Electric potential and method of their evaluations

Laplace equation

The primary task in electrostatics is to find $\vec{E}$ field for a given stationary charge distribution $\rho(\vec{r})$.

We have shown that there are 3 methods:

$$\vec{E}(\vec{r}) = \int \frac{1}{4\pi\epsilon_0} \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} (\vec{r}-\vec{r}\,')\, d\tau' \quad \text{—} ①$$

or

$$V(\vec{r}) = \int \frac{1}{4\pi\epsilon_0} \frac{1}{|\vec{r}-\vec{r}\,'|} \rho(\vec{r}\,')\, d\tau' \quad \text{—} ②$$

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho(\vec{r}) \quad \text{—} ③$$

③ is known as the Poisson's equation.

The integration methods (① & ②) are the most straightforward but they are difficulte to be carry over. Furthermore, in practice, one does not alway know all charge density in advance (for instance, in the presence of conductors, surface charge density is also an unknown.)

Therefore, it's more fruitful to solve the differential form in eq. ③.

In practice, the charge density may not be known completely. We are mostly interested in finding V in regions Ω where $\rho = 0$. In this case, eq. ③ reduces to the Laplace equation

$$\nabla^2 V = 0 \quad --- \quad ④$$

in Ω

All the informations of $\rho(\vec{r})$ are lumped into boundary conditions (BC) of V on $\partial\Omega$. As we shall see,

$$\nabla^2 V = 0 \text{ in } \Omega \\ + \text{BC on } \partial\Omega \quad --- \quad ⑤$$

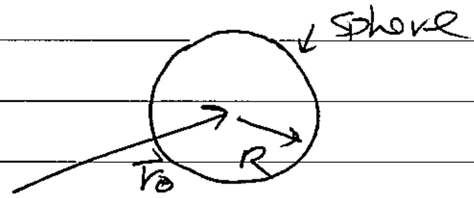
is sufficient to find V in Ω.

The solutions to the Laplace equation are known as harmonic functions in mathematics.

Features of solutions

(i) If V is a solution to $\nabla^2 V = 0$ in Ω,
for any point $\vec{r}_0$ $\in \Omega$, the average of $V(\vec{r})$ on a sphere centered at $\vec{r}_0$ $= V(\vec{r}_0)$, i.e.,

$$V(\vec{r}_0) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V \, da$$



(mean-value property) ← ⑥

(ii) V has no local maxima or minima

∴ The extreme values of V must occur at the boundary

Example in 1D

$\nabla^2 V = 0$ reduces to $\frac{d^2 V}{dx^2} = 0$

∴ $V(x) = mx + b$

(i) a segment at x : $x-a$ —— x —— $x+a$

∴ average of $V(x-a)$ & $V(x+a)$

$$= \frac{1}{2}\left(V(x-a) + V(x+a)\right) = mx + b = V(x)$$

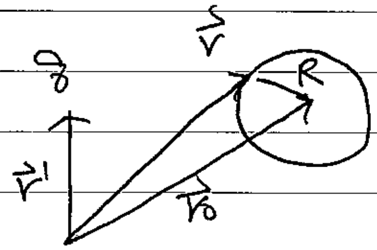
is satisfied!

(ii) $mx+b$ has no local maxima or minima

extreme values of V occur at end points.

Pf. (i) We shall first consider V due to a point charge q at $\vec{r}'$ outside Ω.

$$\therefore \quad V = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}'|}$$



The average of V over sphere

$$\bar{V} = \frac{1}{4\pi R^2}\oint_{\text{sphere}} \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}'|}\,da \quad --- (7)$$

Eq. (7) can be also viewed as the potential at $\vec{r}'$ due to a surface charge

$\sigma(\vec{r}) = \frac{q}{4\pi R^2}$ on the sphere (8)

Since σ is uniform, using the Gauss's theorem, one finds : if $\vec{r}'$ lies outside the sphere

$$\oint_{\text{Sphere}} \frac{\sigma da}{4\pi\epsilon_0|\vec{r}'-\vec{r}|} = \frac{q_{\text{sphere}}}{4\pi\epsilon_0|\vec{r}'-\vec{r_0}|} \quad \text{--- (9)}$$

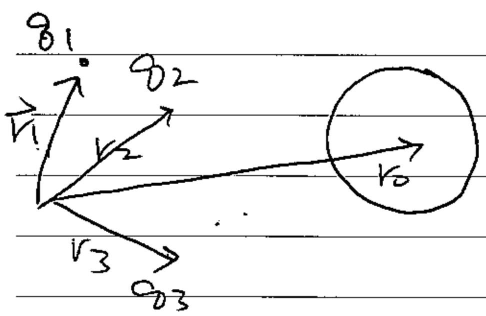
with $q_{\text{sphere}} = \oint \sigma da = \frac{q}{4\pi R^2} \times 4\pi R^2 = q$

Hence
$$\bar{V} = \frac{q}{4\pi\epsilon_0|\vec{r}'-\vec{r_0}|} = \frac{q}{4\pi\epsilon_0|\vec{r_0}-\vec{r}'|}$$

$$\therefore \frac{1}{4\pi R^2}\oint_{\text{sphore}} V_q(r)\, da = \frac{q}{4\pi\epsilon_0|\vec{r_0}-\vec{r}'|} \quad \text{--- (10)}$$

if $\vec{r}'$ lies outside the sphere

Clearly, from the principle of superposition, for general charge distribution, eq. (10) becomes



$$\frac{1}{4\pi R^2}\oint_{\text{Sphore}} V(r)\, da$$

$$= \sum_i \frac{1}{4\pi\epsilon_0} \frac{q_i}{|\vec{r_0}-\vec{r_i}|}$$

= potential due to $\{q_i\}$ at r_0. $= V(r_0)$

$\therefore$ (i) is proved.

* Once (i) is correct, any point $\vec{r_0}$ in Ω can't be local maxima or minima because if $V(\vec{r_0})$ is

a local maxima or minima, $V(r) < V(r_0)$

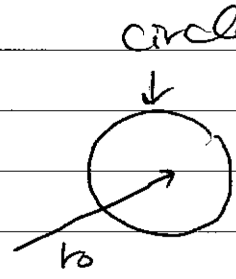
or $V(r) > V(r_0)$ for r lies on sphere,

Eq. (10) can't be satisfied.

Therefore, (ii) is a consequence of (i).
It implies that maxima or minima can only apear on boundary!
* (i) can be generalized to 2D case.

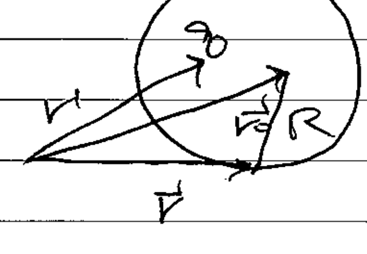
In that case, clearly, one has

$$V(\vec{r}_0) = \frac{1}{2\pi R} \oint_{\text{circle}} V \, dl \qquad \text{--- (11)}$$



* $\vec{r}'$ lies inside the sphere

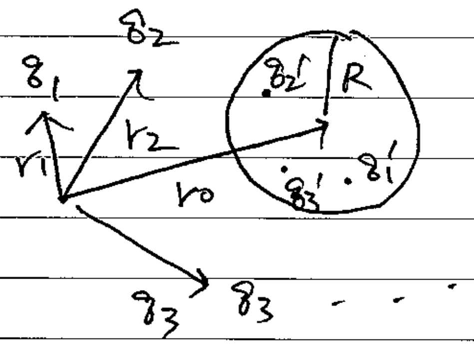
The derivation can be clearly generalized to the situation when $\vec{r}'$ lies inside the sphere.



Using eq. (8), since the potential inside a uniformly charged sphere is a constant $= \frac{q}{4\pi\epsilon_0 R}$,

one gets $$\frac{1}{4\pi R^2} \oint_{\text{sphere}} V_q(r) \, da = \frac{q}{4\pi\epsilon_0 R} \qquad \text{--- (12)}$$

From the principle of superposition, we get



$$\frac{1}{4\pi R^2}\oint_{sphere} V(r)\,da = V_{ext}(\vec{r_0}) + \frac{1}{4\pi\epsilon_0 R}\underbrace{(q_1' + q_2' + \cdots)}_{Q'}$$ --- (13)

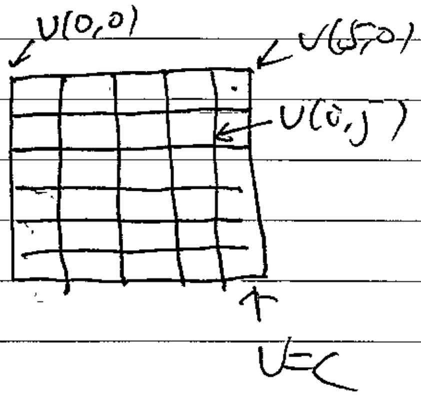
Where $V_{ext}(\vec{r_0})$ is the potential at $\vec{r_0}$ due to external charges $q_1, q_2, \ldots$ (out side sphere) while Q' is the total charge enclosed by the sphere.

Eq. (13) is the generalization of (1).

* method of relaxation

The feature of V in eq. (13) can be used on computer to find solutions to $\nabla^2 V = 0$. This is known as method of relaxation: starting from $V = C$ on the boundary, one calculates
($V = 0$ inside)

V in the next step by:



$$V^{(1)}(i,j) = \frac{1}{4}\left[V(i-1,j) + V(i+1,j) + V(i,j-1) + V(i,j+1)\right]$$

Repeat the above to the next step and so on with

$$V^{(n+1)}(i,j) = \frac{1}{4}\left[V^{(n)}(i+1,j) + V^{(n)}(i-1,j) + V^{(n)}(i,j-1) + V^{(n)}(i,j+1)\right]$$

In $n \to \infty$, the function $V(i,j)$ will converge and satisfies

$$V(i,j) = \frac{1}{4}\left[V(i-1,j) + V(i+1,j) + V(i,j-1) + V(i,j+1)\right] \quad \text{--- (14)}$$

which is a solution to $\nabla^2 V = 0$.

* Earnshaw's theorem

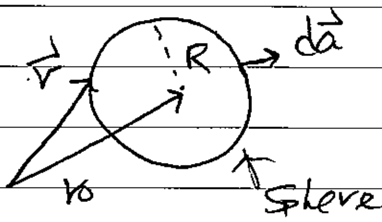
(ii) implies that a charged particle can't be held in a stable equilibrium by electrostatic force alone. This is known as Earnshaw's theorem.

A stable equilibrium position at r_0 must be a local minimum which can not be true if only electrostatic force is involved.

* General proof without using Coulomb interaction form:

$$\nabla^2 V = 0$$

$$\Rightarrow V(\vec{r}_0) = \frac{1}{4\pi R^2}\oint_{sphere} V\, da$$



Pf: We shall use a vector identity

$$\int_V (U\nabla^2 V - V\nabla^2 U)\, d\tau$$

$$= \int_V \nabla\cdot[U\nabla V - V\nabla U]\, d\tau = \oint_S (U\vec{\nabla} V - V\vec{\nabla} U)\cdot d\vec{a} \quad \text{--- (15)}$$

where $\nabla\cdot(U\nabla V - V\nabla U) = \nabla U\cdot\nabla V - \nabla V\cdot\nabla U + U\nabla^2 V - V\nabla^2 U$
$= U\nabla^2 V - V\nabla^2 U$ is used.

Take $U = \frac{1}{|\vec{r}-\vec{r}_0|}$, V satisfies $\nabla^2 V = 0$

$\because \nabla^2 \frac{1}{|\vec{r}-\vec{r}_0|} = -4\pi\delta(\vec{r}-\vec{r}_0)$. $\nabla \frac{1}{|\vec{r}-\vec{r}_0|} = -\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}$

$\therefore$ Left hand of eq (15) $= -4\pi\int V\delta(\vec{r}-\vec{r}_0)\, d\tau$
$= -4\pi V(\vec{r}_0)$

$\therefore$ Take S = sphere centered at r_0, Eq. (15) becomes

$$4\pi V(\vec{r}_0) = \oint_{sphere} V\frac{(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}\cdot d\vec{a} + \oint_{sphere}\frac{1}{|\vec{r}-\vec{r}_0|}(-\nabla V)\cdot d\vec{a} \quad \text{--- (16)}$$

∵ $\vec{E} = -\nabla V$, $\begin{cases} (\vec{r}-\vec{r}_0)\cdot d\vec{a} = R\,da \\ |\vec{r}-\vec{r}_0| = R \text{ on Sphere} \end{cases}$

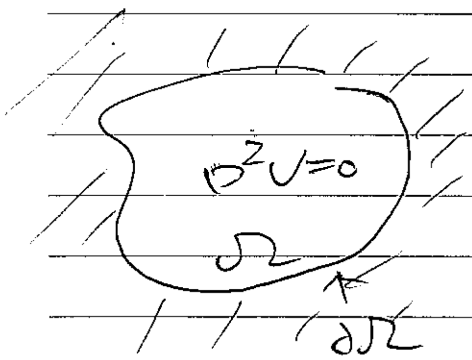
$$\therefore \quad V(\vec{r}_0) = \frac{1}{4\pi R^2}\oint_{\text{Sphere}} V\cdot da + \frac{1}{4\pi R}\oint_{\text{Sphere}} \vec{E}\cdot d\vec{a}$$

$$\because \quad \oint_{\text{Sphere}} \vec{E}\cdot d\vec{a} = \int \nabla\cdot\vec{E}\, dv = 0$$

$$\therefore \quad V(\vec{r}_0) = \frac{1}{4\pi R^2}\oint_{\text{Sphere}} V\,da \qquad \text{Q.E.D.}$$

Boundary conditions & Uniqueness theorem

The Laplace equation $\nabla^2 V = 0$ itself in a region does not determine V.



