Electric fields in Matter

In the previous chapters, the electric fields are mainly discussed in vacuum except $\vec{E}=0$ in perfect conductors.

According to the capability of transport via free electrons, materials are roughly classified as conductors and insulators.

For perfect conductors, there are unlimited supply of free charges.

For insulators, there is no free charge in insulators. All charges are attached to specific atoms or molecules in insulators.

Then, what is the response of insulators in an electric field?

As we shall see, for a neutral material, since there is no monopole, the lowest possible contribution is from the electric dipole.

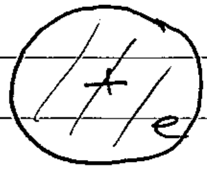
Indeed, insulators are also dielectrics in which dipoles are either induced by external fields or already properties of atoms or molecules.

The electric field is then to stretch the electrons in atoms or rotating dipoles that are already attached in molecules/atoms.

Induced dipoles

To find the response of insulators in $\vec{E}$, one starts from the response of a neutral atom in $\vec{E}$. We shall assume that

When $\vec{E}=0$, the positive charge center (due to nucleus) and negative charge center (due to electrons) coincide so that



$$\vec{P}=\int \rho(\vec{r})\,\vec{r}\,dz$$

$$=\int \rho_+ \vec{r}_+ dz + \int \rho_- \vec{r}_- dz$$

$$= \underset{\substack{\uparrow \\ \text{positive charge center}}}{q\vec{R}_+} - q\vec{R}_- = 0$$

$\leftarrow$ negative charge center

For such atoms,

In the presence of $\vec{E} \neq 0$, $\vec{E}$ pulls electrons and nucleus apart. As a result, electrons experience two forces:

$\overset{F_e}{\longleftarrow}$ $\longrightarrow E$

$-$ $+$

$\overrightarrow{F_+'}$

$(-e)\vec{E}$ & attractive force (F_+') due to positive charges.

Clearly, in equilibrium $-e\vec{E} + \vec{F}_+' = 0$ --- ①

Since $\vec{F}_+'$ is a restoring force, for small separation x, $\vec{F}_+' \propto x$ --- ②

① & ② imply a dipole is induced with the dipole moment $\vec{p} = e\vec{x} \propto \vec{E}$

$\therefore$ $\vec{p} \equiv \alpha \vec{E}$ --- ③

$\alpha \equiv$ atomic polarizability, ($C^2 m/N$, $[qE] = F$) describes the easiness of atom being polarized into a dipole under external field E.

Unit: $[E] = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r^2}\right]$
$[P] = q[r]$
$\left[\frac{\alpha}{4\pi\epsilon_0}\right] = [r^3]$

Example.

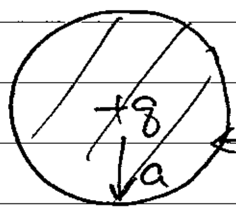
	H	He	Li	C	Cs
$\frac{\alpha}{4\pi\epsilon_0}$ = ($10^{-30} m^3$)	0.667	0.205	24.3	1.67	59.4

Example.

A simple model for a neutral atom

= a point $+q$ charge + an electron cloud $(-q)$ with uniform charge density, $\underbrace{}_{\text{of radius } a}$

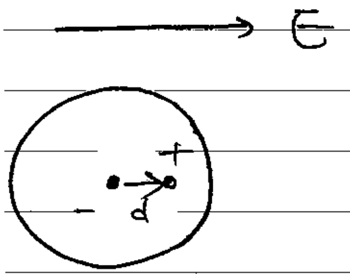
Find $\alpha = ?$



$\rho_e = \frac{(-q)}{\frac{4\pi}{3}a^3}$

Solution: Since nucleus is much heavier, we can assume it to be fixed.

For $\vec{E} \neq 0$, the center of electron clouds moves d.



Relative to the center of cloud, positive charge is at $r=d$

Using Gauss's law, the electric field at $r=d$ is $E_e = \frac{1}{4\pi\epsilon_0} \frac{\frac{4\pi}{3}d^3 \times \rho_e}{d^2}$

$$= \frac{1}{4\pi\epsilon_0} \frac{(-q)d}{a^3}$$

$\therefore$ The force acts on $+q$ due to cloud

$= q E_e$

$\because$ $q E_e + q \underset{(\text{external})}{E} = 0$

$$\therefore \quad E = -E_e = \frac{1}{4\pi\epsilon_0} \frac{qd}{a^3}.$$

$$p = qd = 4\pi\epsilon_0 a^3 E$$

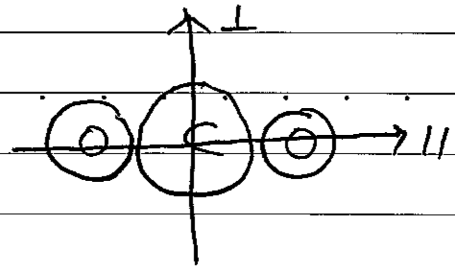
$$\alpha = 4\pi\epsilon_0 a^3$$

which is a crude approximation of $\alpha_{expt.}$ within a factor of 4 for many simple atoms.

Note that nucleus is heavier. What moves is the electron cloud. The force is $-q E_e = \frac{1}{4\pi\epsilon_0} \frac{q^2 d}{a^3}$.

