \ (\vec{f}_{mag} + \vec{f}_{pull}) \cdot d\vec{r} = f_{mag}^{z}\, dl$

$= \vec{f}_{mag} \cdot d\vec{l}$

$\therefore \ W = \oint \vec{f}_{mag} \cdot d\vec{l}$ --- (f) which has the apparent form that $\vec{f}_{mag}$ seems to do the work but actually it is done by $\vec{f}_{pull}$.

Flux rule

It's convenient to note that

$v = -\frac{dx}{dt}$ if x = length that is inside the magnetic field region

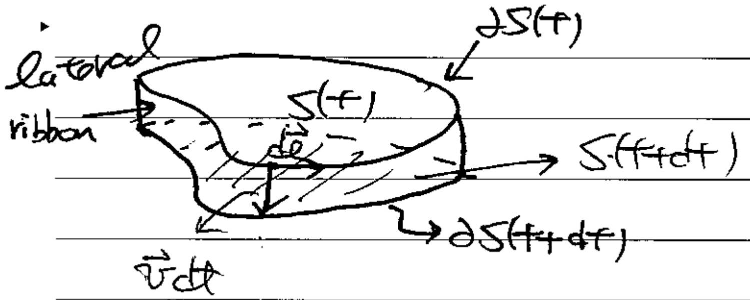
$\therefore \ \mathcal{E} = -Bh\frac{dx}{dt} = -\frac{d\,Bhx}{dt}$ ---- (6)

Realizing $Bhx =$ magnetic flux that the loop encloses $\equiv \Phi = \int_S \vec{B} \cdot d\vec{a}$

$$\therefore \ \mathcal{E} = -\frac{d\Phi}{dt} \quad \text{--- ⑦}$$

which is the flux rule for the motional emf.

The flux rule can be applied to any general loops which can also move in time.



As shown in the left figure, the loop at t is $\partial S(t)$ and it moves to $\partial S(t+dt)$ at $t+dt$.

Let the lateral area (ribbon) be $\Delta S \equiv S_{ribbon}$

$$\because \ \vec{\nabla} \cdot \vec{B} = 0 \quad \therefore \ \Phi(t+dt) = \int_{S(t+dt)} \vec{B}(t+dt) \cdot d\vec{a}$$

$$= \int_{\text{any surface bounded by } \partial S(t+dt)} \vec{B}(t+dt) \cdot d\vec{a}$$

$$= \int_{S(t)} \vec{B}(t+dt) \cdot d\vec{a} + \int_{S_{ribbon}} \vec{B}(t+dt) \cdot d\vec{a}$$

For static field, $\vec{B}$ is independent of t,

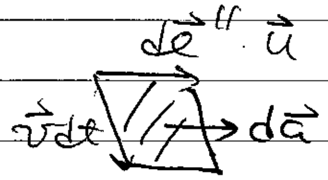
$$\therefore \Phi(t+dt) = \int_{S(t)} \vec{B}\cdot d\vec{a} + \int_{S_{ribbon}} \vec{B}\cdot d\vec{a} \quad \text{--- (8)}$$

$$= \Phi(t) + \int_{S_{ribbon}} \vec{B}\cdot d\vec{a}$$

$$\therefore d\Phi = \Phi(t+dt) - \Phi(t)$$

$$= \int_{S_{ribbon}} \vec{B}\cdot d\vec{a} \quad \text{--- (9)}$$

Now for each point on $\partial S(t)$, it moves with some velocity $\vec{v}$ to $\partial S(t+dt)$



$$\therefore d\vec{a} = \vec{v}dt \times d\vec{\ell} \quad \text{--- (10)}$$

$\therefore$ Eqs (9) & (10) imply

$$\frac{d\Phi}{dt} = \oint \vec{B}\cdot(\vec{v}\times d\vec{\ell}) \quad \text{--- (11)}$$

If the tangential velocity of the charge is $\vec{u}$, $\vec{u} \parallel d\vec{\ell}$, the total velocity of the charge

$$\vec{w} = \vec{u} + \vec{v} \quad \therefore \vec{v}\times d\vec{\ell} = \vec{w}\times d\vec{\ell}$$

$$(\vec{u}\times d\vec{\ell} = 0)$$

Hence

$$\frac{d\Phi}{dt} = \oint \vec{B} \cdot \vec{\omega} \times d\vec{\ell}$$

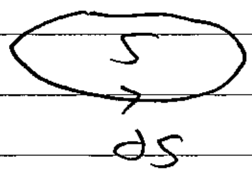
$$= \oint \vec{B} \times \vec{\omega} \cdot d\vec{\ell}$$

$$= -\oint \underbrace{\vec{\omega} \times \vec{B}}_{\text{magnetic force per unit charge}} \cdot d\vec{\ell}$$

$$\therefore \frac{d\Phi}{dt} = -\oint \vec{f}_{mag} \cdot d\vec{\ell} = -\mathcal{E}$$

$$\therefore \mathcal{E} = -\frac{d\Phi}{dt} \text{ is generally correct} \quad \text{--- (12)}$$

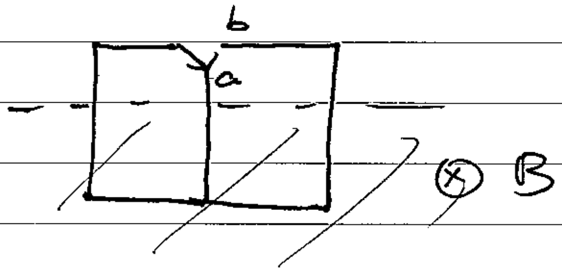
Direction: $\Phi = \int_S \vec{B} \cdot d\vec{a}$ is defined to be positive if $\vec{B} \parallel d\vec{a}$.



$\therefore$ Right-hand rule:
thumb $\doteq d\vec{a}$
4-fingers = orientation of ∂S in $\oint_{\partial S}$

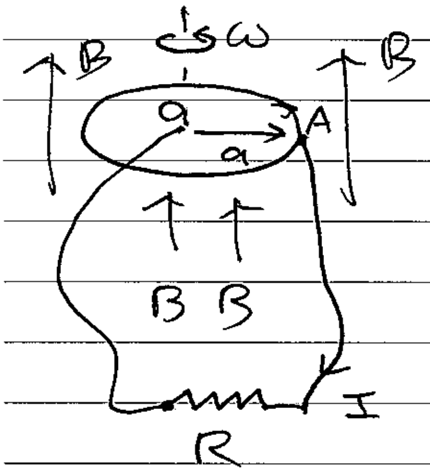
Note that the motional emf must involve the motion of segments in B field. Switches or sliding contacts are not included.

A false example :



Switch a to b doesn't yield any emf !

<u>Example</u> Farady disk



A metal disk of radius a rotates with angular velocity ω in a uniform B field.

Find the current $\hat{I}$ in R.

~~Solution: This problem can not be solved by flux rule as there is no defined~~ conducting loop.

For any point between $\overline{OA}$, $0<\rho<a$, $v=\omega\rho$, the magnetic force per unit charge is $\vec{f}_{mag} = \vec{v}\times\vec{B} = \omega\rho B\hat{\rho}$

$\therefore \mathcal{E} = \int_0^a f_{mag}\,d\rho = \omega B\int_0^a \rho\,d\rho = \frac{1}{2}\omega B a^2$

$$\therefore I = \frac{\mathcal{E}}{R} = \frac{\omega B a^2}{2R}$$

Eddy current

In the case when there is no clear conducting loop; free charge carriers will form their loops by themself. This is the case when one moves a piece of metal (say aluminum) towards or away from a magnet.

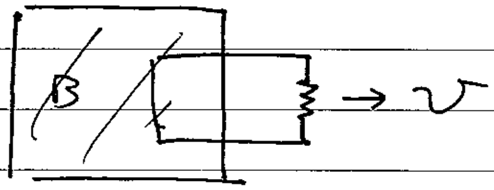
In this case, there are eddy currents in which charges in the metal moves in response to emf generated by motion. (or changing of magnetic fields, which will be discussed later.) The motion of charges forms swirling current, called eddy currents. The eddy current will impose damping to the motion of metal. To avoid generation of eddy currents, one way is to cut the path of current by introducing slots on the metal disk.

Electromagnetic induction

Faraday's law

In 1831, Faraday performed a series of experiments that contains 3 most important setup of experiments that lead to the induction of emf.

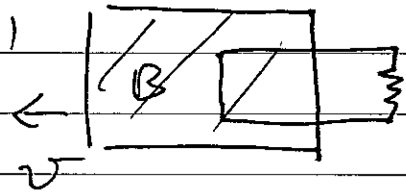
(i)



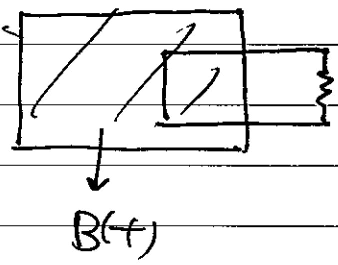
motional emf

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

(ii)



(iii)



(ii) is relative motion version of (i)

∴ the same result $\mathcal{E} = -\frac{d\Phi}{dt}$ is obtained

In (iii), one also obtains $\mathcal{E} = -\frac{d\Phi}{dt}$

For (ii) & (iii), there is no motion of charges. Therefore, it's not due to the magnetic force $q\vec{v}\times\vec{B}$!

What is responsible for $\mathcal{E}$?

Faraday :

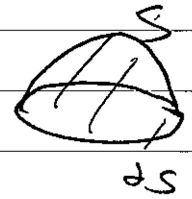
A changing magnetic field induces an electric field --- (13)

The induced electric field $\vec{E}$ obeys

$$\mathcal{E} = \oint_{\partial S} \vec{E}\cdot d\vec{\ell} = -\frac{d\Phi}{dt} \quad \text{--- (14)}$$

Since $\Phi = \int_S \vec{B}\cdot d\vec{a}$, $\frac{d\Phi}{dt} = \int \frac{\partial \vec{B}(\vec{r},t)}{\partial t}\, d\vec{a}$

$$\therefore \oint_{\partial S} \vec{E}\cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t}\, d\vec{a} \quad \text{--- (15)}$$



This is Faraday's law in the integral form. Note that S & ∂S are fixed and don't depend on time !

Using the Stoke's theorem,

$$\oint_{\partial S} \vec{E}\cdot d\vec{\ell} = \int_S \vec{\nabla}\times\vec{E}\cdot d\vec{a}$$

taking $S \to 0$, one gets (in contrast to $\vec{\nabla}\times\vec{E}=0$ in electrostatics)

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{--- (16)}$$

which is the Faraday's law in differential form.

Note that many people called

$$\mathcal{E} = -\frac{d\Phi}{dt}$$ as the

Faraday's law. However, $\mathcal{E} = -\frac{d\Phi}{dt}$ includes (i) – (iii') cases while only eqs (15) & (16) accounts for (ii) (iii).