This reflects the fact that the absolute value of V has no meaning.

From the experience of method of relaxation, clearly one needs to specify V on the the boundary $\partial\Omega$. But what are appropriate boundary conditions (BC) that are sufficient to determine V in Ω?

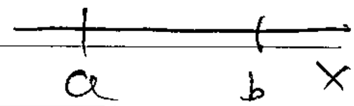
By "sufficient", clearly we want that

BCs can uniquely determine V.

Take 1D as an example,

the general form of

solution $V = \alpha x + \beta$



If $\Omega = [a, b]$, the boundary

$= x=a$ or $x=b$ (two points)

To determine V, one needs to

specify α and β.

Clearly, one can specify $V(a)$ and $V(b)$

so that $\alpha a + \beta = V(a)$

$\alpha b + \beta = V(b)$

determine α & β.

One can also specify the electric field

$E = -\frac{dV}{dx}$ at a & b $\quad E(a) = E(b) = -\alpha$

but it does not specify β.

Or one uses mixed information of V & E.

by specifying $V(a)$ & $E(a)$

$$\therefore \alpha a + \beta = V(a)$$

$$\alpha = -E(a)$$

It turns out that the above boundary conditions obtained in 1D can be generalized to 2D & 3D:

$$U\big|_{\partial\Omega} = \text{Dirichlet boundary condition}$$

$$E\big|_{\partial\Omega} = \text{Neumann boundary condition}$$
$$\left(\frac{\partial U}{\partial n}\Big|_{\partial\Omega}\right)$$

$$U\big|_{\partial\Omega_1},\ \frac{\partial U}{\partial n}\Big|_{\partial\Omega_2} = \text{mixed boundary condition}$$
$$\partial\Omega = \partial\Omega_1 + \partial\Omega_2 \qquad \text{--- (17)}$$

The problem of solving partial differential equations (such as $\nabla^2 V = 0$) with boundary conditions is very typical in physical science and is known as boundary-value problem in mathematics.

The first thing one needs to show is the boundary conditions are sufficient to yield the solution V.

This consists of two parts:

(i) existence of solution

(ii) uniqueness of solution

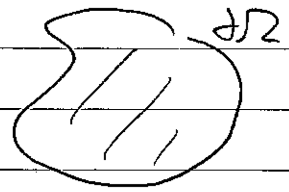
In physics, one often assumes the existence of solution and don't bother in proving it. However, in some cases, one has to make sure that the problem allows solutions.

For instance, in the boundary conditions specified by eq (17), if one specifies both V & $\frac{\partial V}{\partial n}$ on $\partial\Omega$ known as the Cauchy boundary condition, there may not exist a solution due to that V & $\frac{\partial V}{\partial n}$ may be in conflict with each other. Hence the existence does deserve some attention.

The uniqueness theorem allows one to use his imagination/insight to guess the solution. Once the guessed solution is verified to satisfy the equation, it is then the solution!

Therefore, we shall pay more attention to the uniqueness theorem.

First uniqueness theorem



$$U|_{\partial\Omega} = \text{Known}$$

$$\Rightarrow U \text{ is uniquely determined.}$$

pf: Suppose that there are two solutions V_1 and V_2.

$$\therefore \nabla^2 V_1 = 0 \qquad V_1|_{\partial\Omega} = V_2|_{\partial\Omega}$$

$$\nabla^2 V_2 = 0 \qquad = f(\vec{r})$$

Let $\phi = V_1 - V_2$ $\therefore \phi|_{\partial\Omega} = 0$

and $\nabla^2\phi = \nabla^2 V_1 - \nabla^2 V_2 = 0$

Now, according to (ii), ϕ has no maxima & minima in Ω and maxima & minima of ϕ must occur on $\partial\Omega$.

$\because \phi = 0$ on $\partial\Omega$ $\therefore$ max & min of $\phi = 0$

$\therefore \phi = 0$ everywhere in Ω.

More rigorously, one considers a vector identity:

$$\nabla\cdot(\phi\nabla\phi) = \phi\nabla^2\phi + (\nabla\phi)^2 \quad \text{--- (18)}$$

$$\therefore \int_\Omega \nabla\cdot(\phi\nabla\phi)\,d\tau = \int_\Omega \phi\nabla^2\phi + (\nabla\phi)^2\,d\tau$$

$$\underset{\because \nabla^2\phi=0}{=} \int_\Omega (\nabla\phi)^2\,d\tau \quad \text{--- (19)}$$

On the other hand, according to the divergence theorem, $\int_\Omega \nabla\cdot(\phi\nabla\phi)\,d\tau = \int_{\partial\Omega} \phi\vec{\nabla}\phi\cdot d\vec{a}$

$$\underset{\phi=0 \text{ on } \partial\Omega}{=} 0 \quad \text{--- (20)}$$

$\therefore$ Eqs. (19) & (20) imply $\int_V (\nabla\phi)^2\,d\tau = 0$

$\because (\nabla\phi)^2 \geq 0 \quad \therefore \nabla\phi = 0 \quad \phi = \text{Constant}$

Since $\phi=0$ on $\partial\Omega$, $\therefore$ constant.

$\therefore \phi = 0$ everywhere in Ω.

Corollary: V is uniquely determined

if (a) $\rho(\vec{r})$ in Ω is known

(b) $V|_{\partial\Omega}$ known

Pf: $\nabla^2 V = -\rho/\varepsilon_0$

Suppose that there are two solutions V_1 & V_2

$\therefore \nabla^2 V_1 = -\rho/\varepsilon_0 \qquad V_1|_{\partial\Omega}$

$\nabla^2 V_2 = -\rho/\varepsilon_0 \qquad = V_2|_{\partial\Omega}$

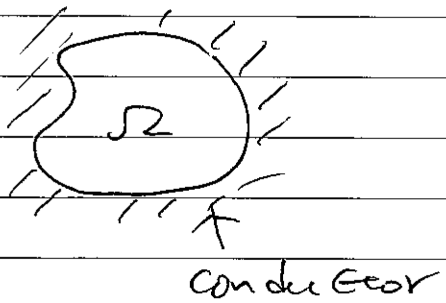
Let $\phi = V_1 - V_2$ again, $\phi|_{\partial\Omega} = 0$

$\nabla^2 \phi = 0$

$\therefore \phi = 0$ everywhere in Ω

$\therefore V_1 = V_2$

Example: $\partial\Omega$ = a conductor



$V|_{\partial\Omega}$ = constant $\equiv C$

$V = C$ in Ω is an obvious solution ($\nabla^2 V = 0$)

according to the uniqueness theorem, $V = C$ is the solution.

<u>General uniqueness theorem</u> for $\Omega = \infty$

V is uniquely determined

if (a) $\rho(\vec{r})$ known in all $\vec{r}$

(b) $V \to 0$ as $r \to \infty$

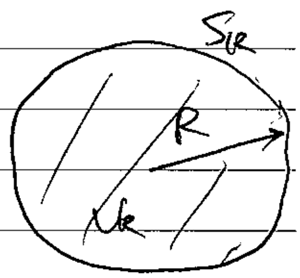
Pf. Suppose that there are two solutions V_1 and V_2.

Then $\Phi \equiv V_1 - V_2$ satisfies $\nabla^2 \Phi = 0$

$\Phi \to 0$ as $r \to \infty$

Using eq. (18) and $\nabla^2 \Phi = 0$, one gets

$$\int_{V_R} (\nabla \Phi)^2 \, d\tau = \oint_{S_R} \Phi \nabla \Phi \cdot d\vec{a} \quad \leftarrow (21)$$



where V_R is the volume of radius R

and S_R is the surface of sphere (radius R)

$$\Phi \nabla \Phi \cdot d\vec{a} = \Phi \frac{\partial \Phi(R,\theta,\phi)}{\partial R} R^2 \underbrace{\sin\theta \, d\theta \, d\phi}_{d\Omega}$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial R}(\Phi^2)\right] R^2 \, d\Omega$$

$\therefore$ Eq. (21) becomes

$$\frac{1}{2} \oint_{\Omega = 4\pi} \frac{\partial \Phi^2}{\partial R} d\Omega - \frac{1}{R^2} \int_{V_R} (\nabla \Phi)^2 d\tau = 0 \quad \leftarrow (22)$$

$\int_0^{R_0}$ eq. (22) yields

$$\frac{1}{2}\oint_{\Omega=4\pi}\left[\Phi^2(R_0,\theta,\phi)-\Phi^2(0,\theta,\phi)\right]d\Omega$$

$$-\int_0^{R_0}\left[\frac{1}{R^2}\int_{V_R}(\nabla\Phi)^2 d\tau\right]dR=0$$

Now, let $R_0\to\infty$, $\because$ $\Phi^2(R_0,\theta,\phi)\to 0$

$\therefore$ one gets

$$\frac{1}{2}\oint_{4\pi}\Phi^2(0,\theta,\phi)d\Omega+\int_0^{\infty}\left(\frac{1}{R^2}\int_{V_R}(\nabla\Phi)^2 d\tau\right)dR=0 \quad \text{--- (23)}$$

$\because$ Both $\Phi^2(0,\theta,\phi)$ and $(\nabla\Phi)^2 \geq 0$,

$\therefore$ Eq. (23) is satisfied only when

$\nabla\Phi=0$ everywhere

and $\Phi(0,\theta,\phi)=0$

Now, $\nabla\Phi=0$ implies $\Phi(R,\theta,\phi)=$ Const.

together with $\Phi(0,\theta,\phi)=0$ $\therefore$ $\Phi=0$ everwhere

$\therefore$ $V_1=V_2$ everywhere.

Conductors and the second uniqueness theorem

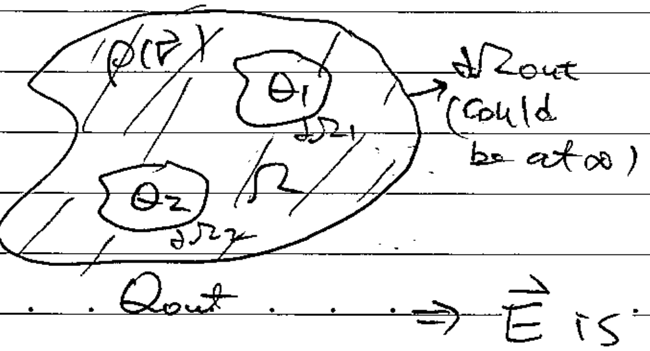
In real situations at laboratory, the boundaries are often set by conductors.

There are two possible ways to set up the conductors that serve as the boundaries: (i) Connected to a battery or grounded, ∴ $V = $ const on the conductor (ii) isolated with total charge Q_i known for the ith conductor. (without even specifying $\sigma(\vec{r})$ on surfaces of conductors)

In this case (ii), is the solution unique?

This is answered in the 2nd uniqueness theorem.

Second uniqueness theorem



$\rho(\vec{r})$ Known in Ω

$\oint_{\partial\Omega_i} (-\nabla V)\cdot d\vec{a} = Q_i/\varepsilon_0$

Known

$\Rightarrow \vec{E}$ is uniquely determined

Pf: Suppose that there are two solutions

$\vec{E}_1$ & $\vec{E}_2$

$\therefore \quad \vec{\nabla}\cdot\vec{E}_1 = \rho/\varepsilon_0$

$\vec{\nabla}\cdot\vec{E}_2 = \rho/\varepsilon_0$

& $\displaystyle\oint_{\partial\Omega_i} \vec{E}_1\cdot d\vec{a} = \frac{Q_i}{\varepsilon_0} = \oint_{\partial\Omega_i} \vec{E}_2\cdot d\vec{a}$

Let $\vec{E} = \vec{E}_1 - \vec{E}_2 = -\nabla\phi$

$\therefore \quad \vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{E}_1 - \vec{\nabla}\cdot\vec{E}_2 = 0 \quad \text{---} \; (24)$

$\displaystyle\oint_{\partial\Omega_i} \vec{E}\cdot d\vec{a} = \oint_{\partial\Omega_i} \vec{E}_1\cdot d\vec{a} - \oint_{\partial\Omega_i} \vec{E}_2\cdot d\vec{a}$

$= 0 \quad \text{---} \; (25)$

Consider the identity $\nabla\cdot(\phi\vec{E}) = \phi\vec{\nabla}\cdot\vec{E} + \nabla\phi\cdot\vec{E}$

$\underset{\uparrow (24)}{=} 0 - E^2$

$\therefore \displaystyle\int_\Omega \vec{\nabla}\cdot(\phi\vec{E})\, d\tau = -\int_\Omega E^2 d\tau \quad \text{---} \; (26)$

$\because \displaystyle\int_\Omega \vec{\nabla}\cdot(\phi\vec{E})\, d\tau = \sum_i \oint_{\partial\Omega_i} \phi\vec{E}\cdot d\vec{a} = 0$

(using eq (25)) $\therefore$ Eq. (26) reduces to $\int_\Omega E^2 d\tau = 0$

$\therefore \vec{E} = 0$, $\vec{E}_1 = \vec{E}_2$ in Ω! There is only one solution

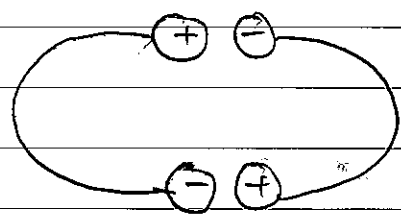
Although the 2nd uniqueness theorem can be proved rigorously, there are examples that are far from intuition. Therefore, the 2nd uniqueness theorem is by no means trivial.