$\underline{\alpha_{ij}}$

For real molecules, the electron cloud may not be isotropic in shape. For instance, for CO_2 molecules,



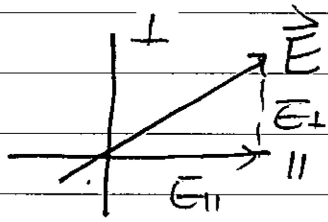
The electron cloud roughly is shown in the left figure

It's anisotropic.

For different directions of E, α's are different.

$\parallel$: $\alpha_{\parallel} = 4.5\times10^{-40}\ C^2m/N$

$\perp$: $\alpha_{\perp} = 2\times10^{-40}$ 〃



For general $\vec{E}$,

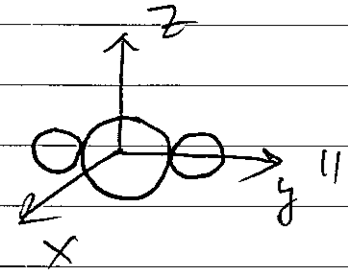
$$\vec{E} = \vec{E}_{\perp} + \vec{E}_{\parallel}$$

$$\vec{p} = \alpha_{\perp}\vec{E}_{\perp} + \alpha_{\parallel}\vec{E}_{\parallel} \quad \cdots \text{④}$$

∴ $\vec{p} \nparallel \vec{E}$ in this case.

If we take ∥ as y axis, ⊥ as x & z axes

Eq. ④ can be written as



$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_{\perp} & 0 & 0 \\ 0 & \alpha_{\parallel} & 0 \\ 0 & 0 & \alpha_{\perp} \end{pmatrix}}_{\overleftrightarrow{\alpha}\ \text{polarizability tensor}} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \quad \cdots \text{⑤}$$

Eq. ⑤ is the case when axes x, y, z are chosen so that CO_2 ∥ axis (symmetric axis of CO_2) is one of x, y, z axes.

In general, x, y, z axes may not coincide with symmetric axes of molecules.
(principal)
One has

$$\vec{p} = \overleftrightarrow{\alpha} \cdot \vec{E} = \underbrace{\begin{pmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{pmatrix}}_{\overleftrightarrow{\alpha}} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

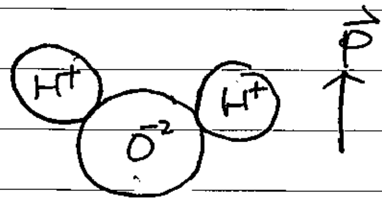
$\overleftrightarrow{\alpha}$ is called polarizability tensor. ⑥

Polar molecules and their response to $\vec{E}$

In the above cases, the dipole moments are induced by $\vec{E}_{ext}$. When $\vec{E}_{ext} = 0$, $\vec{p} = 0$ or simply dielectrics
Materials with this properties are dielectric materials
There are, however, molecules with built-in dipoles even when $\vec{E}_{ext} = 0$

Such molecules are called polar molecules.

For example, the water molecule is

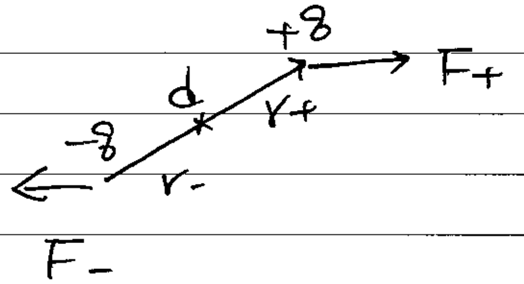


a polar molecule in which + & − charges are separate as a result of chemical bonding

For such molecules in the presence of a uniform $\vec{E}$, the total Force acting on it

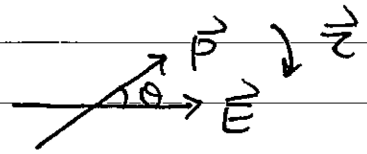
$$\vec{F} = q\vec{E} + (-q)\vec{E} = 0 \quad \cdots (7)$$

However, the torque does not vanish:



$$\vec{\tau} = \vec{r}_+ \times \vec{F}_+ + \vec{r}_- \times \vec{F}_-$$
$$= \frac{\vec{d}}{2} \times q\vec{E} + \left(-\frac{\vec{d}}{2}\right) \times (-q\vec{E})$$
$$= q\vec{d} \times \vec{E}$$
$$= \vec{p} \times \vec{E} \quad \cdots (8)$$

Therefore, $\vec{\tau} \neq 0$. The dipole will rotate in the direction that $\vec{p}$ is trying to be parallel to $\vec{E}$!



If we apply an opposite torque $\vec{\tau}_a = -\vec{p} \times \vec{E}$ so that $\vec{p}$ rotates $d\theta$, the work needs to be done is

$$dw = -\vec{p} \times \vec{E} \cdot d\vec{\theta} \quad (d\vec{\theta} \text{ points out of the plane})$$
$$= pE\sin\theta \, d\theta$$

$$\therefore U(\theta) - U(0) = \int_0^\theta dw = \int_0^\theta pE \sin\theta \, d\theta$$

$$= pE(1-\cos\theta)$$

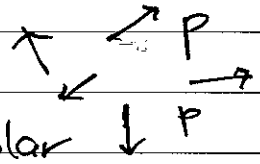
$$\therefore U(\theta) = -pE\cos\theta + \text{Const}$$

$$= -\vec{p}\cdot\vec{E} + \text{const}$$

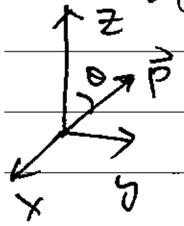
By taking const $=0$ when $\theta = 0$.

$$\therefore U = -\vec{p}\cdot\vec{E} \quad \text{---} \quad (9)$$

The Langevin equation



In real polar materials (gases & liquid polar dielectrics), polar molecules are in thermal equilibrium with other molecules and are under thermal agitation. Each $\vec{p}$ has different directions as shown in the above figure. The probability for a dipole lying in $(\theta, \theta+d\theta)$ & $(\phi, \phi+d\phi)$ is proportional to the Boltzman



factor: $e^{-\frac{U}{k_BT}} = e^{\frac{pE}{k_BT}\cos\theta}$ --- (10)

and : $\sin\theta \, d\theta \, d\phi$

No.
Date.