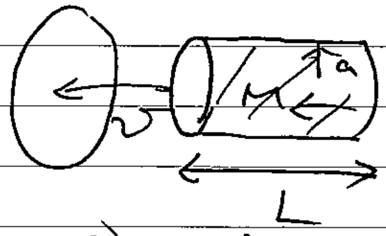
Case (i) has different mechanism and it is still included in $\mathcal{E} = -\frac{d\Phi}{dt}$.

Therefore, eqs. (15) or (16) deserve the name of Faraday's law.

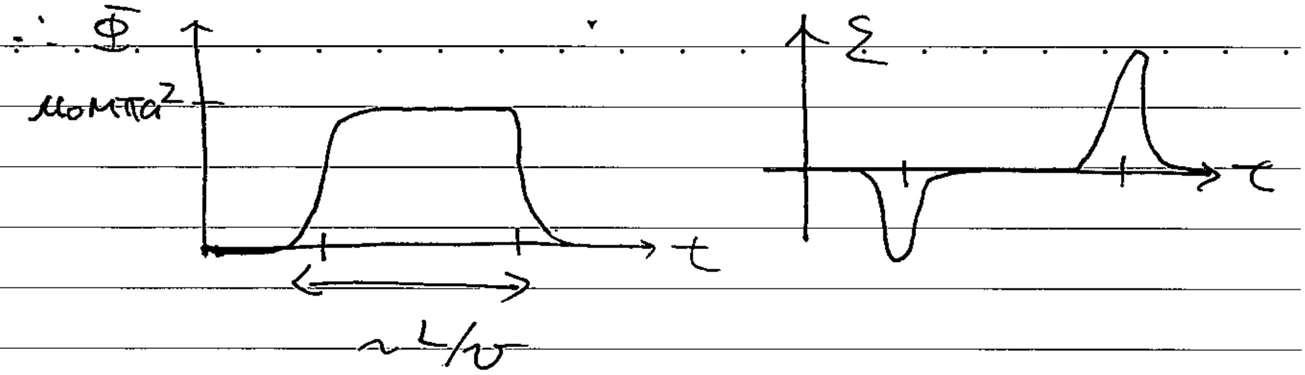
The coincidence of the same form $\mathcal{E} = -\frac{d\Phi}{dt}$ for (i) & (ii) – (iii') has deeper reason related to special relativity.

We will come back to it later in Chapter 12.

Example



$\vec{B} = \mu_0 \vec{M}$, close to ring: $\Phi_B \simeq \pi a^2 \mu_0 M$



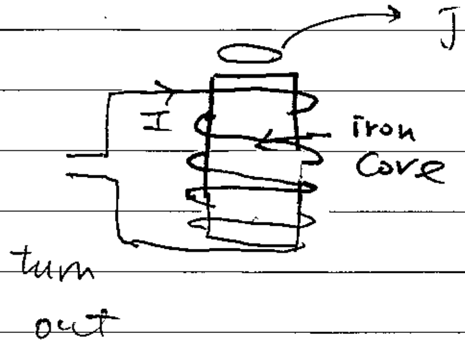
<u>Lenz's law</u> : direction of current flow

In the Faraday's law, it is not easy to keep track of the signs and ~~induced current.~~

Lenz discovered an easy rule to get the right direction. That is,

the induced current tries
to the change of the flux.

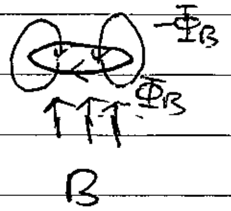
Example : Jumping ring



Before

$\Phi_B = 0$

After



$\rightarrow \Phi_B + (-\Phi_B) = 0$

to keep $\Phi = 0$

The induced Electric field

* In terms of vector potential

Faraday's law generalizes the

electrostatic $\vec{\nabla}\times\vec{E}=0$

to $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$ . ... (17)

$\because$ $\vec{B}$ still satisfies $\vec{\nabla}\cdot\vec{B}=0$

$\therefore$ $\vec{B}=\vec{\nabla}\times\vec{A}$ is still correct even if

$\vec{B}$ depends on time. i.e. $\vec{B}(\vec{r},t)=\vec{\nabla}\times\underline{\vec{A}(\vec{r},t)}$

$\therefore$ $\vec{\nabla}\times\vec{E}=-\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A}=\vec{\nabla}\times\left(-\frac{\partial}{\partial t}\vec{A}\right)$

Therefore, the generalization becomes

$$\vec{\nabla}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=0$$

Hence $\vec{E}+\frac{\partial\vec{A}}{\partial t}=\nabla(\cdots)\equiv-\nabla V$, V= scalar potential

$\therefore$ $\vec{E}=-\nabla V-\frac{\partial\vec{A}}{\partial t}$ ... (18)

which generalizes the relation $\underline{\vec{E}=-\nabla V}$.

Since $\oint_{\partial S}\vec{\nabla}V\cdot d\vec{\ell}=\int_S\vec{\nabla}\times(\nabla V)\cdot d\vec{a}=0$,

 S.

∂S

$$\oint_{\partial S} \vec{E}\cdot d\vec{\ell} = \oint_{\partial S} \left(-\nabla V - \frac{\partial \vec{A}}{\partial t}\right)\cdot d\vec{\ell}$$

$$= \oint_{\partial S} -\frac{\partial \vec{A}}{\partial t}\cdot d\vec{\ell}$$

$$= -\frac{d}{dt}\oint_{\partial S} \vec{A}\cdot d\vec{\ell} = -\frac{d}{dt}\Phi$$

Hence one sees that it is the participation of $\vec{A}$ in $\vec{E}$ so that $\oint \vec{E}\cdot d\vec{\ell} \neq 0$. On one hand, $\nabla\times\vec{A}$ produces $\vec{B}$. On the other hand, $-\frac{\partial \vec{A}}{\partial t}$ gives rise to the induced electric field.

One can also compare the case $\rho = 0$, $J \neq 0$

$$\vec{\nabla}\cdot\vec{E} = 0 \quad \nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \nabla\times\vec{B} = \mu_0\vec{J}$$

$-\frac{\partial \vec{B}}{\partial t}$ plays a similar role to $\vec{J}$.

$$\therefore \quad \vec{E} = \frac{1}{4\pi}\int \frac{-\frac{\partial \vec{B}(r')}{\partial t}\times\hat{\imath}}{\imath^2}\, d\tau'$$

$$= -\frac{d}{dt}\,\frac{1}{4\pi}\int \frac{\vec{B}'\times\hat{\imath}}{\imath^2}\, d\tau'$$

$$\therefore \oint \vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\oint d\vec{r}\cdot\int \frac{\vec{B}(\vec{r}',t)\times(\vec{r}-\vec{r}')}{4\pi|\vec{r}-\vec{r}'|^3}\,d\tau' \quad \text{--- (19)}$$

Now, in the case of magnetostatics,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,d\tau' \quad \text{(Biot-Savart)}$$

$$\oint \vec{B}\,d\vec{r} = \mu_0 I = \mu_0\int \vec{J}\cdot d\vec{a} \quad \text{--- (20)}$$

$$\therefore \oint d\vec{r}\cdot\int \frac{\vec{B}(\vec{r},t)\times(\vec{r}-\vec{r}')}{4\pi|\vec{r}-\vec{r}'|^3}\,d\tau' = \int \vec{B}\cdot d\vec{a} = \Phi$$

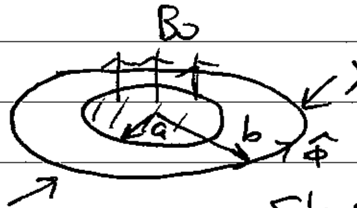
(20) implies

$\therefore$ Eq. (19) implies $\oint \vec{E}\cdot d\vec{\ell} = -\frac{d\Phi}{dt}$

$\therefore$ We see $-\frac{\partial \vec{B}}{\partial t}$ plays a role similar to $\vec{J}$.

Example

B_0 is turned off



suspended

How does the ring rotate?

Final angular momentum?

Solution: Lenz's law: $\circlearrowright E$

Symmetry implies $\vec{E} = E\hat{\phi}$

$$\therefore \oint \vec{E}\cdot d\vec{\ell} = 2\pi b E = -\frac{d\Phi}{dt} = -\pi a^2\frac{dB}{dt}$$

$$\therefore \vec{E} = -\frac{a^2}{2b}\frac{dB}{dt}\hat{\phi}$$

For a segment $d\vec{r}$, it experiences

a torque $d\vec{\tau} = \vec{r} \times [\underbrace{\lambda d\vec{r}\, E}_{d\vec{F}}]$

$\therefore d\tau = \lambda b E\, dl = -\lambda b \cdot \frac{a^2}{2b} \frac{dB}{dt}\, dl$

$\therefore$ total torque

$$= -\lambda b \frac{a^2}{2b} \frac{dB}{dt} \oint dl = -\lambda b \pi a^2 \frac{dB}{dt}$$

$\because$ torque $= \frac{dL}{dt}$

$$\therefore L = \int -\lambda b \pi a^2 \frac{dB}{dt}\, dt = \lambda \pi a^2 \pi B_0$$

The ring gets angular momentum!

Where does it come from?

(We shall discuss it later.)

Quasi-static

In principle, when $\frac{d\Phi}{dt} \neq 0$, $\vec{B}$ is time dependent. In this case, if $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ is still correct. $\vec{J}$ must be time-dependent.

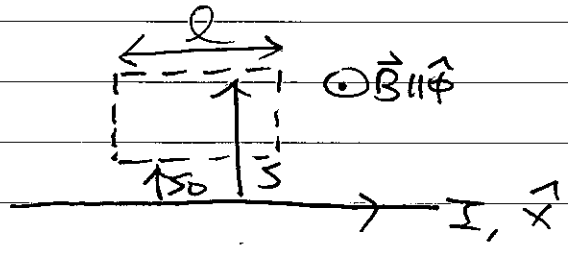
Hence $\frac{\partial \vec{J}}{\partial t} \neq 0$. We are no longer in the steady state!

However, if the change is not extremely rapid, the relation $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ (Ampere's law) & Biot-Savart law are approximately correct with negligible errors.

In this case, we can still apply the Ampere's law & Biot-Savart law.

The regime where magnetostatics is applicable while $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ & $\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0$ are applicable is called quasi-static.

Example. Slow varying current $I(t)$



Find $\underbrace{E(s)}_{\text{the induced}}$

Solution: quasi-static. $\vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$

$\vec{E} \parallel \hat{x}$. Consider a loop shown in the above

$$\oint \vec{E}\cdot d\vec{l} = E(s_0)l - E(s)l = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a}$$

$$= -\frac{\mu_0}{2\pi}\frac{d}{dt} I \int_{s_0}^{s} \frac{1}{s'}ds' \times l$$

$$= -\frac{\mu_0 l}{2\pi}\frac{dI}{dt}\left(\ln s - \ln s_0\right)$$

$$\therefore\ \vec{E}(s) = \Big(\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln s + \underbrace{E(s_0) - \frac{\mu_0}{2\pi}\frac{dI}{dt}\ln s_0}_{K}\Big)\hat{z}$$

K is an s-independent constant but K depends on t.