Example: 4 conductors with charges are arranged in the followings

$+Q \; \oplus \; \ominus \; -Q$

$-Q \; \ominus \; \oplus \; +Q$

Now, one can use conducting wires to connect them as



will charges annihilate each other and get cancelled?

By the 2nd uniqueness theorem, there are now two conductors after connection and their charges $= 0$

$\therefore \vec{E} = 0$ can satisfy $\vec{\nabla}\cdot\vec{E} = 0$

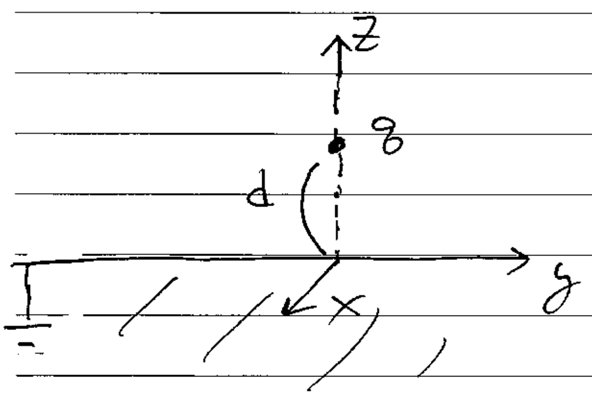
& boundary conditions $\oint_{\partial\Omega_i} \vec{E}\cdot d\vec{a} = 0 \quad i = 1, 2$

$\therefore$ Charges must flow to got cancelled!

Method of images

A typical usage of uniqueness theorem is to find the electric potential in the presence of conductors with certain symmetries. The method of images represent such a usage.

Classic Image problem : In the classic image problem, one tries to find the electric potential for a point charge in front of an infinite grounded conducting plane (shown in the left figure)



Physically, q will attract opposite charges so that there will be surface charge density σ accumulated at $z=0$. However, one doesn't know σ in advance. How can one calculate the potential for $z>0$?

Mathematically, the problem is to solve the Poisson equation:

$$\nabla^2 V = -\frac{q}{\varepsilon_0}\delta(x)\delta(y)\delta(z-d) \quad \text{for } z \geq 0 \qquad \text{--- (27)}$$

With the boundary conditions:

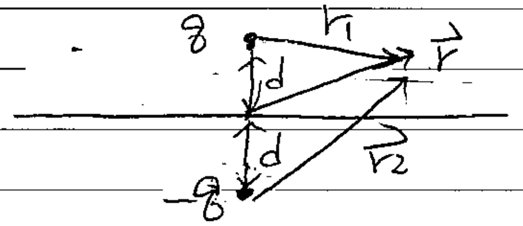
$$\begin{aligned} &1. \quad V=0 \quad \text{at } z=0 \\ &2. \quad V \to 0 \quad \text{for } r=\sqrt{x^2+y^2+z^2} \gg d. \end{aligned} \qquad (28)$$

The 1st uniqueness theorem guarantees that
(+ general " ")
there is only one solution. We can try guess the solution. If the guessed solution satisfies (27) and (28), that is the solution.

In the method of image, one tries to counter the potential due to q at $z=0$ by another charge in $z<0$ so that

$$V(z=0)=0$$

For this purpose, one tries to put $-q$ at $z=-d$, $x=0$, $y=0$ and forget about the conductor for a moment.
If there is no conducting planes, the potential due to $\pm q$ is



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r_1} - \frac{q}{r_2}\right] \quad \leftarrow \cdots (29)$$

Clearly, at $z=0$, $r_1=r_2$, $\therefore V=0$

Furthermore, $V\to 0$ as $r\to\infty$ is satisfied

For $z>0$, there is only charge q at $z>0$

$$\therefore \nabla^2 V = -\frac{q}{\epsilon_0}\delta(x)\delta(y)\delta(z-d)$$

V in eq. (29) satisfies (27) & (28) in $z\geq 0$ $\therefore$ the solution is

$$V(x,y,z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

for $z\geq 0$ $\quad \leftarrow \cdots (30)$

That is, after disregarding $z<0$ in eq. (29),

U in (30) is the solution.

Here $-q$ is termed as the image of q.

as $\pm q$ occupied $z=\pm d$ which appears to be same relation as object and its image in optics.

Induced surface charge

Once U is known, one can compute the surface charge density σ by using

$$\sigma = -\varepsilon_0 \frac{dU}{dn} = -\varepsilon_0 \frac{dU}{dz}\bigg|_{z=0} \quad \cdots (31)$$

From eq (30), we find

$$\frac{dU}{dz} = \frac{1}{4\pi\varepsilon_0}\left[\frac{-q(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{q(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right]$$

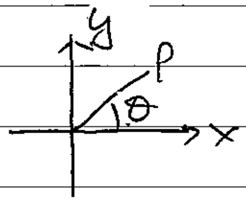
$$\therefore \sigma(x,y) = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}} \quad \cdots (32)$$

Check: one can check that the total induced charge is $-q$ by integrating σ.

$$q_{ind} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx \int dy \; \sigma(x, y)$$

$\because$ σ depends only on $\rho = \sqrt{x^2+y^2}$

$\therefore$ One uses the polar coordinate (ρ, θ)



$\therefore$ q_{ind}

$$= \int_0^{\infty} \rho d\rho \int_0^{2\pi} d\theta \; \frac{-qd}{2\pi(\rho^2+d^2)^{3/2}}$$

$$= -\frac{2\pi qd}{2} \int_0^{\infty} \frac{d\rho^2}{2\pi(\rho^2+d^2)^{3/2}}$$

$$= \left. \frac{qd}{\sqrt{\rho^2+d^2}} \right|_{\rho=0}^{\rho=\infty} = -q$$

Force and energy

For $z>0$, the potential is the same as the sum of potentials due to $\pm q$.

$\therefore$ The electric field is the same as the sum of $\vec{E}$ due to $\pm q$.

Therefore, the force acting on q is

$$\vec{F} = \frac{-1}{4\pi\epsilon_0} \frac{q^2}{(2d)^2} \hat{z} \quad --- \quad (33)$$

However, not everything is the same as the case with $\pm q$ at $z = \pm d$:

The electrostatic energy (work that needs to be done)

is not $-\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d} \pm W$!

$\because$ $E = 0$ for $z < 0$, by symmetry,

$$W = \frac{\epsilon_0}{2}\int_{z>0} E^2 dz = \frac{1}{2}\left(\int \frac{\epsilon_0}{2} E^2 dz\right)$$

($z<0$ & $z>0$ equally contribute)

$$\therefore W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d} \quad \cdots \quad (34)$$

One can also deduce W by using $\vec{F}$ given in eq. (33):

$$W = \int_{\infty}^{z=d} \underbrace{\vec{F}_a}_{\substack{\text{applied force} \\ = -\vec{F}}} \cdot d\vec{r} = \frac{1}{4\pi\epsilon_0}\int_{+\infty}^{d} \frac{q^2}{4z^2} dz$$

$d\vec{r} = dz\,\hat{z}$ ($dz < 0$)

$$= \frac{1}{4\pi\epsilon_0}\left(-\frac{q^2}{4z}\right)_{z=\infty}^{z=d} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}$$

in agree with eq. (34).

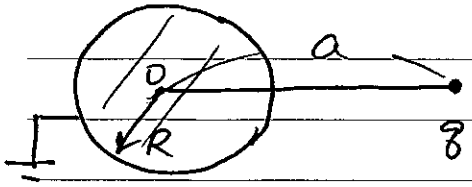
Other image problem

Generally, replacing b.c. by appropriate charges to reproduce correct potential in the domain Ω we are interested is termed as the method of image.

Usually, this works for conductors with symmetries : plane, sphere, ...

In some cases, # of image charges may not be one just as the image problem for optics.

Example. A point charge q is at a distance of a from the center of a grounded conducting sphere of radius R.

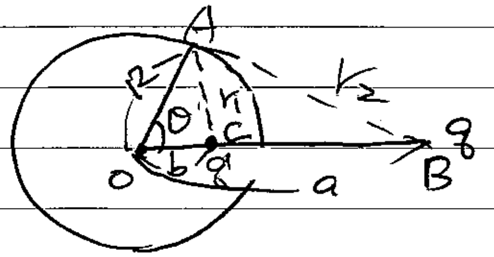


Here $a > R$. Find U outside the sphere

Solution:
From the consideration of symmetry, the image charge q' must lie on the line joining

O. & q. There are two unknowns:

q' & its distance to O (b).



For any point on the sphere (A),

$$V_A = \frac{1}{4\pi\epsilon_0}\left(\frac{q'}{r_1} + \frac{q}{r_2}\right).$$

$$\therefore \quad \frac{q'}{r_1} + \frac{q}{r_2} = 0$$

$$\frac{r_1}{r_2} = -\frac{q'}{q} = \text{constant (independant of } \theta)$$

Now, $r_1^2 = R^2 + b^2 - 2Rb\cos\theta$

$r_2^2 = R^2 + a^2 - 2Ra\cos\theta$

$$\therefore \quad 0 = \frac{d}{d\theta}\left(\frac{r_1^2}{r_2^2}\right) = \frac{1}{r_2^4}\left[\frac{dr_1^2}{d\theta} r_2^2 - r_1^2 \frac{dr_2^2}{d\theta}\right]$$

$$= \frac{1}{r_2^4}\left[2Rb\sin\theta(R^2 + a^2 - 2Ra\cos\theta) - 2Ra\sin\theta(R^2 + b^2 - 2Rb\cos\theta)\right]$$

$$\therefore \quad 2Rb\sin\theta(R^2 + a^2 - \cancel{2Ra\cos\theta}) = 2Ra\sin\theta(R^2 + b^2 - \cancel{2Rb\cos\theta})$$

$$R^2(a-b) = ab(a-b) \qquad a \neq b$$

$$\therefore \quad R^2 = ab$$

It implies

$$\hookrightarrow \frac{\overline{OA}}{\overline{OC}} = \frac{\overline{OB}}{\overline{OA}} \quad \therefore \triangle OAB \sim \triangle OCA$$

$$\frac{\overline{OB}}{\overline{OA}} = \frac{\overline{AB}}{\overline{AC}}$$

$$\frac{R}{a} = \frac{a}{R} = \frac{r_2}{r_1}$$

$$\therefore \frac{r_1}{r_2} = \frac{a}{R} = -q'/q \, .$$

$$\therefore q' = -R/a \, q \, , \quad b = R^2/a$$

The electric potential is thus determined by putting $q' = -R/a \, q$ at $b = R^2/a$.

Method of separation of variables

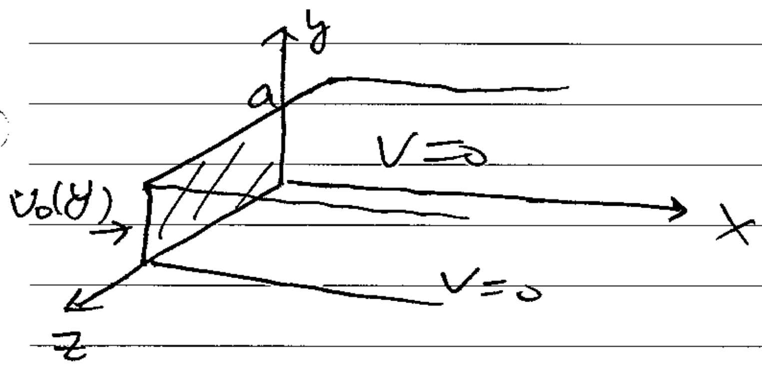
In the region without charges, one needs to solve the Laplace equation. With boundary condition: either V is specified on $\partial\Omega$ or σ is specified on $\partial\Omega$.

It is quite often that boundaries are constant surfaces of some coordinates. In this case, there exists systematic ways of using mathematical series to find the electrostatic potential, known as methods of separation of variables.

We shall illustrate the methods by two examples.

Cartesian Coordinates

example



Two infinite grounded metal plate ∥ xz plane

one at $y=0$, the other at $y=a$.

The left end, $x=0$, is closed off by an infinite strip insulated from two plates, with a specific potential $V_0(y)$

Find V inside the region $0<y<a$, $x>0$.

Solution: The problem is translationally invariant in z direction ∴ V does not depend on z.

$$V = V(x, y)$$

& $$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \text{--- (35)}$$

With BC. (i) $V(x, 0) = 0$

(ii) $V(x, a) = 0$

(iii) $V(0, y) = V_0(y)$ --- (36)

(iv) $V \to 0$ as $x \to \infty$

The most important feature of BC is the separation of x and y.