Hence the average dipole moment.

$$\langle \vec{p} \rangle = \frac{\int \vec{p}\, e^{\frac{pE}{k_B T}\cos\theta} \sin\theta\, d\theta\, d\phi}{\int e^{\frac{pE}{k_B T}\cos\theta} \sin\theta\, d\theta\, d\phi} \quad \text{--- (11)}$$

$\because$ $\vec{p} = p(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$\int_0^{2\pi} \sin\phi\, d\phi = 0 = \int_0^{2\pi} \cos\phi\, d\phi$$

$$\therefore \langle \vec{p} \rangle = \bar{p}\hat{z} = \frac{\int_0^{\pi} p\cos\theta\, e^{\frac{pE}{k_B T}\cos\theta} \sin\theta\, d\theta}{\int_0^{\pi} e^{\frac{pE}{k_B T}\cos\theta} \sin\theta\, d\theta}\hat{z} \quad \text{--- (12)}$$

Let $x = \frac{pE}{k_B T}\cos\theta \equiv u\cos\theta$

$$\bar{p} = \frac{p}{u} \frac{\int_{-u}^{u} x e^x dx}{\int_{-u}^{u} e^x dx}$$

$$= \frac{p}{u} \frac{x e^x - e^x \Big|_{-u}^{u}}{e^x \Big|_{-u}^{u}}$$

$$= \frac{p}{u} \frac{u(e^u + e^{-u}) - (e^u - e^{-u})}{e^u - e^{-u}}$$

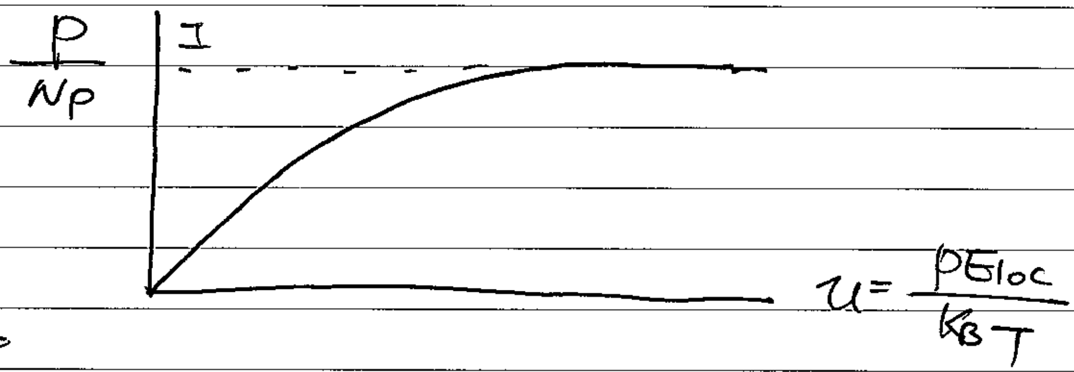
$$= p\left(\coth u - \frac{1}{u}\right) = p\left(\coth \frac{pE}{k_B T} - \frac{k_B T}{pE}\right) \quad \text{--- (13)}$$

If there are N dipoles per volume,

$$P\,(\text{dipolemoment/volume}) = Np\left(\coth\frac{pE}{k_B T} - \frac{k_B T}{pE}\right) \quad \text{--- (14)}$$

where $\vec{E} = \vec{E}_{loc}$ = local electric field acting on $\vec{p}$!

Eq. (14) is known as the Langevin equation



Only in the low temperatures ($T \to 0$, $u \to \infty$)

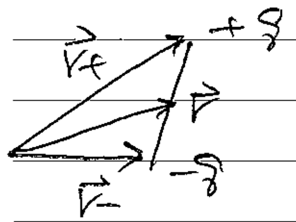
$\bar{p} \to p$ $\quad \bar{P} \to Np$.

Non-uniform field

If $\vec{E}$ is not uniform, $\vec{E} = \vec{E}(\vec{r})$,

The total force on a dipole doesn't vanish.

$$\vec{F} = q\vec{E}(\vec{r}_+) - q\vec{E}(\vec{r}_-)$$



$\because$ $E_i(\vec{r}_+) - E_i(\vec{r}_-)$

$$= (\vec{r}_+ - \vec{r}_-) \cdot \vec{\nabla} E_i \qquad i = x, y, z$$

$$= (\vec{d} \cdot \vec{\nabla}) E_i(\vec{r})$$

$$\therefore \ \vec{F} = (q\vec{d} \cdot \vec{\nabla}) \vec{E}(\vec{r}) = (\vec{p} \cdot \vec{\nabla}) \vec{E}(\vec{r}) \quad \text{--- (15)}$$