K depends on the whole history of I(t) as it is reflected that E(s₀) is not known. That is, to determine E(s), one has to know how I(t) is turned on and changes from the beginning. We shall discuss it later in Chapter 10.

Note that it may seem that $E \to \infty$ as $s \to \infty$. However, as we shall see later, E field can not be propagated faster than the speed of light. Hence the change of I.

Can not propagate faster than c.

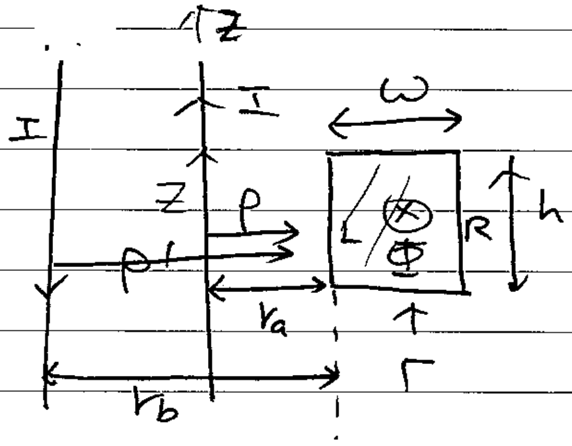
For a given period of Δt after the change of I, ~~for~~

$s > \Delta t\, c$, the point at s does not know that I changes.

Hence, the quasi-static is correct (i.e. point s knows the change of I) only when $s \ll c\tau$ --- (21)

where τ is the characteristic time scale that I changes substantially!

<u>Example</u> I: slow varying, $\frac{dI}{dt}$



Find emf on the loop Γ

$$\mathcal{E} = \oint \vec{E}\cdot d\vec{l}$$

Solution: $\vec{A} = \frac{\mu_0}{4\pi}\int \frac{I\,dz\,\hat{z}}{|\vec{r}-\vec{r}'|} \Leftrightarrow V = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda\, dz'}{|\vec{r}-\vec{r}'|} = \frac{-\lambda}{2\pi\epsilon_0}\ln r + \text{Const}.$

$\therefore \vec{A} = A\,\hat{z}$, $A = \frac{-\mu_0 I}{2\pi}\ln\rho + \text{const}$ for each current line.

$\therefore A = \frac{\mu_0 I}{2\pi} \ln \frac{\rho'}{\rho}$

$\therefore$ On the L side of loop Γ,

$$\vec{A}_L = \frac{\mu_0 I}{2\pi} \ln \frac{r_b}{r_a} \hat{z}$$

On the R side of loop Γ,

$$\vec{A}_R = \frac{\mu_0 I}{2\pi} \ln \frac{r_b+w}{r_a+w} \hat{z}$$

Using $\vec{E} = -\frac{d\vec{A}}{dt}$, one gets

$$\vec{E}_L = -\frac{\mu_0}{2\pi} \frac{dI}{dt} \ln \frac{r_b}{r_a} \hat{z}$$

$$\vec{E}_R = -\frac{\mu_0}{2\pi} \frac{dI}{dt} \ln \left(\frac{r_b+w}{r_a+w} \right) \hat{z}$$

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{\mu_0}{2\pi} \frac{dI}{dt} \ln \frac{r_b}{r_a} h + \frac{\mu_0}{2\pi} \frac{dI}{dt} \ln \frac{r_b+w}{r_a+w} h$$

$$= -\frac{\mu_0 h}{2\pi} \frac{dI}{dt} \ln \left[\frac{r_b}{r_a} \frac{r_a+w}{r_b+w} \right]$$

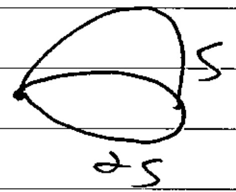
$$= -\frac{d\Phi}{dt}$$

$$\Phi = \frac{\mu_0 I}{2\pi} \left(\int_{r_a}^{r_a+w} \frac{h\,d\rho}{\rho} - \int_{r_b}^{r_b+w} \frac{h\,d\rho}{\rho} \right)$$

$$= \frac{\mu_0 h I}{2\pi} \ln \frac{r_b(r_a+w)}{r_a(r_b+w)}$$

Induce electric field on a moving system.

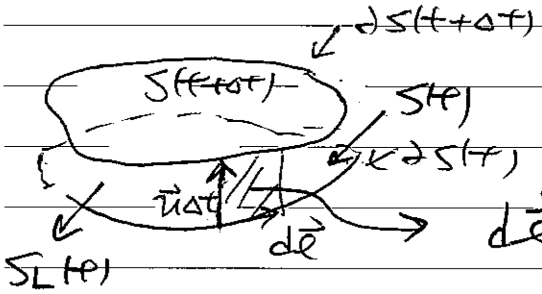
The Faraday law (eq. (15)) is valid when the loop ∂S is fixed, and is not moving:



$$\oint_{\partial S} \vec{E}\cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

$\llcorner$ -- (15)

When the loop ∂S is moving, eq. (15) has to be modified.



$$d\vec{\ell}\times(\vec{u}\Delta t) = d\vec{a}_L$$

First, $$\frac{d\Phi}{dt} = \frac{1}{\Delta t}\left[\int_{S(t+\Delta t)} \vec{B}(t+\Delta t)\cdot d\vec{a} - \int_{S(t)} \vec{B}(t)\cdot d\vec{a}\right]$$

$\llcorner$ - (22)

$\because$ $S(t+\Delta t) + S(t) + S_L(t)$ forms a close surface.

$$\therefore \int_{S(t+\Delta t)} \vec{B}(t+\Delta t)\cdot d\vec{a} - \int_{S(t)} \vec{B}(t+\Delta t)\, d\vec{a} + \int_{S_L(t)} \vec{B}(t+\Delta t)\cdot d\vec{a}_L$$

$$= \int_V \vec{\nabla}\cdot\vec{B}(t+\Delta t)\, d\tau = 0$$

$$\therefore \int_{S(t+\Delta t)} \vec{B}(t+\Delta t)\cdot d\vec{a} - \int_{S(t)} \vec{B}(t+\Delta t)\cdot d\vec{a} + \Delta t \oint_{\partial S(t)} \vec{B}(t+\Delta t)\cdot d\vec{\ell}\times\vec{u} = 0$$

— (23)

Now, $\oint_{\partial S(t)} \vec{B}(t+\Delta t)\cdot d\vec{\ell}\times\vec{u} = \oint_{\partial S(t)} \vec{u}\times\vec{B}(t+\Delta t)\cdot d\vec{\ell}$

$$= \int_{S(t)} \nabla\times(\vec{u}\times\vec{B}(t+\Delta t))\cdot d\vec{a}$$

$$= \int_{S(t)} \nabla\times(\vec{u}\times\vec{B}(t))\cdot d\vec{a} + O(\Delta t)$$

$$\vec{B}(t+\Delta t) = \vec{B}(t) + \Delta t\,\frac{\partial\vec{B}(t)}{\partial t}$$

Eq. (23) becomes

$$\int_{S(t+\Delta t)} \vec{B}(t+\Delta t)\cdot d\vec{a} - \int_{S(t)} \vec{B}(t)\cdot d\vec{a}$$

$$= \Delta t\left[\int_{S(t)} \frac{\partial\vec{B}(t)}{\partial t}\cdot d\vec{a} - \int_{S(t)} \nabla\times(\vec{u}\times\vec{B}(t))\cdot d\vec{a}\right]$$

$\therefore$ Eqs. (22) & (23) imply

$$\frac{d\Phi}{dt} = \int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{a} - \int_S \nabla\times(\vec{u}\times\vec{B}(t))\, d\vec{a}$$

— (24)

Now, Φ is the flux that goes with the loop

- Faraday's law implies that

$-\frac{d\Phi}{dt} = \oint_{\partial S} \vec{E}\cdot d\vec{\ell}$, $\vec{E}$ measured in the frame that moves with $\vec{u}$ --- (25)

Eq. (25) $\Rightarrow$ $\frac{d\Phi}{dt} = -\int_S \vec{\nabla}\times\vec{E}\cdot d\vec{a}$

$\therefore$ Combined with eq. (26), one gets

$$\int_S \vec{\nabla}\times\vec{E}\cdot d\vec{a} = \int_S \left[\vec{\nabla}\times(\vec{u}\times\vec{B}) - \frac{\partial B}{\partial t}\right]\cdot d\vec{a}$$

taking $S\to 0$. one gets

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} + \vec{\nabla}\times(\vec{u}\times\vec{B}) \quad \cdots (26)$$

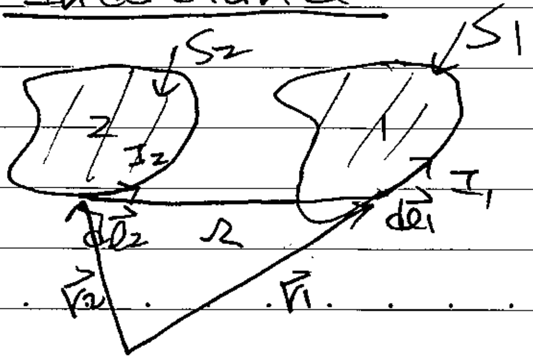
where $\vec{E}$ is measured in the frame that moves with $\vec{u}$ relative to the frame where $\vec{B}$ is measured.

Example. moving conductor :

$\vec{B}$ measured in lab frame

$\vec{E}$ measured in conductor frame

<u>Inductance</u>

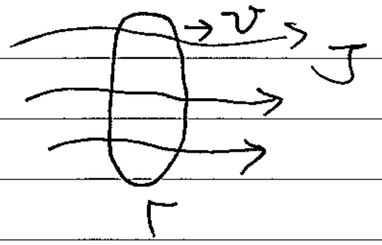


Induced electric field can be induced by other current loops.

Consider two current loops 1 & 2.

Example: Alfven's theorem

In a perfect conducting fluid, the magnetic flux through any closed loop moving with the fluid is constant.



Solution: $\because \vec{J}_f = \sigma(\vec{E} + \vec{v}\times\vec{B})$

For perfect conducting fluids, $\sigma \to \infty$

$$\therefore \vec{E} + \vec{v}\times\vec{B} = 0, \quad \vec{E} = -\vec{v}\times\vec{B}$$

$$\therefore \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \text{ becomes}$$

$$\frac{\partial\vec{B}}{\partial t} - \vec{\nabla}\times(\vec{v}\times\vec{B}) = 0$$

Eq. 24. $\vec{u}$ replaced by $\vec{v}$

$$\therefore \frac{d\Phi}{dt} = 0 \quad \therefore \Phi = \text{constant}.$$

There is no induced electric field if one moves with the ring. As we shall see later, in a moving frame, $\vec{B}$ will generate an electric field $\vec{v}\times\vec{B}$ that cancels $\vec{E} = -\vec{v}\times\vec{B}$.