The surfaces at $y=0$, $y=a$ & $x=0$ only involve either x or y.

This allows the usage of separable of variables:

We look for the solution of product form

$$V(x, y) = X(x)\, Y(y) \quad \text{--- (37)}$$

So that BCs (i) & (ii) impose on X & Y separately: $Y(0) = Y(a)$

Note that the overall solution may not satisfy eq. (37)

We look for generic solutions of the form in eq. (37).

In general, there may exist $V_1, V_2, V_3, \ldots$ that are solutions of the form in eq. (37).
Since $\nabla^2 V_i = 0$, any linear combination

$$V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \cdots$$ is

a solution $\nabla^2 V = 0$ --- (38)

But V is not in the form of eq. (37)!
The superposition, as we shall see, allows V to satisfy $\underline{V(0,y) = V_0(y)}$!

Substituting eq. (37) into eq. (35), we get

$$Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} = 0$$

Dividing the above eq. by V, one find

$$\frac{\frac{d^2 X}{dx^2}}{X} + \frac{\frac{d^2 Y}{dy^2}}{Y} = 0 \quad \text{--- (38)}$$

Now, $\frac{1}{X}\frac{d^2 X}{dx^2}$ is a function of x $(\equiv f(x))$ while $\frac{1}{Y}\frac{d^2 Y}{dy^2}$ " " " " y $(\equiv g(y))$.

∴ We get an equation of the form $f(x) + g(y) = 0$ — (40)

For a fixed x, f(x) is fixed, changing y changes g(y) generally. ∴ Eq. (40) is possible only when g(y) = constant

Similar arguement leads to f(x) = constant as well.

$$\therefore \quad \frac{1}{X}\frac{d^2X}{dx^2} = C_1 \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = -C_1$$

with C_1 being some constant.

We shall set $C_1 = k^2$ (if $C_1 > 0$, $k \in$ real, $C_1 < 0$, k = imaginary).

$$\therefore \quad \frac{d^2X}{dx^2} - k^2 X = 0 \quad \text{--- (41)}$$

$$\frac{d^2Y}{dy^2} + k^2 Y = 0 \quad \text{--- (42)}$$

i.e. X, Y satisfy two ordinary differential equation of 2nd rank.

Hence the partial differential equation reduces.

to solving ordinary differential equations.

This is the advantage of using separation of variables.

Now, the general solutions to eqs. (41) & (42) are

$$\bar{X}(x) = Ae^{kx} + Be^{-kx} \quad --- (43)$$

$$\bar{Y}(y) = C\sin ky + D\cos ky. \quad -- (44)$$

$$\therefore \; V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$$

Until now, we have not specify if C is positive or negative.

This comes from the constraint of BC.

$V(x,0)=0$ $V(x,a)=0$ Clearly imply that we can require

$$Y(0) = Y(a) = 0 \quad --- (45)$$

Hence Y is oscillatory along y. This is possible only when C_1 is positive.

Now, $Y(0)=0$ implies $D=0$

$Y(a)=0$ implies $\sin ka = 0$ $\therefore k = \frac{n\pi}{a}$ $n = 1,2,3 \dots$

(assuming $k>0$, $k<0$ gives the same solution)

Futher requirement from (iv) implies

$A=0$ ($k>0$; for $k<0$, $B=0$)

$\therefore$ one gets a solution of

the form $U_n(x,y) = C e^{-\frac{n\pi}{a}x} \sin\frac{n\pi}{a}y$ --- (46)

with $n = 1, 2, 3, \ldots$

However,

each individual solution V_n does not

satisfy (iii) $V(0,y) = V_0(y)$

Clearly, as implied by eq. (28), one can

try to form a linear combination of

the form

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi}{a}x} \sin\frac{n\pi}{a}y \quad \text{--- (47)}$$

Then the question is if

$$V(0,y) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi y}{a} = V_0(y) \quad \text{--- (48)}$$

can be satisfied?

This is guaranteed if one recognizes it as

the Fourier Series!

As long as $V_0(y)$ has a finite number of discontinuities, the expansion is guaranteed!

To determine C_n, one has first to notice an important relation:

$$\int_0^a \sin\frac{n\pi y}{a}\sin\frac{n'\pi y}{a}\,dy$$

$$= \frac{1}{2}\int_0^a \left[\cos\left(\frac{n-n'}{a}\pi y\right) - \cos\left(\frac{n+n'}{a}\pi y\right)\right] dy$$

$$\because \int_0^a \cos\frac{m}{a}\pi y\,dy = \frac{a}{\pi}\int_0^\pi \cos mz\,dz$$

$$= \begin{cases} 0 & m \neq 0 \\ a & m = 0 \end{cases}$$

$$\therefore \int_0^a \cos\frac{n+n'}{a}\pi y\,dy = 0 \qquad (\because n, n' > 0)$$

$$\int_0^a \sin\frac{n\pi y}{a}\sin\frac{n'\pi y}{a}\,dy = \begin{cases} 0 & n \neq n' \\ \frac{a}{2} & n = n' \end{cases} \quad \text{i.e.} = \frac{a}{2}\delta_{n,n'} \quad \text{--- (49)}$$

Eq. (49) is so-called orthogonal relation.

It's similar to the orthogonality for basis vectors of $\hat{e}_1, \hat{e}_2, \hat{e}_3$: $\hat{e}_n \cdot \hat{e}_{n'} = \delta_{nn'}$ --- (50)

(orthonormal!)

Eq. (50) is similar to Eq. (48) if one denotes the component of $\hat{e}_n$ by $e_n(i)$ $i = x, y, z$, eq. (50) becomes

$$\sum_i e_n(i)\, e_{n'}(i) = \delta_{nn'} \quad \cdots \text{(51)}$$

$\therefore$ i plays the role of y in in eq. (48)

$\underline{\hat{e}_n(i)}$ plays the role of $\underline{\sqrt{\frac{2}{a}} \sin \frac{n\pi}{a} y}$.

Given a vector $\vec{v}$, $\vec{v} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3$ $\cdots$ (52)
is generally true. To find v_n,
one uses (50) and form $\hat{e}_n \cdot \vec{v}$ in eq. (52):
The orthonormal relation (eq. (51))
enables one to write down

$$v_n = \vec{v} \cdot \hat{e}_n \quad \cdots \quad \text{(53)}$$

Eq. (48) is similar to eq. (52) with C_n playing the same role as v_n.
Hence to find C_n, the trick (Fourier trick) is to multiple eq. (48) by $\sin \frac{n'\pi}{a} y$ and integrate from 0 to a:

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin\frac{n\pi y}{a} \sin\frac{n'\pi y}{a}\, dy = \int_0^a V_0(y) \sin\frac{n'\pi}{a} y\, dy$$

L. (54)

The orthogonal relation picks $n = n'$ on the RHS :

$$C_{n'} \cdot \frac{a}{2} = \int_0^a V_0(y) \sin\frac{n'\pi y}{a}\, dy \quad \text{--- (55)}$$

Hence

$$C_n = \frac{2}{a}\int_0^a V_0(y) \sin\frac{n\pi y}{a}\, dy \quad \text{--- (56)}$$

can be found.

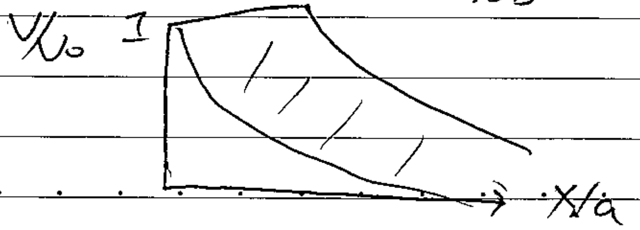
As an example, $V_0(y) = V_0 = \text{constant}$.

$$C_n = \frac{2V_0}{a}\int_0^a \sin\frac{n\pi y}{a}\, dy = \frac{2V_0}{n\pi}\left(-\cos\frac{n\pi y}{a}\right)\Big|_0^{y=a}$$

$$= \frac{2V_0}{n\pi}\left(1 - \cos n\pi\right)$$

$$= \begin{cases} 0 & n = \text{even} \\ \dfrac{4V_0}{n\pi} & n = \text{odd} = 1, 3, 5 \cdots \end{cases}$$

$$\therefore V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\cdots} \frac{1}{n} e^{-n\pi x/a} \sin\frac{n\pi y}{a} \quad \text{--- (57)}$$

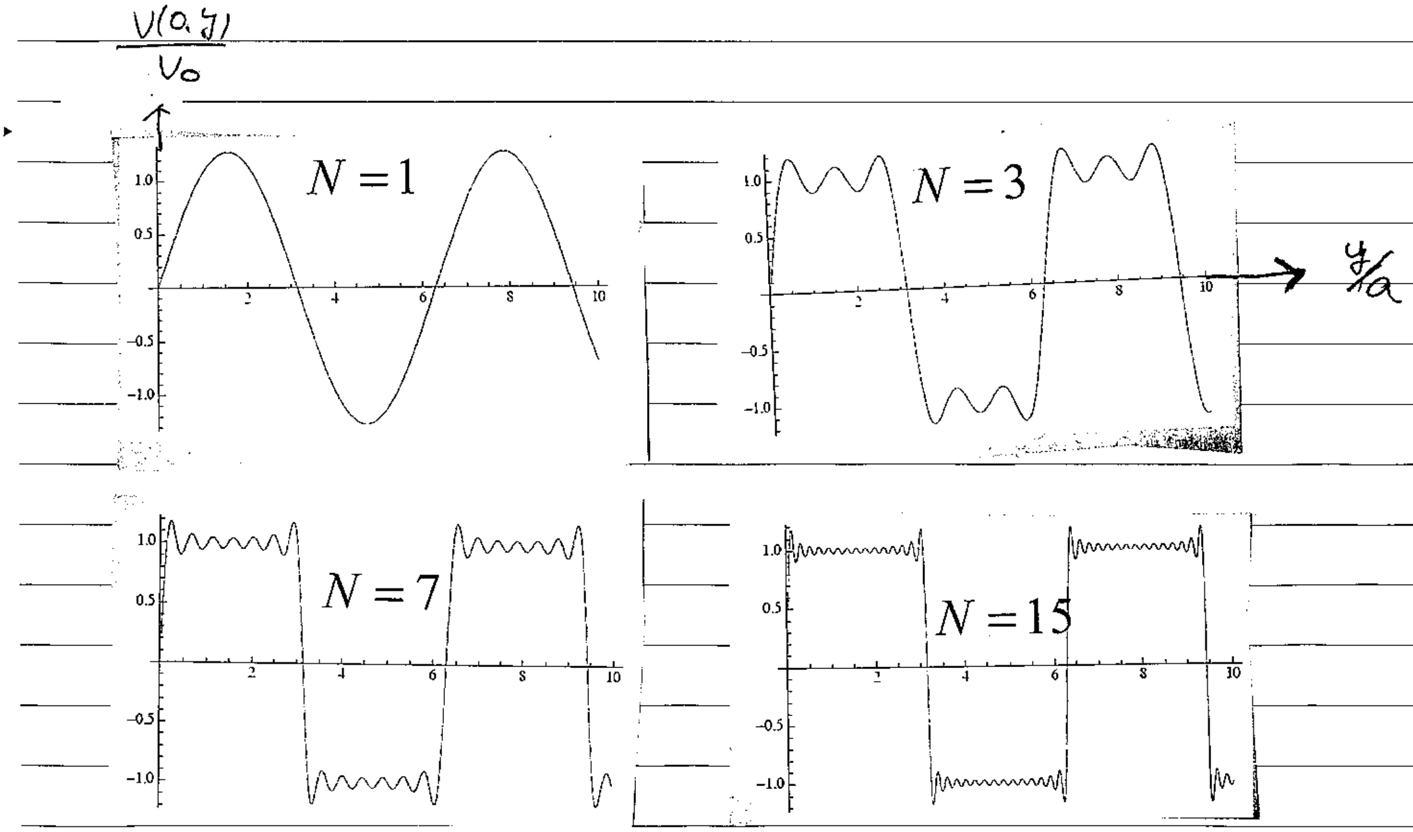


One can check the convergence of the Fourier series at $x=0$.

$$V(0,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5} \frac{1}{n} \sin\frac{n\pi y}{a}$$

$$= \frac{4V_0}{\pi}\left(\sin\frac{\pi y}{a} + \frac{1}{3}\sin\frac{3\pi y}{a} + \frac{1}{5}\sin\frac{5\pi y}{a} + \cdots \right)$$

Using mathematica, one sees that $\frac{V(0,y)}{V_0}$ indeed converges to 1 !



Eq.(57) can be resummed (exercise). One has

$$V(x,y) = \frac{2V_0}{\pi} \tan^{-1}\left(\frac{\sin\frac{\pi y}{a}}{\sinh\frac{\pi x}{a}} \right) \quad \cdots (58)$$

Remarks on completeness & orthogonality :

* The success of method of separation of variable lies on the existence of a set of functions $\{f_n(y)\}$ which are complete and orthogonal to each other:

Complete : if any other function $f(y)$ can be expressed as a linear combination of $f_n(y)$ as

$$f(y) = \sum_{n=1}^{\infty} C_n f_n(y) \quad -- (59)$$

$\{f_n\}$ is said to be complete.