In this case, relative to the center,

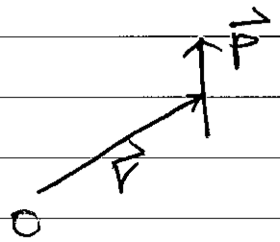
$$\vec{\tau} = \frac{\vec{d}}{2} \times \vec{F}_{+} + \left(-\frac{\vec{d}}{2}\right) \times \vec{F}_{-}$$

$$= q\vec{d} \times \left[\frac{1}{2}\vec{E}(\vec{r}_{+}) + \frac{1}{2}\vec{E}(\vec{r}_{-})\right]$$

$$= q\vec{d} \times \vec{E}(\vec{r}) + O(d^2)$$

$\therefore \vec{\tau} = \vec{p} \times \vec{E}(\vec{r})$ is still correct. -- (16)

For other point (the center is $\vec{r}$),

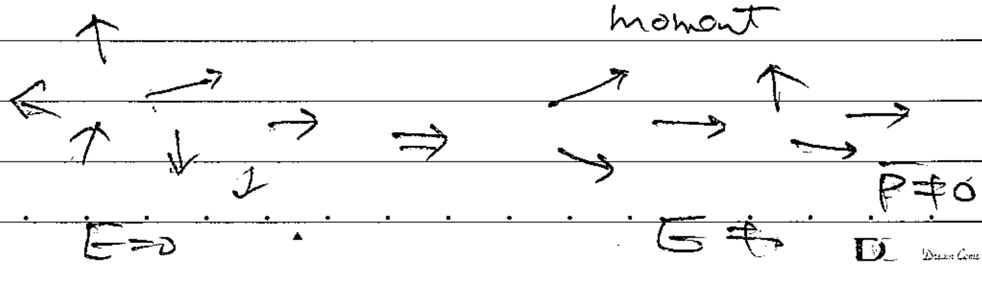


$$\vec{\tau} = \vec{p} \times \vec{E}(\vec{r}) + \vec{r} \times \vec{F} \quad -- (17)$$

<u>Polarization</u>

As we have seen in the above, in the presence of $\vec{E}$ field, there are two effects:

$$\begin{cases} \text{stretch charges} \Rightarrow \text{induced dipoles} \\ \text{rotate} \quad " \quad \Rightarrow \text{net dipole moment} \end{cases}$$



Both effects lead to net dipole moments in the material. The material is said to become polarized.

A convenient way to characterize this effect is to measure the polarization

$$\vec{P} \equiv \text{dipole moment per volume.}$$

As shown in the above, in the Lagenvin equation,

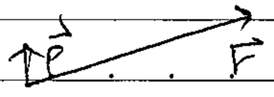
$$P = Np\left(\coth\frac{pE}{k_B T} - \frac{k_B T}{pE}\right) \neq 0$$

for finite temperatures T.

Field due to a polarized object

Given a polarization $\vec{P}$, we shall not discuss the field that may have cause $\vec{P}$. Instead, we would like to know the field it generates.

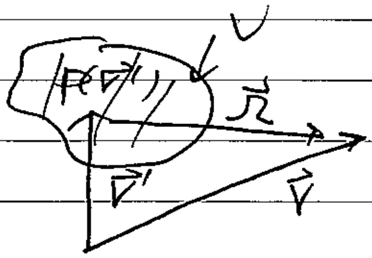
For a single dipole $\vec{p}$, one has



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

for large r!

Therefore, for continuous dipole distribution $\vec{P}(\vec{r})$, the potential outside it is



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\vec{P}(\vec{r}')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau' \quad \text{L}\cdots(18)$$

where $\vec{P}(\vec{r}')d\tau'$ = dipole moment in $d\tau'$

Now, $$\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|} = \vec{\nabla}'\frac{1}{\mathscr{r}}$$

$$\left(= \left(\frac{\partial}{\partial x'}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}, \cdots \right)\right)$$

$$\therefore V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \vec{P}(\vec{r}')\cdot\vec{\nabla}'\frac{1}{\mathscr{r}}\,d\tau'$$

Using $\vec{\nabla}\cdot(f\vec{A}) = \vec{\nabla}f\cdot\vec{A} - f\vec{\nabla}\cdot\vec{A}$, we can rewrite

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\int_V \left(\vec{\nabla}'\cdot\frac{\vec{P}'}{\mathscr{r}}\right)d\tau' - \int_V \frac{1}{\mathscr{r}}(\vec{\nabla}'\cdot\vec{P}')\,d\tau'\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\oint_{\partial V}\frac{\vec{P}'}{\mathscr{r}}\cdot\hat{n}'\,da' + \int_V \frac{(-\vec{\nabla}'\cdot\vec{P}')}{\mathscr{r}}\,d\tau'\right] \cdots (19)$$

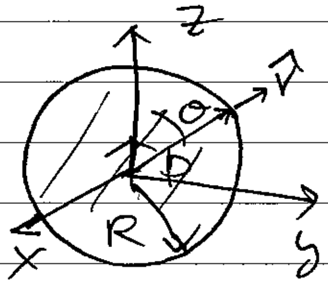
Eq. (19) now provide another form for $V(r)$. It can be re-interpreted as the potential generated by a volume charge density $\rho_b \equiv -\vec{\nabla}\cdot\vec{P}$

and a surface " " $\sigma_b = \vec{P}\cdot\hat{n}$

$$\therefore V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\oint_{\partial V} \frac{\sigma_b(r')}{|\vec{r}-\vec{r}'|} da' + \int_V \frac{\rho_b(\vec{r}')}{|\vec{r}-\vec{r}'|} d\tau'\right]$$

σ_b & ρ_b are known as bound charges, L - (20)
in contrast to free charge, they can't move.
Therefore, to compute the electric potential
due to polarization $\vec{P}(\vec{r})$, one can replace
$\vec{P}(\vec{r})$ by $\rho_b = -\vec{\nabla}\cdot\vec{P}$ & $\sigma_b = \vec{P}\cdot\hat{n}$ and
compute the potential due to ρ_b & σ_b!