The flux of loop 2 due to current in loop 1 is

$$\Phi_{21} = \int_{S_2} \vec{B_2} \cdot d\vec{a_2}$$

$\vec{B_2} = \vec{\nabla} \times \vec{A_2}$ is the magnetic field due to loop 1,

$$\vec{A_2} = \frac{\mu_0 I_1}{4\pi} \oint_1 \frac{d\vec{l_1}}{r}$$

$$\therefore \Phi_{21} = \int_{S_2} \vec{\nabla} \times \vec{A_2} \cdot d\vec{a_2}$$

$$= \oint_2 \vec{A_2} \cdot d\vec{l_2}$$

$$= \frac{\mu_0 I_1}{4\pi} \oint_1 \frac{d\vec{l_1}}{r} \cdot \oint \cdot d\vec{l_2}$$

$$= \left(\frac{\mu_0}{4\pi} \oint_1 \oint_2 \frac{d\vec{l_1} \cdot d\vec{l_2}}{|\vec{r_1} - \vec{r_2}|} \right) I_1 \equiv M_{21} I_1$$

Similarly, $\Phi_{12} = \left(\frac{\mu_0}{4\pi} \oint_2 \oint_1 \frac{d\vec{l_2} \cdot d\vec{l_1}}{|\vec{r_1} - \vec{r_2}|} \right) I_2 \equiv M_{12} I_2$

Clearly, $M_{12} = M_{21} = \frac{\mu_0}{4\pi} \oint_1 \oint_2 \frac{d\vec{l_1} \cdot d\vec{l_2}}{|\vec{r_1} - \vec{r_2}|} \equiv M$ -- (27)

= mutual inductance between two circuit

Eq. (27) is the Neumann equation.

Under quasi-static condition, the change of I_1 will induce emf in loop 2.

$$\mathcal{E}_2 = -\frac{d\Phi_{21}}{dt} = -M\frac{dI_1}{dt}$$

$$\mathcal{E}_1 = -\frac{d\Phi_{12}}{dt} = -M\frac{dI_2}{dt}$$

$\therefore$ M characterizes emf induced between loops 1 & 2.

Self-inductance

Similarly, change of current in one loop also induces change of flux in the same loop. $\therefore$ One expects $\Phi = LI$ -- (28)

$$\mathcal{E} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt} \quad \text{-- (29)}$$

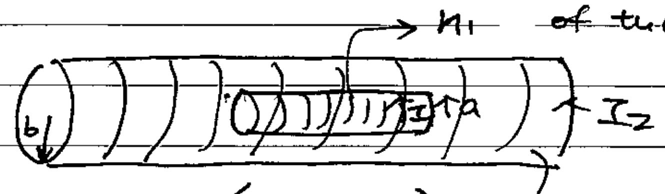
L is called self inductance.

For MKS system, $\mathcal{E}$ = Volt, dI = Ampere

$\therefore$ [L] = Volt · sec / Ampere = Henry

Both M & L are intrinsically positive and are characterized only by geometry of circuits.

Example: Mutual inductance between two coaxial solenoids



Find the mutual inductance

Solution: $B_2 = \mu_0 n_2 I_2$

$$\Phi_{12} = B_2 \cdot \pi a^2 \times n_1 l$$

$$= \mu_0 \pi a^2 n_1 n_2 l \, I_2$$

$$\therefore \; M_{12} = \mu_0 \pi a^2 n_1 n_2 l$$

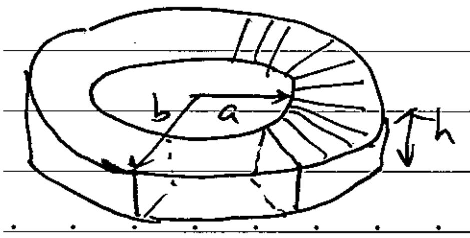
$$B_1 = \mu_0 n_1 I_1$$

$$\Phi_{21} = B_1 \cdot \underline{\underline{\pi a^2}} \cdot n_2 l$$

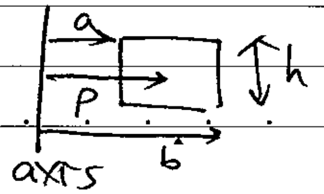
$$= \mu_0 \pi a^2 n_1 n_2 l \, I_1$$

$$\therefore \; M_{21} = \mu_0 \pi a^2 n_1 n_2 l = M_{12}$$

Example Self-inductance of a toroidal coil



From the previous example



$$B = \frac{\mu_0 N I}{2\pi P}$$

N = total # of coil

$$\therefore \Phi = N \times \int \vec{B} \cdot d\vec{a} = \frac{\mu_0 N^2 I}{2\pi} h \int_a^b \frac{d\rho}{\rho}$$

$$= \frac{\mu_0 N^2 I h}{2\pi} \ln b/a = LI$$

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln b/a$$

Energy in magnetic fields

It takes a certain among of energy to set a current on a circuit.

We shall assume that the circuit can be described by ohmic law,

∴ The current is $\vec{J}_f$ and obeys

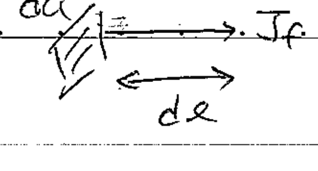
$$\vec{J}_f = \sigma \vec{E}$$

With $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$ --- (30)

Now, $-\frac{\partial \vec{A}}{\partial t}$ is the induced field. To maintain the current, the source (battery) maintains a difference in V

(distance moved on a unit time)

∴ Across $d\vec{l}$, the work done per unit time



$$dw = -\nabla V \cdot d\vec{l} \, \underbrace{J_f \, da}_{I_f}$$

$\because$ $J_f \, d\vec{l} = \vec{J_f} \, dl$

$da \, dl = d\tau$

$\therefore$ total work per unit time done

$$\frac{dW}{dt} = \int -\vec{\nabla} V \cdot J_f \, d\tau$$

$$= \int \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) \cdot \vec{J_f} \, d\tau$$

eq. (30)

$$= \int \frac{J_f^2}{\sigma} d\tau + \int \frac{\partial \vec{A}}{\partial t} \cdot \vec{J_f} \, d\tau$$

... (31)

The first term is the power spent in ohmic or Joule heat.

The 2nd term is the work done by the source against the induced emf, and generates the magnetic field.

$\therefore$ we shall set the work for magnetic field W_m:

$$\frac{dW_m}{dt} = \int \frac{\partial \vec{A}}{\partial t} \cdot \vec{J_f} \, d\tau \quad \cdots (32)$$

In the absence of magnetic materials,

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}_f$$

$$\therefore \frac{dW_m}{dt} = \frac{1}{\mu_0}\int \frac{\partial \vec{A}}{\partial t}\cdot \vec{\nabla}\times\vec{B}\, d\tau$$

Using $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot\vec{\nabla}\times\vec{A} - \vec{A}\cdot\vec{\nabla}\times\vec{B}$,

One gets

$$\frac{dW_m}{dt} = \frac{1}{\mu_0}\int\left[\vec{B}\cdot\nabla\times\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\cdot\left(\frac{\partial\vec{A}}{\partial t}\times\vec{B}\right)\right]d\tau$$

$$= \frac{1}{\mu_0}\int_V \vec{B}\cdot\frac{\partial\vec{B}}{\partial t}d\tau + \oint_S \frac{1}{\mu_0}\vec{B}\times\frac{\partial\vec{A}}{\partial t}\cdot d\vec{a}$$

When $S\to\infty$ $(r\to\infty)$, $\because$ $B\sim\frac{1}{r^3}$, $\frac{\partial A}{\partial t}\sim\frac{1}{r^2}$,
$da\sim r^2$

$$\therefore \oint_S \frac{1}{\mu_0}\vec{B}\times\frac{\partial\vec{A}}{\partial t}\cdot d\vec{a} \to 0$$

$$\therefore \frac{dW_m}{dt} = \frac{1}{\mu_0}\int \vec{B}\cdot\frac{\partial\vec{B}}{\partial t}d\tau$$

$$= \frac{1}{\mu_0}\int\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}\cdot\vec{B}\right)d\tau$$

$$= \frac{d}{dt}\int\frac{1}{2\mu_0}B^2 d\tau \quad \cdots (33)$$

By setting $W_m = 0$ when $B=0$

$$\therefore W_m = \frac{1}{2\mu_0}\int B^2 d\tau \cdots (34)$$

Eq. (34) has a similar form to the electrostatic energy $\int \frac{1}{2}\varepsilon_0 E^2 d\tau$.

When there are magnetic materials,

$$\vec{J_f} = \vec{\nabla}\times\vec{H}$$

One gets $$\frac{dW_m}{dt} = \int \frac{\partial \vec{A}}{\partial t}\cdot\vec{\nabla}\times\vec{H}\, d\tau$$

$$= \int \vec{H}\cdot\vec{\nabla}\times\frac{\partial \vec{A}}{\partial t}\, d\tau$$

$$= \int \vec{H}\cdot\frac{\partial}{\partial t}\vec{B}\, d\tau \quad \text{--- (35)}$$

Hence $$\Delta W_m = \int \vec{H}\cdot\Delta\vec{B}\, d\tau \quad \text{--- (36)}$$

$$W_m = \int d\tau \int_0^B \vec{H}\cdot\Delta\vec{B} \quad \text{--- (36)-1}$$

For linear media, $\vec{H} = \frac{1}{\mu}\vec{B}$,

One gets $$\frac{dW_m}{dt} = \int \frac{1}{\mu}\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}\, d\tau$$

$$= \frac{\partial}{\partial t}\int \frac{1}{2\mu} B^2\, d\tau$$

$$\therefore\ W_m = \int \frac{1}{2\mu(\vec{r})}\vec{B}^2(\vec{r},t)\, d\tau \quad \text{--- (37)}$$

which has a similar form to

$$W_E = \int \frac{1}{2}\epsilon E^2\, d\tau$$

$\therefore$ Eq. (37) includes the energy that generates $\vec{M}$!

Other forms

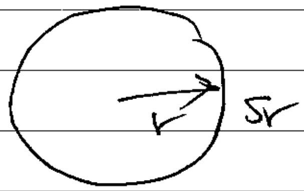
Eq. (34) can be converted into other forms.