$\{\sin \frac{n\pi y}{a}, n=1,3 \ldots\}$ are complete in the interval $0 \le y \le a$

orthogonal : $\{f_n\}$ is said to be orthogonal if $\int_0^a f_n(y) f_{n'}(y) dy = 0$ for $n \ne n'$

orthonormal : if $\int_0^a f_n(y) f_{n'}(y) dy = \delta_{nn'}$

Eigenfunctions : Eq. (42) $\frac{d^2 \bar{Y}}{dy^2} + k^2 \bar{Y} = 0$ with k^2 to be determined by BC is said to be an eigenvalue problem with k^2 being the

eigenvalue and the corresponding functions
$\left(k = \frac{n\pi}{a},\ n = 1, 3, 5 \ldots\right)$

$\left(\sin \frac{n\pi}{a} y,\ n = 1, 3, \ldots\right)$ being eigenfunctions.

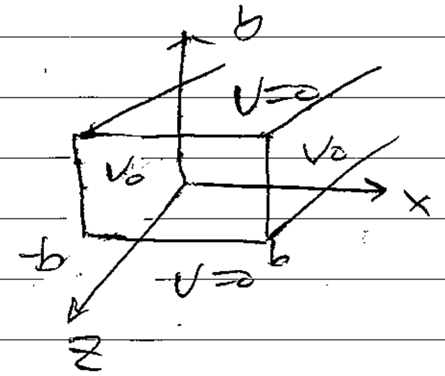
It can be shown that with appropriate conditions, eigenfunctions are complete and orthonormal to each other!

This is the deep reason why method of separation of variables works.

More example

Two infinitely-long grounded metal plates at $y=0$ & $y=a$ are connected at $x=\pm b$ by metal strips maintained at constant potential V_0.

Find V inside the rectangular pipe



solution: V independent of z

$$\therefore \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

BC (i) $V(x, 0) = 0$

(ii) $V(x, a) = 0$

(iii) $U(b,y) = V_0$

(iv) $U(-b,y) = V_0$

Eqs (43) & (44) still work.

$$\therefore U(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$$

However, one can't set $A=0$ as x does not go to ∞ and $U(x\to\infty) \to 0$ is not required!

Nonetheless, one recognizes that the problem is symmetric with respect to $\pm x$

$\therefore A = B$ $\therefore$ $U = \cosh kx\,(C\sin ky + D\cos ky)$ --- (60)

$A=B$ absorbed into C & D

Now (i) implies $D=0$.

(ii) " $k = \frac{n\pi}{a}$

$$\therefore U(x,y) = C\cosh\frac{n\pi}{a}x\,\sin\frac{n\pi}{a}y \quad \text{--- (61)}$$

Now U is even in x $\therefore$ U needs to satisfy either (iii) or (iv) by superposition

$$V(x,y) = \sum_{n=1}^{\infty} C_n\cosh\frac{n\pi x}{a}\sin\frac{n\pi}{a}y \quad \text{--- (62)}$$

$\because\ V(b, y) = V_0$

$$\therefore\quad V_0 = \sum_{n=1}^{\infty} C_n \cosh \frac{n\pi b}{a} \sin \frac{n\pi y}{a} \quad \text{--- (63)}$$

This is the same problem as solving eq. (48)

with $V_0(y) = V_0$ & $C_n \cosh \frac{n\pi b}{a} = C_n$ in

eq. (48)

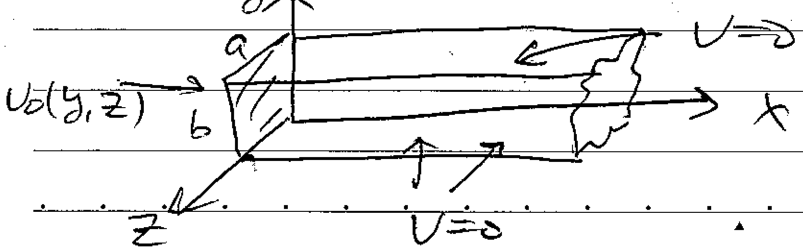
$\therefore$ We conclude

$$C_n \cosh \frac{n\pi b}{a} = \begin{cases} 0 & n = \text{even} \\ \frac{4V_0}{n\pi} & n = \text{odd} = 1, 3, 5, \ldots \end{cases}$$

$$\therefore\ V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5} \frac{1}{n} \frac{\cosh \frac{n\pi x}{a}}{\cosh \frac{n\pi b}{a}} \sin \frac{n\pi y}{a} \quad \text{--- (64)}$$

<u>Example</u> An infinitely long rectangular metal pipe (sides a and b) is grounded but one end at $x=0$ is maintained at a specified potential $V_0(y, z)$.

Find V inside the pipe



Solution: In this case, there is no translationally invariant direction any more. One needs to solve

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}=0 \quad \cdots \quad (65)$$

With BCs (i) $U(x,0,z)=0$

(ii) $U(x,a,z)=0$

(iii) $U(x,y,0)=0$

(iv) $U(x,y,b)=0$

(v) $U\to 0$ as $x\to\infty$

(vi) $U(0,y,z)=U_0(y,z)$

Try separation of variable.

$$U(x,y,z)=X(x)\,Y(y)\,Z(z)$$

Substituted in eq. (65) and divide it by U, one gets

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\cdot\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0$$

Hence $\frac{1}{X}\frac{d^2X}{dx^2}=C_1 \quad \frac{1}{Y}\frac{d^2Y}{dy^2}=C_2$

$\frac{1}{Z}\frac{d^2Z}{dz^2}=C_3$ with $C_1+C_2+C_3=0$

Since U is not periodic in x direction, from the above experience, one sets $C_1 > 0$ & C_2 & $C_3 < 0$

$\therefore$ One can set $C_2 = -k^2$, $C_3 = -l^2$

$$C_1 = -(C_2 + C_3) = k^2 + l^2 > 0$$

Therefore, one gets

$$\frac{d^2 X}{dx^2} = (k^2 + l^2) X \quad \text{--- (66)}$$

$$\frac{d^2 Y}{dy^2} = -k^2 Y \quad \text{--- (67)}$$

$$\frac{d^2 Z}{dz^2} = -l^2 Z \quad \text{--- (68)}$$

Hence one obtains

$$X = A e^{\sqrt{k^2+l^2}\,x} + B e^{-\sqrt{k^2+l^2}\,x}$$

$$Y = C \sin ky + D \cos ky$$

$$Z = E \sin lz + F \cos lz$$

BC (v) implies $A = 0$

(i) " $D = 0$ (ii) implies $k = \frac{n\pi}{a}$

(iii) " $F = 0$ (iv) " $l = \frac{m\pi}{b}$ $n, m = 1, 2, 3 \ldots$

$$\therefore U(x, y, z) = C e^{-\pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}\, z} \sin \frac{n\pi y}{a} \sin \frac{m\pi z}{b} \quad \text{--- (69)}$$

The most general solution is a linear combination of the one given in eq. 69.

One has

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\, e^{-\pi\sqrt{(\frac{n}{a})^2+(\frac{m}{b})^2}\,x} \sin\frac{n\pi y}{a} \sin\frac{m\pi z}{b} \quad \text{--- (70)}$$

which needs to satisfy (vi)

$$\therefore V(0,y,z) = V_0(y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \sin\frac{n\pi y}{a} \sin\frac{m\pi z}{b} \quad \text{--- (71)}$$

Using the orthogonal relation, but now we have two sinusoidal functions, multiply eq. 71 by $\sin\frac{n'\pi}{a}y \sin\frac{m'\pi}{b}z$ and integrate, we get

$$\int_0^a\int_0^b V_0(y,z) \sin\frac{n'\pi}{a}y \sin\frac{m'\pi}{b}z \, dy\, dz$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \int_0^a \sin\frac{n\pi y}{a} \sin\frac{n'\pi}{a}y \, dy$$

$$\times \int_0^a \sin\frac{m\pi z}{a} \sin\frac{m'\pi}{a}z \, dz$$

$$= C_{n'm'} \cdot \left(\frac{a}{2}\right)\left(\frac{b}{2}\right)$$

$$\therefore C_{n,m} = \frac{4}{ab}\int_0^a\int_0^b V_0(y,z) \sin\frac{n\pi y}{a} \sin\frac{m\pi z}{b}\, dy\, dz \quad \text{--- (72)}$$

As an example, $V_0(y,z) = V_0 = $ constant

$$C_{n,m} = \frac{4V_0}{ab}\int_0^a \sin\frac{n\pi y}{a}\,dy \int_0^b \sin\frac{m\pi z}{b}\,dz$$

$$\begin{cases} = 0 & \text{if } n \text{ or } m = \text{even} \\ = \dfrac{16V_0}{\pi^2 nm} & \text{if } n \,\&\, m \text{ are odd} \end{cases}$$

$$\therefore\ V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{\substack{n,m \\ =1,3,5,\ldots}} \frac{1}{nm} e^{-\pi\sqrt{(\frac{n}{a})^2+(\frac{m}{b})^2}\,x} \sin\frac{n\pi y}{a}\sin\frac{m\pi z}{b}$$

Note that $\because$ $e^{-\pi\sqrt{(\frac{n}{a})^2+(\frac{m}{b})^2}\,x}$ decays quickly as $n, m \uparrow$ $\therefore$ a reasonable approximation is obtained by keeping only the first few terms.

Spherical Coordinates

In the examples shown in the above, the boundaries are planes so that one can adopt Cartesian coordinates. If the boundaries are round, spherical coordinates are more natural. In this case, as we have shown

No.
Date.

$$\nabla V = \underbrace{\frac{\partial V}{\partial r}}_{v_r}\hat{r} + \underbrace{\frac{1}{r}\frac{\partial V}{\partial \theta}}_{v_\theta}\hat{\theta} + \underbrace{\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}}_{v_\phi}\hat{\phi}$$

and $\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_r)$

$$+ \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$\therefore \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)$$

$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} \quad --- (73)$$

The general solutions to eq. (73) are known as spherical harmonic functions.

We shall consider special cases when the problem has azimuthal symmetry so that V is independent of ϕ.

$\therefore \frac{\partial^2 V}{\partial\phi^2} = 0$, one gets

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0 \quad -- (74)$$

We seek the solution of the form

$$V(r,\theta) = R(r)\,\Theta(\theta) \quad -- (75)$$

Substituting eq. (75) into eq. (74) and dividing it by V, One gets

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0$$

L - (76)

∵ The first term depends only r and the
2nd term " only θ.

∴ The first term must be a constant C

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = C \quad \text{--- (77)}$$

$$\therefore \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -C \quad \text{-- (78)}$$

Eq. (77) can be rewritten as

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - CR = 0 \quad \text{-- (79)}$$

which is a 2nd Ordinary differential equation
(ODE).

It has a specifical property: degree (power)
of each term is the same.

if $R = r^n$, $\frac{d^2R}{dr^2} = n(n-1)\, r^{(n-2)}$ $\quad r^2\frac{d^2R}{dr^2} = n(n-1) r^n$

$\frac{dR}{dr} = n r^{n-1}$ $\quad 2r\frac{dR}{dr} = 2n r^n$

$CR = C r^n$

Therefore, $C = n(n-1) + 2n = n(n+1)$ ---- (80)

$\because$ any $R(r)$ can be expanded in the form $\sum_n a_n r^n$ $\therefore$ eq. (80) is generally true.

There are two possible solution for a given $C = n(n+1)$: r^n

or $\frac{1}{r^{n+1}}$ $[-(n+1)][-(n+1)+1] = (-n)(-(n+1)) = C$

Chang the notation $n \to \ell$,

$$\therefore R(r) = A r^{\ell} + \frac{B}{r^{\ell+1}} \quad \ell = 0, 1, 2, 3 \ldots \quad (81)$$

With $C = \ell(\ell+1)$, eq. (78) becomes

$$\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\sin\theta\,\Theta = 0 \quad \text{---} (82)$$

It's convenient to change variables by setting $x = \cos\theta$

$$\frac{d\Theta}{d\theta} = \frac{d\Theta}{dx}\frac{dx}{d\theta} = -\sin\theta\frac{d\Theta}{dx} = -\sqrt{1-x^2}\,\frac{d\Theta}{dx}$$

Eq. (82) becomes

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + \ell(\ell+1)\,\Theta = 0$$

$$\ell = 0, 1, 2, 3, \ldots \quad \text{--- } (83)$$

Eq. (83) is the Legendre's equation.

Which is a 2nd ODE.

Any general 2nd ODE can be written

in the form

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)\, y = f(x) \quad \text{--- (84)}$$

It can be shown that

the general solution $y(x)$ can be generally

expressed as

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + y_p$$

Where y_1 & y_2 satisfy the homogeneous

solution $$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \quad \text{--- (85)}$$

and are independent from each such that

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0$$

(Wroskian)

y_p is a partiaclar solution found by whatever

method and satisfies

$$\frac{d^2y_p}{dx^2} + a\frac{dy_p}{dx} + b\, y_p = f \quad \text{--- (86)}$$

For Eq. (83), it's homogeneous. There are two solutions for every l: $\Theta_1(x)$, $\Theta_2(x)$.