Example : Find the electric field due to
a uniformly polarized sphere of
radius R



$\because \vec{P}$ is uniform.

$\therefore \vec{\nabla}\cdot\vec{P} = -\rho_b = 0$

Only surface charge $\sigma_b = \vec{P}\cdot\hat{r}$
$= P\cos\theta$

exists on the surface.

This is the same problem we discussed for a conducting sphere in an $\vec{E}_0$ field,

with $P = 3\epsilon_0 E_0$

In that case, $\because \vec{E}_0 + \vec{E}_{ind} = 0$ for $r < R$

$$\therefore \vec{E}_{ind} = -E_0 \hat{z} = -\frac{\vec{P}}{3\epsilon_0} \quad \text{for } r \leq R \quad \text{--- (21)}$$

i.e. $$V(r,\theta) = \frac{P}{3\epsilon_0} z = \frac{P}{3\epsilon_0} r\cos\theta \quad \text{for } r \leq R. \quad \text{--- (22)}$$

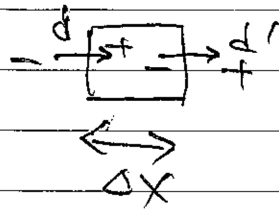
For $r > R$, the sphere acts as a dipole with $\vec{p} = \frac{4\pi}{3} R^3 \vec{P}$

$$\therefore V(r) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\vec{r}}{r^3} \quad \text{for } r > R \quad \text{--- (23)}$$

<u>Physical picture</u>

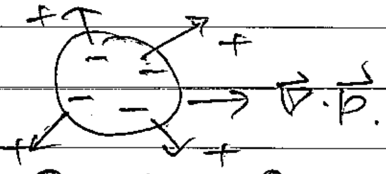
The arising of σ_b & ρ_b is due to the boundary and non-uniformity of displacement of charges.

Consider a small volume with width Δx,



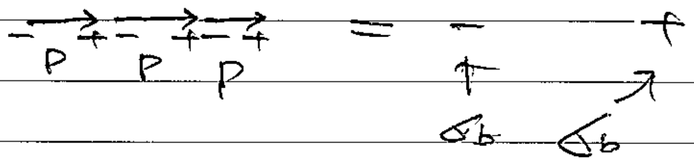
shown in the left figure, if $d < d'$, it means that there are more charges moving in than that moves out.

Hence $\vec{\nabla}\cdot\vec{p} = \frac{\partial P_x}{\partial x}$ in this case measures the accumulated charge.



One can also image a line of dipoles.

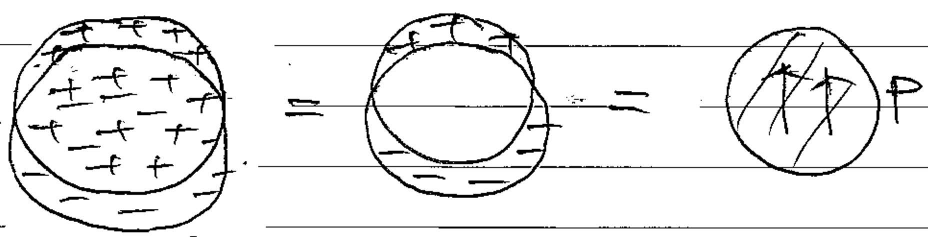
Tail of dipoles essentially get cancelled by head of dipoles as follows:



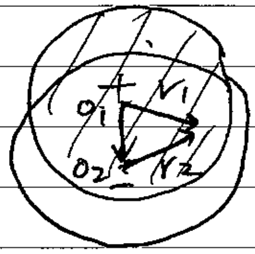
Therefore, $\vec{P}$ push charges to the boundary resulting in σ_b

Example : One can use the above picture to analyze the $\vec{E}$ field inside a uniformly polarized sphere.

One can think $\vec{P}$ results from the relative displacement of + charged sphere and − charged sphere as follows



∴ For a uniformly charge sphere, the Electric field at the distance $\vec{r}$ to the center is



$$\vec{E} = \frac{\rho}{3\epsilon_0}\vec{r}$$

Now, let origins $(O_1 \,\&\, O_2)$ of + sphere and − sphere to any point in the overlap region be $\vec{r}_1$ and $\vec{r}_2$

$$\therefore \vec{E} = \frac{1}{3\epsilon_0}\left(\rho_+ \vec{r}_1 + \rho_- \vec{r}_2\right)$$

$$= \frac{\rho}{3\epsilon_0}\left(\vec{r}_1 - \vec{r}_2\right) = \frac{\rho}{3\epsilon_0}\overrightarrow{O_1O_2}$$

$$= \frac{\rho}{3\epsilon_0}(-\vec{d}) = \frac{1}{3\epsilon_0}\frac{q}{\frac{4\pi}{3}R^3}(-\vec{d})$$

$$\therefore \vec{E} = -\frac{1}{4\pi\epsilon_0}\frac{q\vec{d}}{R^3} = -\frac{1}{4\pi\epsilon_0}\frac{1}{R^3}\left[\frac{4\pi}{3}R^3\vec{P}\right]$$

$$= -\frac{1}{3\epsilon_0}\vec{P} \quad \text{for } r \le R$$

Which reproduces eq. (21)!