Using $\vec{B} = \nabla \times \vec{A}$, one has

$$\vec{B}^2 = \vec{B} \cdot \nabla \times \vec{A} = \nabla \cdot (\vec{A} \times \vec{B}) + \vec{A} \cdot \nabla \times \vec{B}$$

$$\therefore W_m = \frac{1}{2\mu_0} \int \left[\nabla \cdot (\vec{A} \times \vec{B}) + \vec{A} \cdot \nabla \times \vec{B} \right] d\tau$$

$$\int \nabla \cdot (\vec{A} \times \vec{B})\, d\tau = \int_{S_r} \vec{A} \times \vec{B} \cdot d\vec{a} \to 0 \quad \text{as } r \to \infty$$

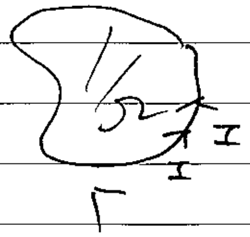


$$\therefore W_m = \frac{1}{2\mu_0} \int \vec{A} \cdot \nabla \times \vec{B}\, d\tau$$

$$= \frac{1}{2} \int \vec{A} \cdot \vec{J}_f\, d\tau \qquad (38)$$

When the current is in a circuit Γ, one gets

$$W_m = \frac{1}{2} \oint_\Gamma \vec{A} \cdot d\vec{l}\; I$$



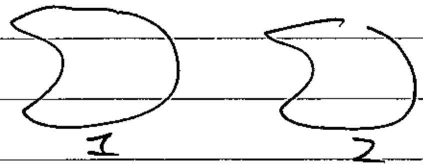
Now $\oint \vec{A} \cdot d\vec{l} = \int_\Omega \nabla \times \vec{A} \cdot d\vec{a}$

$$= \Phi_B = L I$$

L = self inductance of loop Γ.

$$\therefore W_m = \frac{1}{2} L I^2, \quad \frac{dW_m}{dt} = L I \frac{dI}{dt} = -\mathcal{E} I \qquad (39)$$

In general, if there are more than two circuits, say, Γ_1 & Γ_2,



$$W_m = \sum_n \frac{1}{2} \oint_{\Gamma_n} \vec{A} \cdot d\vec{\ell} \, I_n$$

$$\oint_{\Gamma_n} \vec{A} \cdot d\vec{\ell} = \Phi_n = \sum_m \Phi_{nm} = L_n I_n + \sum_{m \neq n} M_{nm} I_m$$

$$\therefore W_m = \sum_n \frac{1}{2} L_n I_n^2 + \sum_{\substack{n,m \\ n \neq m}} \frac{1}{2} I_n M_{nm} I_m \quad \cdots (40)$$

where L_n is the self inductance of loop n.

and M_{nm} is the mutual inductance between loop n & m.

For two loops, one gets

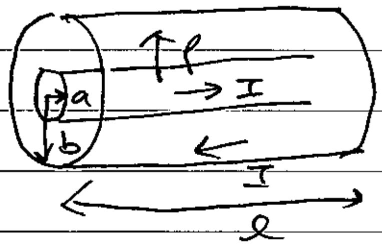
$$\oint_{\Gamma_1} \vec{A} \cdot d\vec{\ell} = L_1 I_1 + M I_2$$

$$\oint_{\Gamma_2} \vec{A} \cdot d\vec{\ell} = L_2 I_2 + M I_1$$

$$\therefore W_m = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + \frac{1}{2} I_1 M I_2 + \frac{1}{2} I_2 M I_1$$

$$= \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2$$

Example. A long coaxial cable carries current I as shown in the left figure. Find the self inductance.



Solution: L can be found by calculating the energy $W_m = \frac{1}{2} L I^2$

According to Ampere's law,

$$\vec{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\phi} \quad \text{for } a < \rho < b$$

$$\therefore \frac{1}{2\mu_0}B^2 = \frac{\mu_0}{8\pi^2}\frac{I^2}{\rho^2}$$

$$\because d\tau = 2\pi\rho\, d\rho \cdot \ell$$

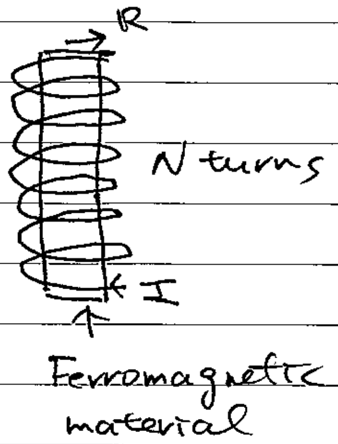
$$\therefore W_m = \int \frac{1}{2\mu_0} B^2 d\tau$$

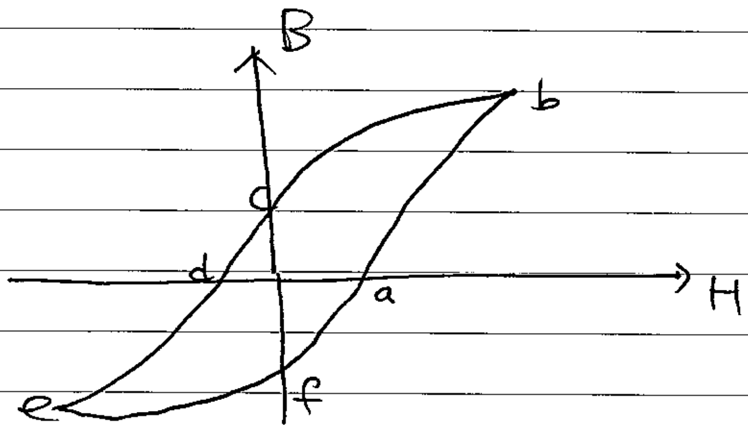
$$= 2\pi\ell \cdot \frac{\mu_0}{8\pi^2} I^2 \int_a^b \frac{\rho\, d\rho}{\rho^2}$$

$$= \frac{\mu_0 I^2}{4\pi}\ell \ln b/a = \frac{1}{2} L I^2$$

$$\therefore L = \frac{\mu_0 \ell}{2\pi} \ln b/a$$

Example. Energy dissipation in a hysteresis cycle





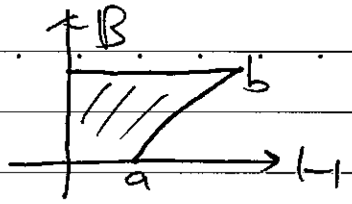
Solution: By increasing I, $H \uparrow$, one polarizes M inside the coil. One needs to do work against the opposed emf $\left(-N\frac{d\Phi}{dt}\right)$ due to Lenz's law

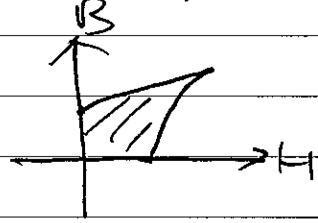
$$\therefore \frac{dW}{dt} = I N \frac{d\Phi}{dt}, \quad \Phi = \pi R^2 B$$

$$\therefore \frac{dW}{dt} = \underbrace{\frac{IN}{l}}_{H=nI} \underbrace{\pi R^2 l}_{V} \frac{dB}{dt}$$

$$\therefore \frac{dW}{dt} = V H \frac{dB}{dt}.$$

$W = V \int H\,dB$ in consistent with eq. 36

$\therefore W_{a\to b} = U \cdot$ Shaded area 

$W_{b\to c} = U \cdot$ area of 

$\therefore$ total work $= U \cdot \oint H\, dB$

$= U \cdot$ area of 

$\therefore$ For each cycle, one needs to do the work in proportional to the area of hysteresis loop.

Maxwell's equations

Inconsistency of Ampere's law & other laws

Up to now, we have obtained 4 equations for $\vec{E}$ & $\vec{B}$:

(i) $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ (Gauss's law)

(ii) $\vec{\nabla}\cdot\vec{B} = 0$

(iii) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's law)

(iv.) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ (Ampere's law).

(ii) & (iii) are consistent as one can check

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) = -\vec{\nabla}\cdot\frac{\partial\vec{B}}{\partial t}$$

$$\underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E})}_{=0} = -\frac{\partial}{\partial t}\underbrace{\vec{\nabla}\cdot\vec{B}}_{\overset{(ii)}{=}0} = 0$$

$\therefore$ $0=0$ (ii) & (iii) are consistent.

However, $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ is only valid

When $\vec{\nabla}\cdot\vec{J}=0$: $\underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})}_{=0} = \mu_0\vec{\nabla}\cdot\vec{J}$ ... (41)

$\therefore$ $\vec{\nabla}\cdot\vec{J}$ must vanish.

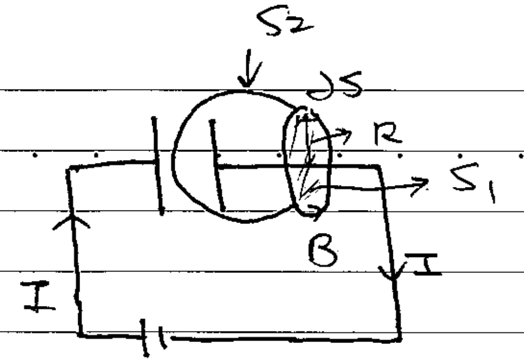
From the conservation of charge,

$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$ which does not

generally vanish!

Hence Ampere's law is not consistent with the continuity equation.

Another way to see that the Ampere's law is not complete is to consider the charging of a capacitor as shown in the below.



Applying the Ampere's law to ∂S, one gets

$$\oint_{\partial S} \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enclosed} = \mu_0 I \quad \cdots (42)$$

However, using the Stoke's theorem, one can rewrite

$$\oint_{\partial S} \vec{B}\cdot d\vec{\ell} = \int_{S_1} \vec{B}\cdot d\vec{\ell} \quad \cdots (43)$$

where ∂S is the boundary of S_1.

There are many ways to choose S_1 as long as it is bounded by ∂S.

$\therefore$ S_2 is equally correct:

$$\therefore \oint_{\partial S} \vec{B}\cdot d\vec{\ell} = \int_{S_2} \vec{\nabla}\times\vec{B}\cdot d\vec{a}$$

$$\underset{(iv)}{=} \int_{S_2} \mu_0 \vec{J}\cdot d\vec{a}$$

but $\vec{J}=0$ on S_2 ! One gets a conflict result $\oint_{\partial S} \vec{B}\cdot d\vec{\ell} = 0$!

Fixed Ampere's law.

Maxwell fixed the conflict by noting that

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial t}(\varepsilon_0\vec{\nabla}\cdot\vec{E})$$

$$= -\vec{\nabla}\cdot\varepsilon_0\frac{\partial\vec{E}}{\partial t}.$$

Hence, if one adds $\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ to Right hand side of (iv), eq. (41) becomes

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0\vec{\nabla}\cdot\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

both LHS & RHS vanish now!

$$\therefore \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \quad \cdots (44)$$

can resolve the conflict shown in eq. (41)

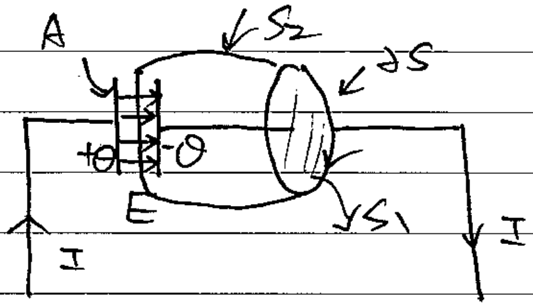
$\therefore$ Changing electric field contributes a current and induces magnetic field!