It turns out that one of the solutions blow up at $\theta=0$ & $\theta=\pi$ ($x=1$ & -1)

For instance, $l=0$,

$\Theta_1(x) = \text{Const}$ is an obvious solution.

A second solution satisfies $\sin\theta \dfrac{d\Theta}{d\theta} = \text{const} = C$

$$\therefore \frac{d\Theta}{d\theta} = \frac{C}{\sin\theta}$$

$$\Theta = \int \frac{C}{\sin\theta} d\theta = C\int \frac{d\theta}{2\sin\theta/2 \cos\theta/2}$$

$$= C\int \frac{d\theta/2}{\sin\theta/2 / \cos\theta/2 \ \cos^2\theta/2} = C\int \frac{\sec^2\theta/2 \ d\theta/2}{\tan\theta/2}$$

$$= C\int \frac{d\tan\theta/2}{\tan\theta/2} = C \ln \tan\theta/2$$

$\therefore$ $\Theta_2 = \ln\tan\theta/2$ which blows up at $\theta=0$ & $\theta=\pi$

The solution that does not blow up is known as Legendre polynomials

$$\Theta(x) = P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \quad (87)$$

(Rodrigue formula) □

which is a polynomial of degree being ℓ.

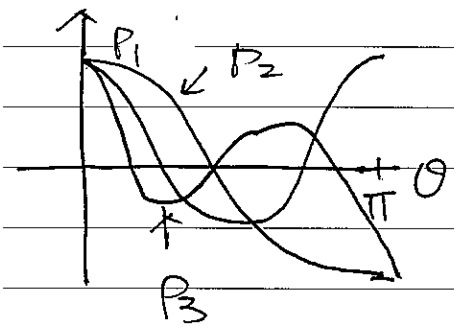
Furthermore, $\because (x^2-1)^\ell = (x+1)^\ell (x-1)^\ell$

At $x=1$ only $\frac{d^\ell}{dx^\ell}$ acts on $(x-1)^\ell$ survives

One finds $\left.\frac{d^\ell}{dx^\ell}(x-1)^\ell\right|_{x=1} = \ell!$

$\therefore P_\ell(1) = 1$

The first few Legendre polynomials are



$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2}(5x^3-3x)$

$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

The Legendre polynomials is an example of set of functions that are complete & orthogonal. They satisfy

$$\int_{-1}^{1} dx\, P_n(x) P_m(x) = \frac{2}{2n+1} \quad n=m$$
$$= 0 \quad n\neq m$$

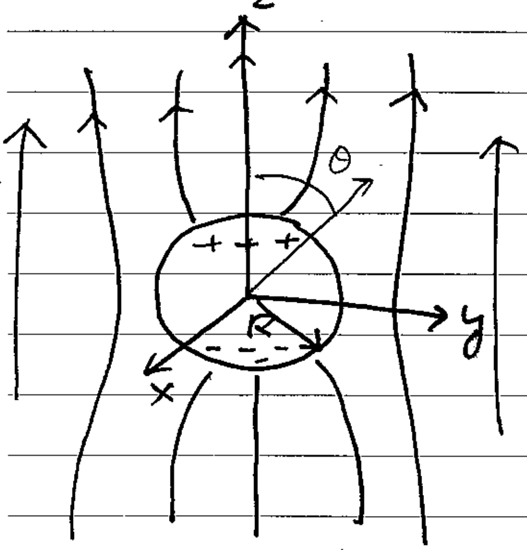
88

Altogether, we have that the general solution is

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta) \quad \text{--- (89)}$$

where A_l & B_l are fixed by boundary conditions using eq. (88).

Example: Conducting sphere in a uniform Electric field



An uncharged metal sphere of radius R is placed in a uniform electric field $\vec{E} = E_0 \hat{z}$

Find the potential outside the sphere and the charge density on the sphere.

Solution: The potential outside the sphere is the summation of original U and U_{in} due to the induced charge density on the sphere. By symmetry, U_{in} doesn't depend on on ϕ ∴ The potential is azimuthal symmetric
$V = V(r,\theta)$.

Far from the sphere, $U_m \to 0$,

U is solely due to $E_0 \hat{z}$

$\because \vec{E} = -\nabla U$

$\therefore E_0 = -\frac{\partial U}{\partial z}$ as $r \to \infty$, $z \to \infty$

$\therefore U = -E_0 z +$ Constant as $z \to \pm\infty$

By symmetry, U is symmetric w.r.t. $z=0$.

and there is (E_x, E_y) in xy plane

$\therefore$ xy plane is an equipotential plane.

If we set the constant voltage on the metal sphere to zero, $U(z=0) = 0$

(surface of metal sphere must be equipotential).

Hence Constant $= 0$

$\because z = r\cos\theta$

$\therefore$ BCs are (i) $V = 0$ when $r = R$

(ii) $V \to -E_0 r\cos\theta$ for $r \geq R$ --- (90)

Now, the general solution of U is given by (88). To satisfy (i), one gets

$$0 = \sum_{l=0}^{\infty} \left(A_l R^l + \frac{B_l}{R^{l+1}}\right) P_l(\cos\theta) \quad \text{--- (91)}$$

Now, one multiply eq. (P1) by $P_n(\cos\theta)$

and integrate $\int_{-1}^{1} d\cos\theta$.

Using eq. (P1), only $l = n$ contributes.

One get $\frac{2}{2n+1}\left[A_l R^l + \frac{B_l}{R^{l+1}}\right] = 0$

$$\therefore B_l = -A_l R^{2l+1}$$

$$\therefore V(r,\theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}}\right) P_l(\cos\theta)$$

└ (92)

A_l is to be fixed by using (ii)

$\because r \gg R$, $V \to -E_0 r\cos\theta$, $\frac{1}{r^{l+1}} \to 0$

$\therefore$ Eq. (92) reduces to $\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta$

$\therefore$ Only $A_1 \neq 0$ $(l=1)$, for other l, $A_l = 0$

$\because P_1(\cos\theta) = \cos\theta$, $\therefore A_1 = -E_0$

we obtain $\rightarrow V_{in} = \frac{E_0 R^3}{r^2}\cos\theta$

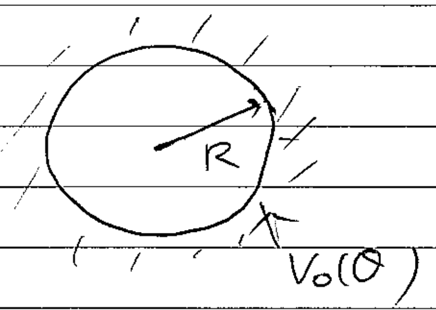
$$\therefore V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta \quad \text{--- (93)}$$

The surface charge density is given by

$$\sigma(\theta) = -\varepsilon_0 \frac{\partial V}{\partial r}\Big|_{r=R} = \varepsilon_0 E_0\left(1 + 2\frac{R^3}{r^3}\right)\cos\theta\Big|_{r=R}$$

$$= 3\varepsilon_0 E_0 \cos\theta \quad \text{--- (94)}$$

Example : The potential $V_0(\theta)$ is given on the surface of a hollow sphere of radius R. Find the potential inside the sphere



Solution. $V_0(\theta)$ is azimuthal symmetric (doesn't depend on ϕ)

$\therefore$ $V = V(r,\theta)$ for $r \leq R$ with B.C. $V(R,\theta) = V_0(\theta)$

$\because$ $V(r=0,\theta)$ is well-defined.

$\therefore$ All $B_\ell = 0$ in eq. 89

$$\therefore \quad V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \quad \text{--- 95}$$

$$V_0(\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) \quad \text{--- 96}$$

Multiplying eq. 96 by $P_n(\cos\theta)$ and integrate, one gets (using eq. 88)

$$A_n R^n \times \frac{2}{2n+1} = \int_0^\pi \sin\theta d\theta \, V_0(\theta) P_n(\cos\theta)$$

$$\therefore A_n = \frac{2n+1}{2R^n} \int_0^\pi V_0(\theta) P_n(\cos\theta) \sin\theta \, d\theta \quad \text{--- 97}$$

Therefore, once $V_0(\theta)$ is known, $V(r,\theta)$ can be found.

For instance, if $V_0(\theta) = k \sin^2 \theta/2$,

$$\because V_0(\theta) = \frac{k}{2}(1-\cos\theta) = \frac{k}{2}(P_0(\cos\theta) - P_1(\cos\theta))$$

$$\therefore A_n = \frac{2n+1}{2R^n}\int_{-1}^{1} dx\, \frac{k}{2}\left[P_0(x) - P_1(x)\right] P_n(x)$$

$$= \frac{2n+1}{2R^n}\cdot\frac{k}{2}\left[\delta_{n,0}\cdot 2 - \delta_{n,1}\frac{2}{2\cdot 1+1}\right]$$

$\therefore$ Only $A_0 = k/2$ & $A_1 = -k/2$ survive.

Hence $V(r,\theta) = \frac{k}{2}\left(r^0 P_0(\cos\theta) - \frac{r^1}{R} P_1(\cos\theta)\right)$

$$= \frac{k}{2}\left(1 - \frac{r}{R}\cos\theta\right)$$

Example. The surface charge density on a sphere shell of radius R is $\sigma_0(\theta) = k\cos\theta$. Find the potential inside and outside the sphere.

Solution: method (i): direct integration

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma_0(\theta')}{|\vec{r}-\vec{r}'|} da'$$

(ii) Use separation of variable.

$V = V(r,\theta)$. (azimuthal symmetry)

$r \le R$ $\because V$ well-defined at $r=0$ $\therefore B_\ell = 0$

$$\therefore V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \quad --- (98)$$

$r \ge R$, $\because V \to 0$ as $r \to \infty$, $\therefore A_\ell = 0$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \quad --- (99)$$

The above two solutions must be joined at $r=R$ by appropriate boundary conditions:

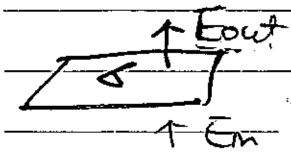
<u>V is continuous</u>

$$\therefore \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta) \quad --- (100)$$

$$\therefore A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}}, \quad B_\ell = A_\ell R^{2\ell+1} \quad --- (101)$$

<u>Surface charge $\sigma_0(\theta)$</u>

$$\left.\frac{dV_{out}}{dr} - \frac{dV_{in}}{dr}\right|_{r=R} = -\frac{1}{\varepsilon_0}\sigma_0(\theta)$$



$$\because (\vec{E}_{out} - \vec{E}_{in}) \cdot \hat{n} = \frac{\sigma}{\varepsilon_0}$$

$$\therefore -\sum_{\ell=0}^{\infty} (\ell+1) \frac{B_\ell}{R^{\ell+2}} P_\ell(\cos\theta) - \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos\theta)$$

$$= \frac{-1}{\varepsilon_0}\sigma_0(\theta) \quad --- (102)$$

Using eq.(101), eq.(102) becomes

$$\sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l(\cos\theta) = \frac{1}{\varepsilon_0}\sigma(\theta) \quad \text{--- (103)}$$

Multiplying eq.(103) by $P_n(\cos\theta)$ and integrate, one gets

$$(2n+1) A_n R^{n-1} \cdot \frac{2}{2n+1} = \frac{1}{\varepsilon_0}\int_0^{\pi} \sigma(\theta) P_n(\cos\theta) \times \sin\theta \, d\theta$$

$$\therefore A_n = \frac{1}{2\varepsilon_0 R^{n-1}} \int_0^{\pi} \sigma(\theta) P_n(\cos\theta) \sin\theta \, d\theta \quad \text{--- (104)}$$

For $\sigma_0(\theta) = K\cos\theta = K P_1(\cos\theta)$

$$\int_0^{\pi} \sigma_0(\theta) \, P_n(\cos\theta) \sin\theta \, d\theta$$

$$= K \int_{-1}^{1} dx \, P_1(x) \, P_n(x)$$

$$= K \frac{2}{2\cdot 1+1} \delta_{n,1}$$

$\therefore$ Only $A_1 = \frac{1}{2\varepsilon_0} \cdot \frac{2}{3} K$ does n't vanish.

$$\because B_1 = A_1 R^3 = \frac{K}{3\varepsilon_0} R^3$$

$$\therefore U(r,\theta) = \frac{KR^3}{3\varepsilon_0} \frac{1}{r^2}\cos\theta \qquad r > R,$$

$$= \frac{K}{3\varepsilon_0} r\cos\theta \qquad r \leq R.$$

Take $K=3\epsilon_0 E_0$, $\sigma(\theta)$ is the surface charge density induced on the metal sphere by $\vec{E}=E_0\hat{z}$

In this case $V_{in}(r,\theta)=E_0 \underbrace{r\cos\theta}_{z}$ for $r\le R$

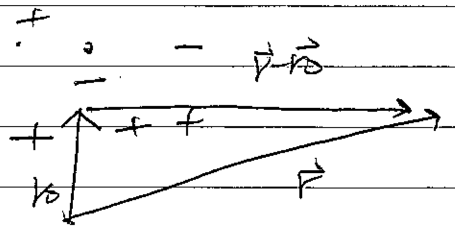
$\therefore \vec{E}_{in} = -\vec{\nabla} V = -E_0\hat{z}$ which cancels $\vec{E}=E_0\hat{z}$ exactly $\therefore \vec{E}_{total}=0$ inside the metal sphere.