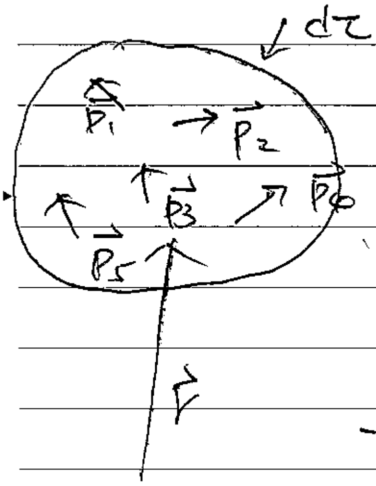
i.e. $$\vec{E} = -\nabla\frac{1}{4\pi\epsilon_0}\int_{\text{Sphere}}\frac{\vec{P}\cdot(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,d\tau' = -\frac{1}{3\epsilon_0}\vec{P}$$

for $r \le R$ -- (24)

Field inside a polarized material

Microscopic v.s. macroscopic fields

As we indicated, in a polarized material, each molecule may possess different direction of $\vec{p}_i$ but the polarization $\vec{P}$ is the average dipole moment per volume. This picture is shown in the following figure.

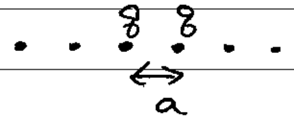


~~Therefore,~~ fields $\vec{E}(\vec{r})d\tau$ that are based on $\vec{P}(\vec{r})$ are averaged fields.

The volume $d\tau$ is the coarse-grained volume and has to be much larger than sizes of molecules but small enough that one can replace $\sum_i f(r_i)d\tau$ by $\int f(r)d\tau$.
Such fields are called macroscopic fields

The macroscopic fields are for macroscopic purposes by ignoring molecular structures and treat the materials as continuum.

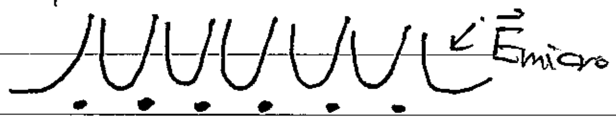
An example was given regarding the continuous charge distribution.



As we show, for the charge distribution in the left figure, if one does not need to know detailed fields between charges, one can replace the system by $\lambda = q/a$ and compute E field as it is a constant charge density.

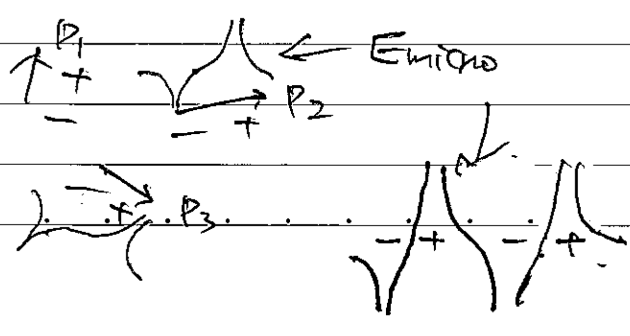
The E field we computed is a macroscopic field.

Microscopically, however, $\vec{E}_{micro}$ varies up & down near each q as follows



because $E_{micro} \to \infty$ as one approaches each q.

Similarly, in polarized materials, microscopically

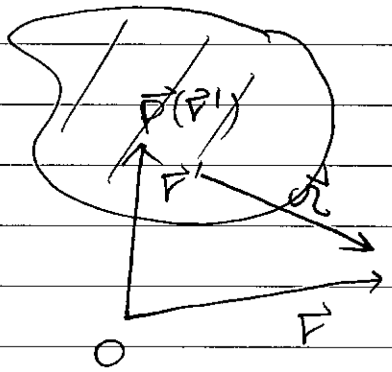


E_{micro} also varies dramatically as one goes from one dipole to another.

These fields are called microscopic fields.

They are the fields that act on each individual dipole and are often termed as local fields $\vec{E}_{loc}$!

Macroscopic field in polarized material

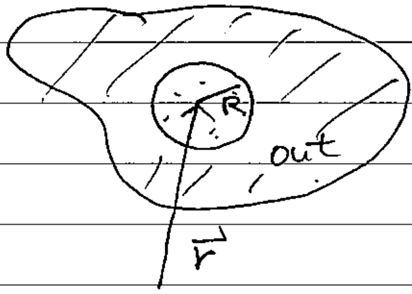


Given a polarized material, the macroscopic field outside the material can be found by using the polarization $\vec{P}(\vec{r})$

$\therefore$ r outside, $$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\vec{P}(\vec{r}\,')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau' \quad \text{--- } (25)$$

(25) is valid, as long as $\vec{r}$ is not microscopically close to the material, i.e., $|\vec{r}-\vec{r}\,'| \gg$ molecular sizes. Eq. (25), however, may not be correct when $\vec{r}$ is inside the material, because in this case, $\vec{r}$ is close to dipoles, the potential $\frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$ no longer works!

To obtain the macroscopic field $\vec{E}$ inside the material, we separate the contribution into two parts by introducing a sphere of radius R centered at $\vec{r}$ as shown in the left figure.



The volume of sphere is macroscopically small but is mucher than size of molecules.