The term $\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ is called the displacement current

$$\vec{J}_D = \varepsilon_0\frac{\partial\vec{E}}{\partial t} \quad \cdots \quad (45)$$

The inclusion of $\vec{J}_D$ also resolves the conflict.

of charging a capacitor:



$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

$$\therefore \epsilon_0 \frac{dE}{dt} = \frac{1}{A}\frac{dQ}{dt} = \frac{I}{A}$$

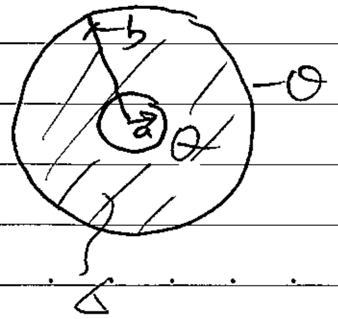
$$\therefore \oint_{\partial S} \vec{B}\cdot d\vec{l} = \oint_{S_2} \vec{\nabla}\times\vec{B}\cdot d\vec{a}$$

$$= \oint \mu_0 \underbrace{\epsilon_0 \frac{d\vec{E}}{dt}}_{I/A} \cdot d\vec{a}$$

$$= \mu_0 I/\cancel{A} \times \cancel{A}$$

Which is now consistent with the calculation of S_1.

Example — Find $\vec{J}$ & $\vec{B}$ in $a<r<b$.



$$\vec{J} = \sigma\vec{E} = \sigma \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

$$I = \int \vec{J}\cdot da = \frac{\sigma Q}{\epsilon_0} \quad (46)$$

$$I = -\dot{Q}$$

By symmetry and $\vec{\nabla}\cdot\vec{B}=0$, $\oint \vec{B}\cdot d\vec{a}=0$

$$\therefore B\cdot 4\pi r^2 = 0 \qquad B=0$$

Why I doesn't generate B ?

Solution : $\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{1}{4\pi r^2}\frac{dQ}{dt}\hat{r}$

$$= \frac{1}{4\pi r^2}\left(-\frac{\sigma Q}{\epsilon_0}\right)\hat{r}$$

eq. (6)

$$\therefore \vec{J}+\vec{J}_d = 0 \Rightarrow \vec{\nabla}\times\vec{B}=0.$$

together with $\vec{\nabla}\cdot\vec{B}=0$

$$\therefore \vec{B}=0$$

Maxwell equations

After the important work by Maxwell, the equations that characterize $\vec{E}$ & $\vec{B}$ ~~in most general situations~~ are the

(i) $\vec{\nabla}\cdot\vec{E}=\rho/\epsilon_0$ Gauss's law

(ii) $\vec{\nabla}\cdot\vec{B}=0$

(iii) $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$ Faraday's law

(iv) $\vec{\nabla}\times\vec{B}=\mu_0(\vec{J}+\epsilon_0\frac{\partial\vec{E}}{\partial t})$ Ampere-Maxwell law

These equations are now complete and called Maxwell's equations.

Together with $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$,

they describe the entire classical electrodynamics.

Note that $\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$ is a consequence
the continuity equation
of eq.(iv) if one takes $\vec{\nabla}\cdot$(iv).

In the above form, it's tempting to conclude that $\vec{E}$ can be produced by either ρ or $-\frac{\partial\vec{B}}{\partial t}$ and $\vec{B}$ can be produced either by $\vec{J}$ or $\frac{\partial\vec{E}}{\partial t}$.

This, however, is quite correct as all $\vec{E}$ & $\vec{B}$ are produced by ρ & $\vec{J}$.

Hence it's more logical to write

(i) $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ (ii) $\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$

(iii) $\vec{\nabla}\cdot\vec{B} = 0$ (iv) $\vec{\nabla}\times\vec{B} - \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J}$ -- (47)

so that all sources are on the right hand side.

Magnetic monopole

The Maxwell equations are symmetric in $\vec{E}$ & $\vec{B}$ in the absence of matters:

$$\vec{\nabla}\cdot\vec{E}=0 \quad (+\rho/\varepsilon_0) \qquad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\times\vec{B}=\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \quad (+\mu_0\vec{J})$$

Change $(\vec{E},\vec{B})$ by $(\vec{B},-\varepsilon_0\mu_0\vec{E})$

exchange the first pair of equations into the second pair of equations.

This symmetry is spoiled by matters when ρ & $\vec{J}$ are in present!

The spoil of the symmetry between $\vec{E}$ & $\vec{B}$ is easily traced to be due to the absence of magnetic charges (monopoles).

If we had

(i) $\vec{\nabla}\cdot\vec{E} = \frac{\rho_e}{\epsilon_0}$ (iii) $\vec{\nabla}\times\vec{E} = -\underline{\mu_0 J_m} - \frac{\partial \vec{B}}{\partial t}$

(ii) $\vec{\nabla}\cdot\vec{B} = \underline{\mu_0 \rho_m}$ (iv) $\vec{\nabla}\times\vec{B} = \mu_0 J_e + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$

$\llcorner$ -- (48)

with ρ_m = density of magnetic charges
$\vec{J}_m$ = current of magnetic charges

& Both charges are conserved

$$\vec{\nabla}\cdot\vec{J}_m = -\frac{\partial \rho_m}{\partial t}, \quad \vec{\nabla}\cdot\vec{J}_e = -\frac{\partial \rho_e}{\partial t},$$

$\llcorner$ -- (49)

, then $(\vec{E}, \vec{B}) \rightarrow \left(\frac{\vec{B}}{\sqrt{\epsilon_0\mu_0}}, -\sqrt{\mu_0\epsilon_0}\,\vec{E}\right)$

$(\rho_e, \rho_m) \rightarrow \left(\sqrt{\epsilon_0\mu_0}\,\rho_m, -\frac{1}{\sqrt{\mu_0\epsilon_0}}\rho_e\right)$

$(\vec{J}_e, \vec{J}_m) \rightarrow \left(\sqrt{\epsilon_0\mu_0}\,\vec{J}_m, -\frac{1}{\sqrt{\mu_0\epsilon_0}}\vec{J}_e\right)$

Eqs (48) (49) are invariant.

In fact, there are more transformations that leave eq. (48) (49) invariant. (problem 7.64)

The Maxwell equations seem to be more complete if there are magnetic charges.
In spite of this, however, up to now, no one

has ever found any magnetic monopole.

Maxwell's equations in Matter

The Maxwell's equations, eq. (47), are complete. However, inside materials, it's more convenient to separate responses of materials in ρ & $\vec{J}$ and express everything in terms of ρ_f & $\vec{J}_f$.

For static case, we have

$$\rho_b = -\vec{\nabla}\cdot\vec{P} \quad \text{(bound charge)}$$

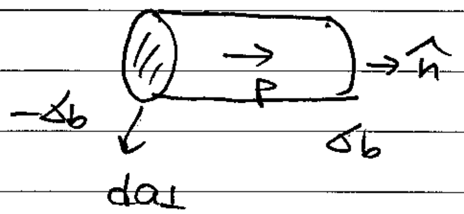
$$\vec{J}_b = \vec{\nabla}\times\vec{M} \quad \text{(bound current)}.$$

When we consider non-static situations, changes in ρ_b & $\vec{J}_b$ will result in a new quantity:

Polarization current $\vec{J}_p$

Change of $\vec{P}$ in time results in the polarization current $\vec{J}_p = \frac{d\vec{P}}{dt}$. --- (50)

This can be understood by considering a small volume with $\vec{P}$. Clearly, there are



$\pm\sigma_b$ at two ends

with $\sigma_b = \vec{P}\cdot\hat{n} = P$

When P increases, σ_b increases. Therefore, there are charges moving from the left end to the right end

The current $I = \frac{dQ}{dt} = \frac{d}{dt}(\sigma_b \cdot da_\perp)$

$$= \frac{dP}{dt} \cdot da_\perp$$

$$\therefore \vec{J}_p = \frac{d\vec{P}}{dt}$$

Microscopically, when $\vec{P}$ increases,

$+$ moves right, $-$ moves left, therefore there is a net move of charges to the right. This is the origin of $\vec{J}_p$.

Continuity equation: $\vec{\nabla}\cdot\vec{J}_p = \vec{\nabla}\cdot\frac{d\vec{P}}{dt}$

$$= \frac{d}{dt}\vec{\nabla}\cdot\vec{P} = -\frac{d}{dt}\rho_b$$

$$\therefore \frac{\partial \rho_b}{\partial t} + \vec{\nabla}\cdot\vec{J}_p = 0 \quad --- \quad (51)$$

With $\vec{J}_p$, the general decompositions of ρ & $\vec{J}$ are

$$\rho = \rho_f + \rho_b = \rho_f - \vec{\nabla}\cdot\vec{P} \quad -- \quad (52)$$

$$\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p$$

$$= \vec{J}_f + \vec{\nabla}\times\vec{M} + \frac{\partial \vec{P}}{\partial t} \quad -- \quad (53)$$

$\therefore$ Gauss's law

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} = \frac{\rho_f}{\epsilon_0} - \frac{1}{\epsilon_0}\vec{\nabla}\cdot\vec{P}$$

becomes $\vec{\nabla}\cdot\vec{D} = \rho_f$

with $\vec{D} \equiv \epsilon_0\vec{E} + \vec{P}$

While for the Ampere's law,

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$$

$$= \mu_0\left(\vec{J}_f + \vec{\nabla}\times\vec{M} + \frac{\partial \vec{P}}{\partial t}\right) + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J}_f + \frac{\partial}{\partial t}\left(\epsilon_0\vec{E} + \vec{P}\right)$$

$$\therefore \vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

with $\vec{H} \equiv \vec{B}/\mu_0 - \vec{M}$

Hence, in terms of ρ_f & $\vec{J}_f$, the Maxwell's equations become

(i) $\vec{\nabla}\cdot\vec{D} = \rho_f$ (ii) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(iii) $\vec{\nabla}\cdot\vec{B} = 0$ (iv) $\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$ . ... (54)

which have to be supplemented by the constitution relations:

$$\vec{D} = \epsilon\vec{E}, \quad \vec{H} = \vec{B}/\mu$$

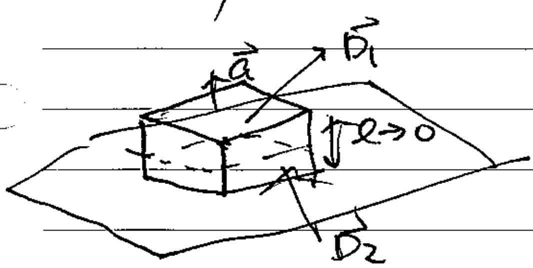
$$\vec{P} = \epsilon_0\chi_e\vec{E}, \quad \vec{M} = \chi_m\vec{H}$$

where $\epsilon = \epsilon_0(1+\chi_e)$, $\mu = \mu_0(1+\chi_m)$.