For $r\ge R$, $V_{out}=\frac{E_0R^3}{r^2}\cos\theta$ is consistent with eq. (93).

Multipole expansion

In many situations, charges that one considers are localized in space.

In these cases, the most dominant contribution is obviously the net charge Q



$$V(r)\sim \frac{Q}{4\pi\epsilon_0|r-r_0|}$$

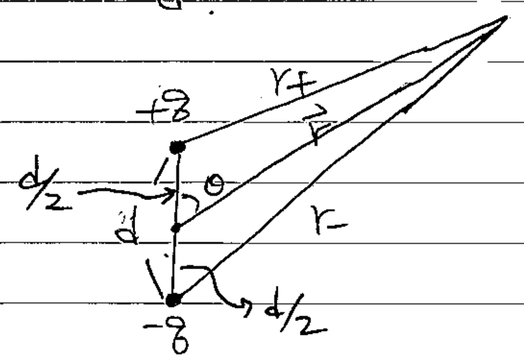
$$Q=\sum_i q_i \quad \text{--- (105)}$$

A natural question arises: what happens if $Q=0$?

Eq. (105) is too crude to answer this question. The appropriate method is the multiplole expansion. We shall first explore a few simple configurations.

The electric dipole

The simplest charge configuration with net charge $=0$ is an electric dipole which consists of two equal but opposite charges ($\pm q$) separted by a distance d.



Let $r_\pm$ be the distance to $\pm q$ respectively

Then

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right) \quad (106)$$

Now,

$$r_\pm^2 = r^2 + (d/2)^2 \mp 2 \cdot r \cdot \frac{d}{2}\cos\theta$$

$$= r^2\left[1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right]$$

$\therefore$ For $r \gg d$, one can neglect $\frac{d^2}{4r^2}$

$$\frac{1}{r_\pm} \simeq \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-\frac{1}{2}} + O\left(\frac{d^2}{r^3}\right)$$

$$(1+x)^{-\frac{1}{2}} = 1 + C^{-\frac{1}{2}}_1 x + C^{-\frac{1}{2}}_2 x^2 + \cdots \simeq 1 + \frac{x}{2}$$

$$\therefore \frac{1}{r_\pm} \simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) + O\left(\frac{d^2}{r^3}\right)$$

$$\frac{1}{r_+} - \frac{1}{r_-} \simeq \frac{d}{r^2}\cos\theta + O\left(\frac{d^4}{r^3}\right) \quad \left(\frac{d^2}{r^3} \text{ get cancelled}\right)$$

Hence, to the leading term

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q d \cos\theta}{r^2} \quad \cdots (107)$$

One defines the dipole moment $\vec{p} = q\vec{d}$

$\vec{d}$ = vector from $-q$ to $+q$.

Eq. (107) can be written as

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} \quad \cdots (108)$$

which decays as $\frac{1}{r^2}$ as $r \to \infty$ (faster than that due to the net charge).

<u>Electric field</u>

The electric field due to an electric dipole can be found by using (in spherical coordinate)

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{\partial V}{r\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\phi}$$

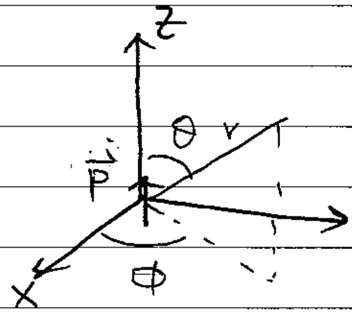
$$\because V = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}$$

$$\therefore \frac{\partial V}{\partial\phi} = 0, \quad \frac{\partial V}{\partial r} = \frac{-2}{4\pi\epsilon_0}\frac{p\cos\theta}{r^3}$$

$$\frac{\partial V}{\partial\theta} = \frac{-1}{4\pi\epsilon_0}\frac{p\sin\theta}{r^2}$$

$$\therefore E_r = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}$$

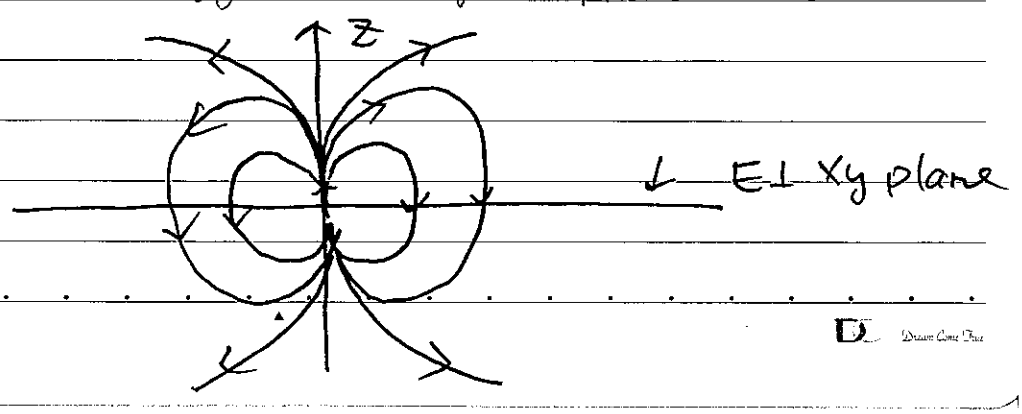
$$E_\theta = \frac{p\sin\theta}{4\pi\epsilon_0 r^3} \qquad E_\phi = 0$$



$$\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right] \quad \text{--- (109)}$$

$\therefore$ For $r \gg d$, E decays as $\frac{1}{r^3}$ as $r \to \infty$.

The following is the configuration for field lines:



Equation for field lines can be found

by solving $\frac{dr}{E_r} = \frac{r d\theta}{E_\theta}$, i.e. $\frac{r d\theta}{dr} = \frac{E_\theta}{E_r}$

$\because \frac{E_\theta}{E_r} = \frac{\sin\theta}{2\cos\theta}$

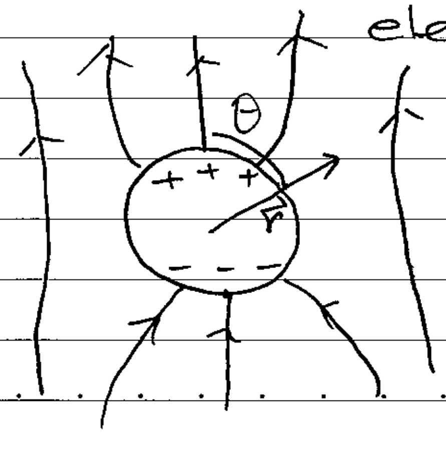
$\therefore \frac{r d\theta}{dr} = \frac{\sin\theta}{2\cos\theta} \quad \therefore \frac{dr}{r} = \frac{2\cos\theta d\theta}{\sin\theta} = \frac{2 d\sin\theta}{\sin\theta}$

$\therefore \ln r = 2\ln\sin\theta + \text{Const.}$

$r = A\sin^2\theta$

The electric dipole is the leading possible configuration of a charge system when the net charge vanishes. One can use it to understand the electric field generated in many systems.

Example Conducting sphere $(V=0)$ in a uniform electric field. $\vec{E} = E_0\hat{z} \quad z \to \pm\infty$



As we have solved using method of separable of variable

$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta$ (eq. 93)

The potential due to induced charge

is $$V_{in}(r,\theta) = E_0 \frac{R^3 \cos\theta}{r^2}$$

which can be viewed as fields

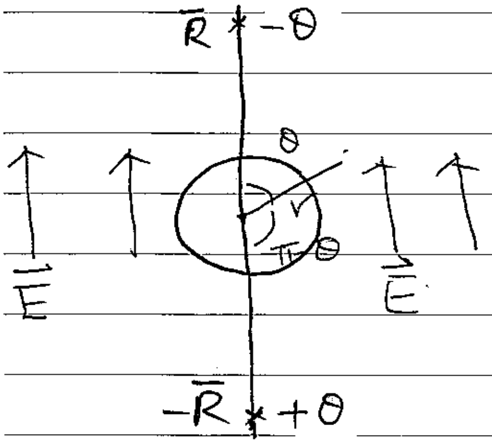
of an electric dipole

$$\vec{p} = 4\pi\epsilon_0 E_0 R^3 \hat{z} \quad \text{--- (110)}$$

$$V_{in}(\vec{r}) = \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2} \quad \text{--- (111)}$$

The dipole picture is clearly consistent with the induced charge distribution that one expects.

<u>Electric dipole</u> & <u>method of image</u>



The uniform electric on the sphere can be generated by putting $\mp Q$ at $\pm\bar{R}$ and take $R\to\infty$

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{\sqrt{\bar{R}^2+r^2-2\bar{R}r\cos(\pi-\theta)}} - \frac{Q}{\sqrt{\bar{R}^2+r^2-2\bar{R}r\cos\theta}}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{\sqrt{\bar{R}^2+r^2+2\bar{R}r\cos\theta}} - \frac{Q}{\sqrt{\bar{R}^2+r^2-2\bar{R}r\cos\theta}}\right]$$

Using the same expansion, one has
(in deriving V for a dipole)

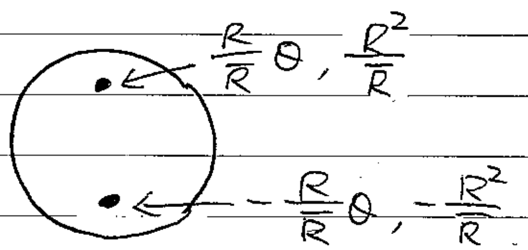
$$\frac{1}{\sqrt{\bar{R}^2 + r^2 \pm 2\bar{R} r\cos\theta}} \cong \frac{1}{\bar{R}} \mp \frac{r}{\bar{R}^2}\cos\theta$$

$$\therefore V(r,\theta) = \frac{-1}{4\pi\epsilon_0}\frac{2Q r\cos\theta}{\bar{R}^2} = -\frac{Q}{2\pi\epsilon_0 \bar{R}^2} z$$

The uniform electric field imposed on the conducting sphere is $E_0 = \frac{Q}{2\pi\epsilon_0 \bar{R}^2}$ --- (112)

Now, according to method of images used for a conducting sphere at $V=0$, the images charges for $\pm Q$ are located at $z = \mp \frac{R^2}{\bar{R}}$ with charges $\mp \frac{R}{\bar{R}} Q$

$\bar{R}$ • Q



$-\bar{R}$ • Q

$\therefore$ The induced electric dipole moment

$$\vec{p} = 2\frac{R^2}{\bar{R}}\hat{z} \times \frac{R}{\bar{R}} Q$$

$$= 2\frac{R^3}{\bar{R}^2} Q \hat{z}$$

Using eq. (112), one obtains

$$\vec{p} = 4\pi\epsilon_0 E_0 R^2 \hat{z}$$

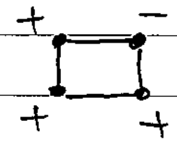
in agreement with eq. (110). Hence the response of a conducting sphere to a uniform $\vec{E}$ generates a dipole.

Generalization of electric dipole

The net charge of an electric dipole $= 0$ and it's potential decays as $\frac{1}{r^2}$, faster than that of a point charge.

The reason is due to cancellation of positive & negative charges for $1/r$:

$$U_{dipole} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} - \frac{q}{r}\right) + \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} \quad (r \to \infty)$$

Similarly, one can put $\pm\vec{p}$ together so that the $\frac{1}{r^2}$ potential gets cancelled at $r \to \infty$. Such a configuration is called a quadrupde $\uparrow\vec{p} \; \downarrow\vec{p} =$ 

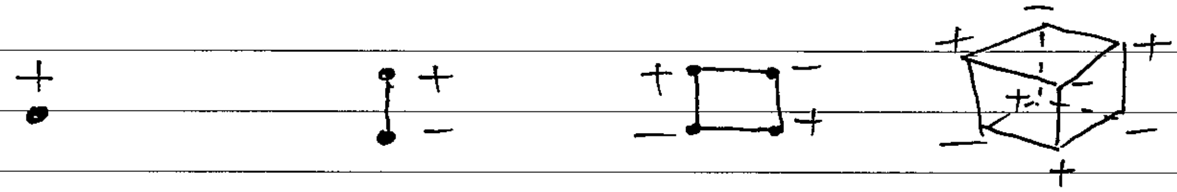
$$V_{quadrapole} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} - \frac{q}{r}\right) + \frac{1}{4\pi\epsilon_0}\left(\frac{\vec{p}\cdot\hat{r}}{r^2} - \frac{\vec{p}\cdot\hat{r}}{r^2}\right)$$

$$+ O\left(\frac{1}{r^3}\right) \quad \text{as } r \to \infty$$

$\therefore V_{quadrupole} \to \frac{1}{r^3}$ as $r \to \infty$

Clearly, this can be further generalized to octopole. We have the following.

Configuration :



monopole	dipole	quadrupole	octopole
$V \sim \frac{1}{r}$	$V \sim \frac{1}{r^2}$	$V \sim \frac{1}{r^3}$	$V \sim \frac{1}{r^4}$.