Then, $\vec{E}$ can be written as

$$\vec{E} = \vec{E}_{out} + \vec{E}_{in} \quad \text{--- (26)}$$

where $\vec{E}_{in} = \frac{1}{\frac{4\pi}{3}R^3} \int_{r \in sphere} \vec{E}_{micro}(\vec{r})\, d\tau$

$= \text{average of } \vec{E}_{micro} \text{ in the sphere}$

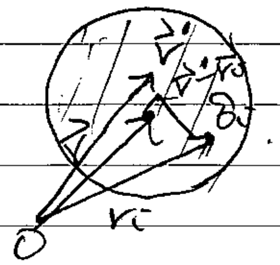
and $\vec{E}_{out}$ is the electric field due to $\vec{P}$ outside the sphere.

Since $R \gg$ sizes of molecules, for $\vec{P}$ outside the sphere, one can use eq. (25). We have

$$\vec{E} = -\nabla V_{out}$$

$$V_{out}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{outside} \frac{\vec{P}(\vec{r}')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau' \quad \text{--- (27)}$$

To find $\vec{E}_m$, we first consider the average of $\vec{E}_{micro}$ due to a point charge q_i at $\vec{r}_i$.



$$\vec{E}_m = \frac{1}{\frac{4\pi}{3}R^3}\int_{|\vec{r}'-\vec{r}|<R} \frac{q_i(\vec{r}'-\vec{r}_i)}{4\pi\epsilon_0|\vec{r}'-\vec{r}_i|^3}\, d\tau' \quad \text{--- (28)}$$

The integration term in eq. (28) can be viewed as the electric field at $\vec{r}_i$ due to a uniform charge sphere with

$$\rho = -q_i \quad \text{--- (29)}$$

$$\therefore \int_{|\vec{r}'-\vec{r}|<R} \frac{(\vec{r}_i-\vec{r}')\rho\, d\tau'}{4\pi\epsilon_0|\vec{r}_i-\vec{r}'|^3} = \frac{1}{3\epsilon_0}\rho(\vec{r}_i-\vec{r}) \quad \text{--- (30)}$$

($\vec{r}$ = displacement vector of center)

Combining eqs. (30) & (28), we get

$$\vec{E}_m = \frac{1}{\frac{4\pi}{3}R^3}\frac{1}{3\epsilon_0}(-q_i)(\vec{r}_i-\vec{r})$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{\vec{p}_i}{R^3} \quad \text{--- (31)}$$

where $\vec{p}_i = q_i(\vec{r}_i-\vec{r})$ is the dipole moment relative to the center

By using principle of superposition, one gets

$$\vec{E}_{in} = -\frac{1}{4\pi\epsilon_0}\frac{1}{R^3}\sum_i \vec{P}_i$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{1}{R^3}\Big(\underbrace{\frac{4\pi}{3}R^3\,\vec{P}(\vec{r})}_{\vec{p}}\Big)$$

$$= -\frac{1}{3\epsilon_0}\vec{P}(\vec{r}) \quad --- \; (32)$$

i.e. $V_{in} = \frac{1}{3\epsilon_0}\vec{p}\cdot\vec{r}$ (treating $\vec{p}$ as a constant vector)

However, eq. (32) is exactly the field generated by assuming that eq. (25) is valid for computing fields inside the sphere :

As shown in the previous example,

$$\frac{1}{4\pi\epsilon_0}\int_{\text{sphere}}\frac{\vec{p}\cdot\hat{\mathscr{r}}'}{\mathscr{r}^2}\,d\tau' = \frac{p}{3\epsilon_0}\,r\cos\theta \qquad r\le R$$

$$-\nabla\frac{1}{4\pi\epsilon_0}\int_{\text{sphere}}\frac{\vec{p}\cdot\hat{\mathscr{r}}'}{\mathscr{r}^2}\,d\tau' = -\frac{1}{3\epsilon_0}\vec{p}\,,$$

hence eq. (32) implies $\vec{E}_{in} = -\nabla\frac{1}{4\pi\epsilon_0}\int_{\text{sphere}}\frac{\vec{p}\cdot\hat{\mathscr{r}}'}{\mathscr{r}^2}\,d\tau'$ --- (33)

Therefore, we conclude that the macroscopic field $\vec{E} = -\nabla U$ with

$$U(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\vec{P}(\vec{r}')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau' \quad \text{--- (34)}$$

where $\vec{P}(\vec{r})$ is a coarse-grained smooth polarization. and is taken to be constant inside $d\tau$ (with characteristic size $R \gg$ size of molecules).

Electric displacement $\vec{D}$

In general, in addition to the bound charge $\rho_b = -\vec{\nabla}\cdot\vec{P}$, there are other charges in presence such as ions embedded in dielectrics or electrons on a conductor. Such charges are not due to polarization and are termed as free charges with charge density being denoted as ρ_f

$$\therefore\ \vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}(\rho_f + \rho_b) \quad \text{--- (35)}$$

$\because \rho_b = -\vec{\nabla}\cdot\vec{P}$,

$\therefore$ Eq. (35) can be re-written as

$$\vec{\nabla}\cdot\vec{E} + \frac{1}{\epsilon_0}\vec{\nabla}\cdot\vec{P} = \frac{1}{\epsilon_0}\rho_f .$$

i.e. $\vec{\nabla}\cdot(\vec{E}+\vec{P}/\epsilon_0) = \frac{1}{\epsilon_0}\rho_f$

It's convenient to define the

electric displacement $\vec{D} = \epsilon_0\vec{E}+\vec{P}$

so that $\vec{\nabla}\cdot\vec{D} = \rho_f$ --- (36).