In this context, $\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$ is the displacement current.

<u>Boundary conditions</u>

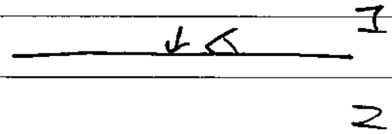
In the presence of surface charge density σ & surface current density $\vec{K}$, $\vec{E}, \vec{B}, \vec{D}$ & $\vec{H}$ may be discontinuous.



(i) using a pillar box near surface with $l \to 0$

$$\oint \vec{D}\cdot d\vec{a} = \vec{D}_1\cdot\vec{a} - \vec{D}_2\cdot\vec{a} = \sigma_f a$$

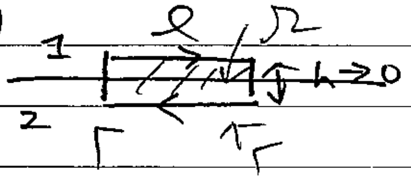
$\vec{D}\cdot\vec{a} = D^{\perp}a$



$\therefore \quad D_1^{\perp} - D_2^{\perp} = \sigma_f \quad \cdots (55)$

(ii) Similarly, $\because \vec{\nabla}\cdot\vec{B}=0$, using the same pillar box, one gets $B_1^{\perp} - B_2^{\perp} = 0 \quad \cdots (56)$

(iii)



Consider an Amperian loop shown in the left figure.

$h \to 0$

$$\oint \vec{E}\cdot d\vec{l} = \int_{\Omega} \nabla\times\vec{E}\cdot d\vec{a}$$

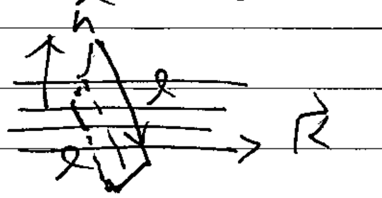
$$= \int_{\Omega} -\frac{\partial \vec{B}}{\partial t}\cdot d\vec{a} \longrightarrow 0 \text{ as } h\to 0$$

$\& \ \frac{\partial \vec{B}}{\partial t}$ is finite

But $\oint \vec{E}\cdot d\vec{l} = \vec{E}_1\cdot\vec{l} - \vec{E}_2\cdot\vec{l} \quad (h\to 0)$

$\therefore \quad E_1^{\parallel} - E_2^{\parallel} = 0 \quad \cdots (57)$

(iv) Using the same loop as (iii) but with $\vec{l} \perp \vec{K}$



$$\oint \vec{H}\cdot d\vec{l} = \int \vec{\nabla}\times\vec{H}\cdot d\vec{a}$$

$$= \int \left(\vec{J}_f + \frac{\partial \vec{D}}{\partial t}\right)\cdot d\vec{a}$$

$$= \vec{K}_f\cdot \hat{n}\times\vec{l}$$

$$\vec{K}_f \times \hat{n} \cdot \vec{l}$$
$$\shortparallel$$

$$\therefore \vec{H}_1 \cdot \vec{l} - \vec{H}_2 \cdot \vec{l} = \vec{K}_f \cdot (\hat{n} \times \vec{l}) \quad \text{for any } \vec{l}$$

$$\therefore \vec{H}_1^{\parallel} - \vec{H}_2^{\parallel} = \vec{K}_f \times \hat{n} \quad \text{-- (58)}$$

Eqs. (55) - (58) are the general boundary conditions for electrodynamics with materials.

For linear media, one can express them in terms of $\vec{E}$ & $\vec{B}$:

(i) $\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f$

(ii) $\vec{E}_1^{\parallel} - \vec{E}_2^{\parallel} = 0$

(iii) $B_\perp^1 - B_\perp^2 = 0$

(iv) $\frac{1}{\mu_1} \vec{B}_1^{\parallel} - \frac{1}{\mu_2} \vec{B}_2^{\parallel} = \vec{K}_f \times \hat{n}$

These equations will be the basis for the theory of electrodynamics in the materials.

Quasi-static approximations & AC circuit theory

Slowly time-varying charges & quasi-electrostatic approximation

In the following, we shall use l to characterize the typical size of the system considered $\therefore \nabla \sim 1/l$, and use ω to characterize the typical time scale. $\therefore \omega \sim 1/T$.

If the EM fields are generated primarily by a slowly time-changing charge density $\rho(\vec{r},t)$, one can write

$$\vec{E} = \underset{\text{"Coulomb"}}{\vec{E}_c} + \underset{\text{"Faraday"}}{\vec{E}_{in}} \quad \text{with } \vec{\nabla}\times\vec{E}_c = 0,\ \vec{\nabla}\cdot\vec{E}_{in} = 0$$

, $\vec{E}$ is primarily $\vec{E}_c$

$\therefore \quad \vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{E}_c = \rho/\varepsilon_0 \ , \quad E_c/l \sim \rho/\varepsilon_0 \ , \quad E_c \sim \frac{l}{\varepsilon_0}\rho$

$\vec{\nabla}\times\vec{E}_{in} = -\frac{\partial \vec{B}}{\partial t} \ , \quad E_{in}/l \sim \omega B \ , \quad E_{in} \sim \omega B l$

$\because \quad \vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0 \quad \therefore \quad J/l \sim \omega\rho \ , \quad J \sim \omega l \rho$

$$J_D = \varepsilon_0\frac{\partial E_c}{\partial t} + \varepsilon_0\frac{\partial E_{in}}{\partial t} \sim \varepsilon_0\omega(E_c + E_{in})$$

$\therefore$ J_D is at least $\varepsilon_0\omega E_c$, $J_D \gtrsim \varepsilon_0\omega E_c$

$$\therefore \quad \frac{J}{J_D} \lesssim \frac{\omega\rho l}{\varepsilon_0\omega E_c} \sim \frac{\omega\rho l}{l\rho\omega} \sim 1$$

From $\vec{\nabla}\times\vec{B} = \mu_0(\vec{J}+\vec{J_0}) \sim \mu_0\vec{J}$

$$\therefore \quad B/l \sim \mu_0 J$$

$$B \sim \mu_0 J l \sim \mu_0 \omega \rho l^2 \quad \text{--} (59)$$

Hence $\quad E_{in} \sim \omega B l \sim \mu_0 \omega^2 l^3 \rho$

$$\frac{E_{in}}{E_c} \sim \frac{\mu_0 \omega^2 l^3 \rho}{l/\varepsilon_0 \rho} \sim \varepsilon_0 \mu_0 \omega^2 l^2$$

$$= \frac{\omega^2 l^2}{c^2} \ll 1 \quad \text{---} (60)$$

if $l \ll c/\omega$ or $\omega \ll c/l$.

This is quai-electrostatic approximation. One has

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \varepsilon_0 \frac{\partial\vec{E}}{\partial t}\right)$$

(E_F can be neglected. $\vec{\nabla}\times\vec{E_c} = 0$)

$\because \vec{\nabla}\times\vec{E} = 0 \quad \therefore \quad \vec{B} = \vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}',t)\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} d\tau'$

$(\vec{\nabla}\times\vec{J_0} = 0)$

$$\vec{E} = -\nabla \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r',t)}{|\vec{r}-\vec{r}'|} d\tau'$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}',t)}{|\vec{r}-\vec{r}'|} d\tau' \quad \text{---} (61)$$

In this case, taking the static limit, one gets electrostatic

Slowly time-varying currents & quasi-magnetostatic approximation

If the EM fields are generated by slowly time-changing currents $\vec{J}(\vec{r},t)$, the analysis is different. ($\because$ ρ is induced by $\vec{\nabla}\cdot\vec{J}$.)

In this case, we start by estimating the magnetic field generated by $\vec{J}$ via the Ampere's law.

$$\vec{\nabla}\times\vec{B}_A = \mu_0 \vec{J}$$

$$\therefore\ B_A \sim \mu_0 J l$$

which in turn generates

$$\vec{\nabla}\times\vec{E}_{in} = -\frac{\partial\vec{B}_A}{\partial t}$$

$$\therefore\ E_{in} \sim l\omega B_A \sim \mu_0 \omega l^2 J$$

$\therefore\ \vec{J}_D = \epsilon_0 \frac{\partial \vec{E}_{in}}{\partial t}$, $\quad J_D \sim \epsilon_0 \omega E_{in} \sim \epsilon_0 \omega \mu_0 \omega l^2 J$

$$\frac{J_D}{J} \sim \mu_0\epsilon_0\omega^2 l^2 = \frac{\omega^2 l^2}{c^2} \ll 1 \quad \text{--- (62)}$$

if $l \ll c/\omega$ or $\omega \ll c/l$

This is the quasi-magnetostatic approximation in which J_D is neglected.

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \quad , \quad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad , \quad \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

$$\because\ \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0 \qquad \therefore\ \vec{\nabla}\cdot\vec{J} = 0$$

$\rho = \rho(\vec{r})$ one can choose $\rho = 0$

so that in the static limit, one recovers the magnetostatics.

$$\vec{\nabla}\cdot\vec{E} = 0 \quad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

In the quasi-magnetostatic approximation,

$$\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}$$

$$= \vec{\nabla}\times\left(\frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}',t)}{|\vec{r}-\vec{r}'|} d\tau'\right)$$

$$\therefore\ \vec{E}(\vec{r},t) = -\frac{\mu_0}{4\pi}\int \frac{\frac{\partial \vec{J}(\vec{r}',t)}{\partial t}}{|\vec{r}-\vec{r}'|} d\tau'$$

$$= -\frac{\partial \vec{A}}{\partial t} \qquad \cdots \quad (63)$$

In general, when both charges and currents are slowly-changing, it is the quasi-static limit. (or quasi-stationary)

In this case, when $l \ll c\tau$
(or $\omega \ll c/l$, $l \ll c/\omega$
$l \ll \lambda$)

(i) both $\vec{J}$ & $\vec{J}_D$ are present,

One can neglect $\vec{J}_D$ $\therefore \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$
$\vec{\nabla} \cdot \vec{B} = 0$

(ii) both $\vec{E}_m$ and $\vec{E}_c$ are present

one can neglect $\vec{E}_m$

$$\therefore \vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0, \quad \vec{\nabla} \times \vec{E} = 0$$

$\therefore$ There is no direct coupling between $\vec{E}$ and $\vec{B}$ in quasi-static approximation.