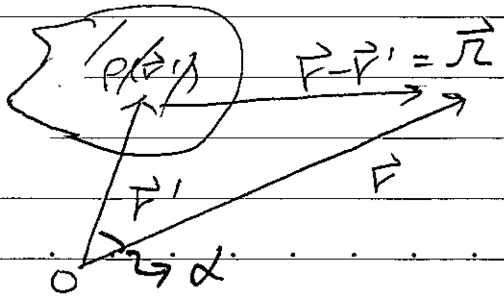
General (localized) charge distribution

The above definitions of dipole, quadrupole & octopole are for special charge configurations.

The definitions can be generalized.

For instance, for a charge distribution, it may possess dipole moment but it may not just consist of $\pm q$ only!

To get general definitions, we start by consider a charge distribution $\rho(\vec{r})$ which is limited to some finite region.



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \, d\tau' \quad \text{--- (113)}$$

For $\vec{r}'$ is confined to limited region,

$|\vec{r}-\vec{r}'| >> r'$ if $r \to \infty$.

Therefore, one may perform Taylor's expansion of $\mathscr{r}$

$$\frac{1}{\mathscr{r}} = \frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} + (\vec{r}'\cdot\vec{\nabla}')\frac{1}{\mathscr{r}}\Big|_{r'=0} + \frac{1}{2!}(\vec{r}'\cdot\vec{\nabla}')^2\frac{1}{\mathscr{r}}\Big|_{r'=0} + \cdots$$

$\llcorner \cdots (114)$

Clearly, the first term gives

$$U_{mon} = \frac{1}{4\pi\epsilon_0}\int \frac{1}{r}\rho(r')d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r}\underbrace{\int \rho(r')d\tau'}_{Q}$$

Which is the contribution of monopole

$$Q = \int \rho(\vec{r}')d\tau'$$

$\because \frac{\partial}{\partial x'}\frac{1}{|\vec{r}-\vec{r}'|} = \frac{\partial}{\partial x'}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} = +\frac{x-x'}{|\vec{r}-\vec{r}'|^3}$

$\therefore \vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|}\Big|_{r'=0} = +\frac{\vec{r}}{r^3} = +\hat{r}/r^2 \quad \cdots (115)$

$\therefore$ The 2nd term in eq. (114) gives

$$U_{dip} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}\cdot\int \rho(\vec{r}')\vec{r}'d\tau' \quad \cdots (116)$$

Which decays as $\frac{1}{r^2}$ and is the generalization of eq. dipole potential.

The generalized dipole moment for $\rho(\vec{r})$

is $\vec{p} \equiv \int \rho(\vec{r}\,')\, \vec{r}\,' d\tau'$ ... (117)

so that $V_{dip} = \frac{1}{4\pi\epsilon_0} \frac{\hat{r}\cdot\vec{p}}{r^2}$

In general, there may have many point charges $q_1, q_2, q_3, ..$ located at $\vec{r}_1\,', \vec{r}_2\,', ...$

$$\vec{p} = \sum_{i=1}^{n} q_i \vec{r}_i\,' \quad ... (118)$$

which reduces to $q \cdot \vec{d}$ when there is only two charges $\pm q$ at $\pm \vec{d}/2$. Hence Eq. (117) is the most general definition of dipole moment.

The expansion in eq. (114) can be done further. However, to be more systematic, it's more convenient to use angle α, via the law of cosines:

$$r^2 = r^2 + r'^2 - 2rr'\cos\alpha$$

$$= r^2\Big[1 + \underbrace{\frac{r'}{r}\left(\frac{r'}{r} - 2\cos\alpha\right)}_{\varepsilon}\Big]$$

For large r, $\varepsilon \ll 1$ ∴ one may expand

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\varepsilon)^{-\frac{1}{2}} = \frac{1}{r}\left[1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \cdots\right]$$

$$\uparrow \; C_1^{-\frac{1}{2}} \qquad \uparrow \; C_2^{-\frac{1}{2}} \qquad \uparrow \; C_3^{-\frac{1}{2}}$$

$$C_n^{-\frac{1}{2}} = \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-(n-1))}{n!}$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\frac{r'}{r}\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\alpha\right)^3 + \cdots\right]$$

Collect terms according $\left(\frac{r'}{r}\right)^n$, one finds

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\alpha + \left(\frac{r'}{r}\right)^2 \frac{3\cos^2\alpha - 1}{2} + \left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\alpha - 3\cos\alpha}{2}\right) + \cdots\right]$$

The coefficient of $(r'/r)^n$ turns out to be the Legendre polynomials. $P_n(\cos\alpha)$!

$$\therefore \; V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\rho(r')d\tau'$$

L. (119)

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(r')d\tau' + \frac{1}{r^2}\int r'\cos\alpha\, \rho(r')d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(r')d\tau' + \cdots\right]$$

L (120)

The first term is the same as the one we got before $(n=0)$

For the second term, $\because$ $r'\cos\alpha = \hat{r}\cdot\vec{r}'$

$\therefore$ It's the same as eq. (17). $(n=1)$

For $n=2$, one obtains

$$V_{quad} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int d\tau' \left[\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right] r'^2 \rho(r') \quad \text{--- (21)}$$

which can be rewritten as

$$V_{quad} = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^5}\int d\tau' \left[3(\vec{r}\cdot\vec{r}')^2 - r'^2\right]\rho(r')$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{2r^5}\sum_{i,j} x_i\, Q_{ij}\, x_j \to \frac{1}{r^3} \text{ as } r\to\infty$$

with $\vec{r} = (x_1, x_2, x_3)$

$$Q_{ij} = \int d\tau' \left(3x_i' x_j' - r'^2\delta_{ij}\right)\rho(r')$$

being the quadrupole momentum tensor (matrix)
(Ex 3.52)

Eq. (20) is the multipole expansion for localized charge distribution, valid for $r \gg$ range of charge distribution.

Example of quadrupole

The quadrupole momentum is generally characterized by a matrix

$$\overleftrightarrow{Q} = \begin{pmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{yx} & Q_{yy} & Q_{yz} \\ Q_{zx} & Q_{zy} & Q_{zz} \end{pmatrix}$$

with $Q_{ij} = Q_{ji}$

$$Q_{xx} = \int d\tau' \rho(\vec{r}') (3x'^2 - r'^2)$$

$$Q_{yy} = \int d\tau' \rho(\vec{r}') (3y'^2 - r'^2)$$

$$Q_{zz} = \int d\tau' \rho(\vec{r}') (3z'^2 - r'^2)$$

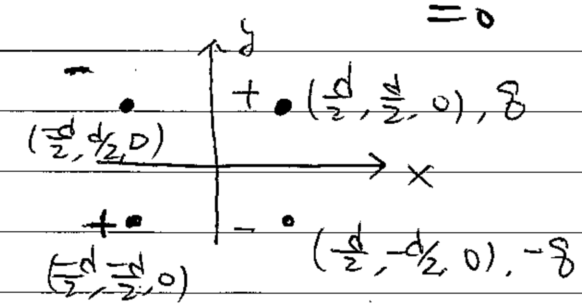
$$Q_{xy} = \int d\tau' \rho(\vec{r}')\, 3x'y'$$

$$Q_{yz} = \int d\tau' \rho(\vec{r}')\, 3y'z'$$

$$Q_{xz} = \int d\tau' \rho(\vec{r}')\, 3x'z'$$

$$\mathrm{Tr}\, Q = Q_{xx} + Q_{yy} + Q_{zz} = \int d\tau' \rho(r') \underbrace{[3r'^2 - 3r'^2]}_{=0}$$

A typical example



$$Q_{xx} = q \cdot [3(\tfrac{d}{2})^2 - 2(\tfrac{d}{2})^2] + (-q)[3(\tfrac{d}{2})^2 - 2(\tfrac{d}{2})^2]$$
$$+ q[3(\tfrac{d}{2})^2 - 2(\tfrac{d}{2})^2] + (-q)[3(\tfrac{d}{2})^2 - 2(\tfrac{d}{2})^2] = 0$$

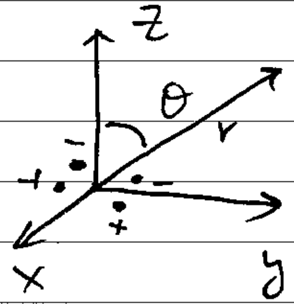
Similarly, $\Theta_{yy}=0$. $\Theta_{zz}=0$

$\because$ $z'=0$ for all charges. $\therefore$ $\Theta_{xz}=\Theta_{yz}=0$

Only $\Theta_{xy}\neq 0$

$$\Theta_{xy} = 3\cdot\left(\frac{d}{2}\right)\left(\frac{d}{2}\right)\times q + 3\left(\frac{d}{2}\right)\left(-\frac{d}{2}\right)(-q) + 3\left(-\frac{d}{2}\right)\left(\frac{d}{2}\right)\times(-q) + 3\left(-\frac{d}{2}\right)\left(-\frac{d}{2}\right)\times q$$

$$= 3dq^2$$



$$U(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^5}\left(xy\cdot 3dq^2\right)$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{3}{2}dq^2\right)\frac{xy}{r^5}$$

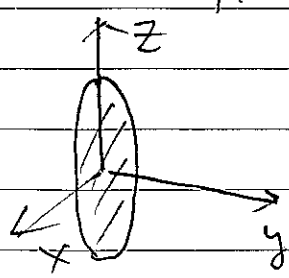
Cylindrical symmetric charge distribution

In general, if the charge distribution is cylindrically symmetric. $Q_{xx}=\Theta_{yy}$, $\Theta_{zz}\equiv\Theta$

$\Theta_{xy}=\Theta_{yz}=\Theta_{zx}=0$

$\because$ $\Theta_{xx}+\Theta_{yy}+\Theta_{zz}=0$

$\therefore$ $\Theta_{xx}=\Theta_{yy}=-\Theta/2$



$$U_{quad} = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^5}\left[x^2\Theta_{xx}+y^2\Theta_{yy}+z^2\Theta_{zz}\right]$$

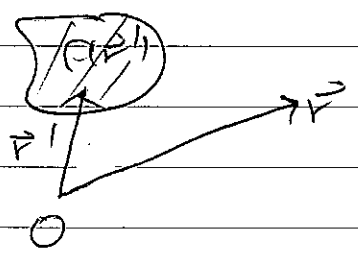
$$= \frac{1}{4\pi\epsilon_0}\frac{1}{2r^5}\left[Q z^2 - \frac{Q}{2}(x^2+y^2)\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q}{4r^5}\left[3z^2 - (x^2+y^2+z^2)\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q}{4r^3}(3\cos^2\theta - 1)$$

$Q = \Theta_{zz}$ is referred as the quadrupole momentum for charge distribution with azimuthal symmetry.

Origin dependence of multipole expansion



The origin O of multipole expansion may not lie inside $\rho(\vec{r}')$.

Hence for monopole, $V = \frac{1}{4\pi\epsilon_0}\frac{Q_{total}}{r}$ the distance is r. That is, Q_{total} is placed at O outside $\rho(\vec{r}')$

This appears to be odd. but this is an artifat of multipole expansion

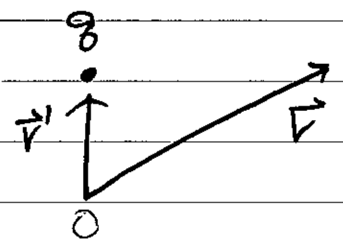
Eespecially, when the monopole Q_{total} doesn't vanish, the terms in multipole expansion do.

not represent real multipoles' contribution.

For example, consider a point charge q at O, this is termed as "pure" monople.

its potential at r is exactly

$$V(r)=\frac{1}{4\pi\epsilon_0}\frac{q}{r}.$$



Hower, for a point charge at $\vec{r}'=d\hat{y}$, the multipole expansion gets

monopole q at O

dipole $\vec{p}=qd\hat{y}$

$\vdots$

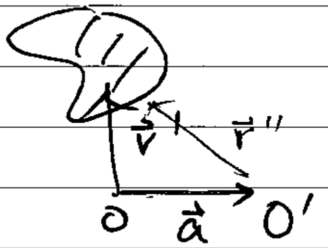
these are just resulted from the expansion of the exact potential $V(\vec{r})=\frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}'|}$ and the expansion depends position of O except the monopole q.

Therefore, the multipole expansion generally depends on the position of origin.

Explicitly, for the dipole moment,

$\vec{p} = \int \rho(\vec{r}\,') \vec{r}\,' d\tau'$, if we shift

O to O'.



relative to O'

$$\vec{p}\,' = \int \rho(\vec{r}\,'') \vec{r}\,'' d\tau''$$

$$= \int \rho(\vec{r}\,') [\vec{r}\,' - \vec{a}] d\tau''$$

$$= \vec{p} - \vec{a} \int \rho(\vec{r}\,') d\tau''$$

$$= \vec{p} - \vec{a} Q$$

$\therefore$ the dipole momentum gets shifted by $-\vec{a} Q$!

Therefore, unless $Q = 0$, the multipole expansion generally depends on coordinates of O, and one has to specify the coordinate before performing the expansion.