The quasi-static approximation underlies the most application of alternating-current (AC) circuit when sizes of circuit $l \ll c/\omega$ with ω = frequency of AC

Quasi-static approximation for good conductors

Charge relaxation in conductors

Inside a conductor, $\vec{J}_f = \sigma \vec{E}$

$\because \epsilon \vec{\nabla}\cdot\vec{E} = \rho_f$

$\therefore \vec{\nabla}\cdot\vec{J}_f = \sigma \vec{\nabla}\cdot\vec{E} = \frac{\sigma}{\epsilon}\rho_f$ -- (64)

The continuity equation $\vec{\nabla}\cdot\vec{J}_f + \frac{\partial \rho_f}{\partial t} = 0$

becomes $\frac{\partial \rho_f}{\partial t} + \frac{\sigma}{\varepsilon}\rho_f = 0$ -- (65)

$\therefore \rho_f(\vec{r},t) = \rho_f(\vec{r},0)\, e^{-t/\tau_E}$ -- (66)

$\tau_E = \epsilon/\sigma$

$\therefore$ for $t \gg \tau_E = \epsilon/\sigma$, $\rho_f = 0$

$\therefore \vec{\nabla}\cdot\vec{J}_f = 0$ --- (67)

$\because t \sim 1/\omega$ $\therefore$ For $\omega \ll \sigma/\varepsilon$, $\vec{\nabla}\cdot\vec{J}_f = 0$
$(\omega\tau_E \ll 1)$

Now, $\vec{J}_D = \epsilon \frac{\partial \vec{E}}{\partial t} \sim \varepsilon \omega E$

$\vec{J}_f = \sigma E$

$\therefore \frac{J_D}{J_f} = \frac{\varepsilon\omega}{\sigma} = \omega\tau_E$ -- (68)

Similar to quasi-magnetostatic approximation,

In principle, there are $\vec{J}_{ext}$ due to external sources. The induced $\vec{J}_0$ due to $\vec{J}_{ext}$ satisfies

$$\frac{J_0}{J_{ext}} \sim \frac{\omega^2 \ell^2}{c^2} \quad \text{-- (69)}$$

(cf. eq. (62))

Hence for $\underline{\omega \ll c/\ell \ \& \ \omega \ll \sigma/\omega}$, one can ignore $\vec{J}_0$.

In the absence of $\vec{J}_{ext}$, one gets the quasi-static approximation for good conductors:

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu\sigma\vec{E}$$

By taking $\nabla\times(\vec{\nabla}\times\vec{E})$, one gets

$$\nabla^2\vec{E} = \mu\sigma\frac{\partial\vec{E}}{\partial t}$$

Similarly, taking $\vec{\nabla}\times(\vec{\nabla}\times\vec{B})$, one gets

$$\nabla^2\vec{B} = \mu\sigma\frac{\partial\vec{B}}{\partial t}$$

as well. Hence both $\vec{E}$ & $\vec{B}$ satisfy the diffusion equation: $\frac{\partial}{\partial t}\psi = D\nabla^2\psi$, $D = \frac{1}{\mu\sigma}$

$\psi = \vec{E}$ or $\vec{B}$, -- (70)

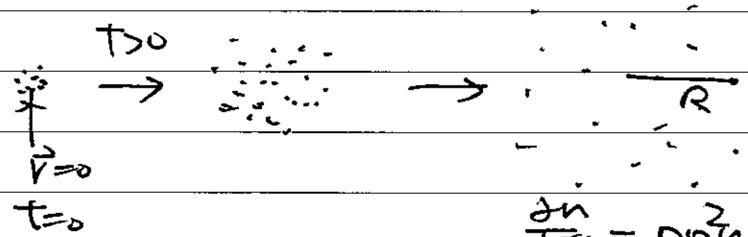
The diffusion equation originates the description for the spread of particles in the air or water.

$\frac{\partial n}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$ $\quad$ n = particle density / volume

$\vec{J} = -D\nabla n$ $\quad$ D = diffusion constant

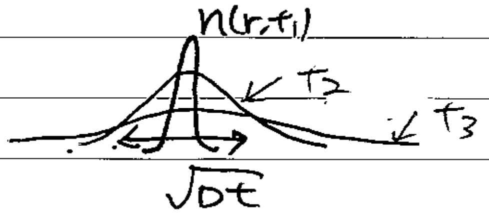
$\frac{\partial n}{\partial t} = D\nabla^2 n$.

If at $t=0$, $n = f(\vec{r})$, n evolves as follows:



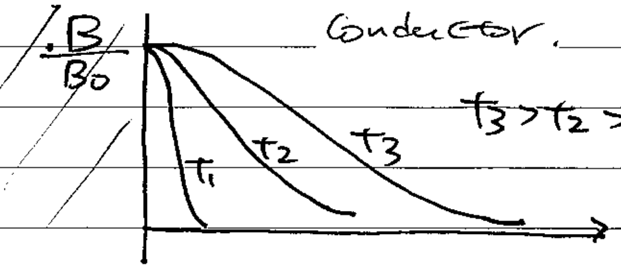
$R \sim \sqrt{Dt}$

$\frac{\partial n}{\partial t} = D\nabla^2 n \Rightarrow \frac{n}{Dt} \sim \frac{n}{R^2}$



$t_3 > t_2 > t_1$

∴ $\vec{E}$ & $\vec{B}$ fields in the quasi-static approximation enter into a good conduction in the way diffusion in conductor. ∵ $D \propto \frac{1}{\sigma}$. ∴ if $\sigma \to \infty$, $\vec{E}$ & $\vec{B}$ can't penetrate a perfect conductor. σ impedes the diffusion of $\vec{B}$ into a conductor.



$t_3 > t_2 > t_1$

In the quasi-static approximation, one has the Kirchhoff's laws for circuits.

Here one sets $I = I(t)$, $\mathcal{E} = \mathcal{E}(t)$

$$V = V(t)$$

(i) Kirchhoff's current law :

In general, $\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \vec{J}_D)$

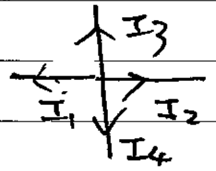
$\therefore \vec{\nabla} \cdot (\vec{J} + \vec{J}_D) = 0$

$\therefore \vec{J}$ itself would not conserve.

However, in quasi-static limit, $\vec{J}_D$ can be neglected (i.e. there is no appreciable charge accumulation, $\vec{\nabla} \cdot \vec{J}_D = \frac{\partial \rho}{\partial t}$)

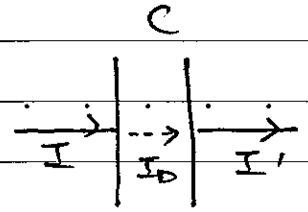
$\therefore \vec{J}$ is conserved, i.e. $\vec{\nabla} \cdot \vec{J} = 0$

$\therefore \quad \sum_i I_i = 0$ for each node in circuit.



$I_1 + I_2 + I_3 + I_4 = 0$

Note that the Kirchhoff's law does not imply the conservation of current when the circuit breaks such as the current enter = the current leave the capacitor.



, which is a consequence of conservation of $\vec{J}+\vec{J_D}$: $I = I_D$, $I_D = I'$ $\therefore I = I'$

(iii) Kirchhoff's voltage law

In quasi-electrostatic approximation, $\vec{\nabla}\times\vec{E} = 0$

$$\therefore \oint \vec{E}\cdot d\vec{l} = 0 \quad \text{--- (ii)}$$

$$\text{i.e. } \oint \vec{f_E}\cdot d\vec{l} = 0$$

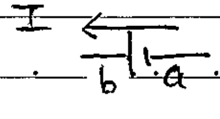
$\vec{f_E}$ = force per unit charge due to $\vec{E}$.

In applying the above approximation to AC circuits, one needs to introduce emf force $\vec{f_s}$ per unit charge either due to batteries or induced emf.

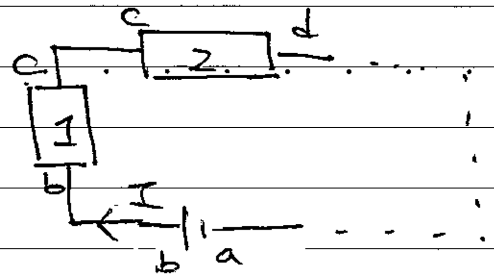
Inside emf devices, $\sigma = \infty$. $\therefore f = \frac{J}{\sigma} = 0$

$$\therefore \vec{f_s} + \vec{f_E} = 0 \quad \left(\text{for inductors, } \vec{f_E} = \vec{E} = -\frac{\partial\vec{A}}{\partial t} = -\vec{f_s} \quad \therefore f_s = \frac{\partial\vec{A}}{\partial t}\right)$$

$$\therefore \int_a^b \vec{f_E}\cdot d\vec{l} = -\underbrace{\int_a^b \vec{f_s}\cdot d\vec{l}}_{\mathcal{E}} = \int_a^b \vec{E}\cdot d\vec{l} = V_a - V_b$$



$\therefore V_a - V_b = -\mathcal{E}$; $V_b - V_a = \mathcal{E} > 0$



Therefore, (ii) implies $\underbrace{V_a - V_b}_{\Delta V_{\mathcal{E}}} + V_b - V_c + V_c - V_d + \cdots = 0$

$$\therefore \quad -\mathcal{E} + \Delta V_1 + \Delta V_2 + \cdots = 0$$

where except for emf $\mathcal{E}$, $\Delta V_i = \overline{V_i}$ = voltage drop across ith circuit element ($V_{enter} - V_{leave}$)

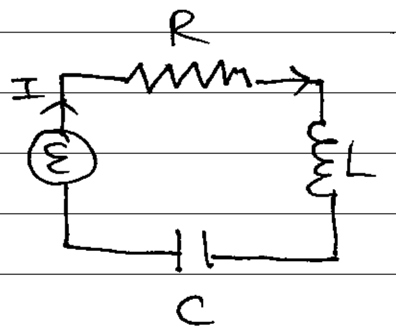
$\mathcal{E} = V_{leave} - V_{enter}$.

$\therefore$ One has the kirchhoff's voltage law

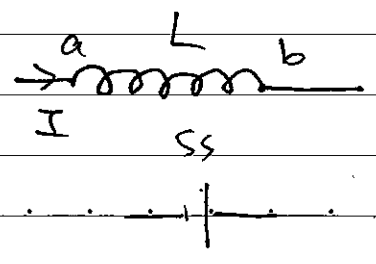
$$\sum_i \overline{V_i} = 0$$

$\overline{V_i}$ = voltage drop across ith circuit element.

Example: RLC circuit.



$$-\mathcal{E} + \underbrace{IR}_{\text{voltage drop across } R} + \overbrace{L\frac{dI}{dt}}^{-\mathcal{E}_L} + \underbrace{\frac{Q}{C}}_{\text{drop across } C} = 0$$



$\mathcal{E}_L = -L\frac{dI}{dt}$ = emf of inductance

$= V_b - V_a$

$$\therefore \quad \mathcal{E} = L\frac{dI}{dt} + IR + Q/C$$