That is, it appears that $\vec{\nabla}\cdot\vec{D}$ is entirely determined by ρ_f (free charges)

Since ρ_b is response of the materials to external fields (non-polar materials) but ρ_f is not, the advantage of eq. (36) is that $\vec{\nabla}\cdot\vec{D}$ is entirely determined by external charges in dielectrics or free charges which are under our control!

Gauss' law

Eq. (36) implies that the Gauss's law in the presence of polarization is replaced by

$$\oint \vec{D} \cdot d\vec{a} = Q_f \quad \cdots \quad (37)$$

However, since $\vec{\nabla} \times \vec{D} \neq 0$ in general, one can't conclude

$$\vec{D}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \rho_f(\vec{r}\,') d\tau'$$

$$(\times). \quad \cdots \quad (38)$$

Example A thick spherical shell $(a<r<b)$ is made of dielectric material with $\vec{P}(\vec{r}) = \frac{k}{r}\hat{r}$

Find $\vec{E}$ for $0<r<a$
$a<r<b$
and $r>b$.

Solution $\because$ $\rho_f = 0$ everywhere

$\therefore \oint \vec{D} \cdot d\vec{a} = 0$, $\vec{\nabla} \cdot \vec{D} = 0$ everywhere + $\vec{D} \parallel \hat{r}$

$\vec{D} = 0$ everywhere (this conclusion is not rigorous)

$\because$ $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ $\therefore$ For $0<r<a$, $r>b$, $(\vec{P}=0)$.

No.
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$\vec{E} = 0$

For $a<r<b$, $\vec{P} = \frac{k}{r}\hat{r}$, $\vec{E} = -\frac{\vec{P}}{\epsilon_0} = -\frac{k}{\epsilon_0 r}\hat{r}$.

Check by calculating $\vec{E}$ directly :

$$\oint \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0} Q_b \qquad \sigma_b = \vec{P}\cdot\hat{n} = k/b \quad \text{at } r=b$$

$$= -k/a \quad \text{at } r=a$$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = \frac{-1}{r^2}\frac{d}{dr}\left(r^2\frac{k}{r}\right)$$

$$= -k/r^2$$

$0<r<a$, $Q_b = 0$ $\therefore E = 0$ (Symmetry, $\vec{E} \parallel \hat{r}$)

$(\vec{D} = 0)$

$r>b$, $Q_b = \int_a^b \rho_b \cdot 4\pi r^2 dr + k/b \times 4\pi b^2$

$$+ (-k/a) \times 4\pi a^2$$

$$= -k\cdot 4\pi\,(b-a) + 4\pi bk - 4\pi k a = 0$$

$$\therefore E = 0 \quad (\vec{D} = 0)$$

$a<r<b$. $Q_b = \frac{-k}{a} \times 4\pi a^2 + \int_a^r \left(-\frac{k}{r^2}\right) 4\pi r^2 dr$

$$= -4\pi k r$$

$$\therefore E = \frac{1}{4\pi r^2}\frac{1}{\epsilon_0} Q_b = -\frac{k}{\epsilon_0 r}$$

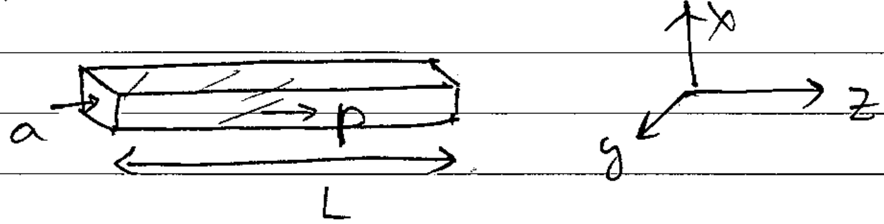
$$\vec{E} = -\frac{k}{\epsilon_0 r}\hat{r} = -\vec{P}/\epsilon_0 \qquad \vec{D} = 0$$

In general, eqs. (36) & (37) are not sufficient to determine $\vec{D}$ ($\because \vec{\nabla}\times\vec{D} \neq 0$ generally) unless particular symmetry is evolved as the direction of $\vec{D}$ is known.

In other words, $\vec{\nabla} \times \vec{D} = \nabla \times \vec{P} \neq 0$ (generally) -- (39)

(i.e. $\vec{\nabla} \times \vec{P} \neq 0$) so that eq. (38) is not correct.

Example. Polarized bar



$\because \rho_f = 0 \quad \therefore \vec{\nabla} \cdot \vec{D} = 0$ every where

$\oint \vec{D} \cdot d\vec{a} = 0$

Can one conclude $\vec{D} = 0$?

No! Because there is no symmetry, direction of $\vec{D}$ is not fixed.

In fact, $\vec{\nabla} \times \vec{P} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & P(x,y,z) \end{vmatrix}$

$= \left(\frac{\partial P}{\partial y}, -\frac{\partial P}{\partial x}, 0\right) \neq 0$

$\therefore \vec{\nabla} \times \vec{D} \neq 0$. $\vec{D}$ is not soly determined by ρ_f.

For small a, $\because \vec{\nabla} \cdot \vec{P} = 0$, only $\sigma_b = \pm P$ at two ends contribute. $\therefore \vec{E}$ is due to two point charges $\pm pa$ at $z=0$ & $z=L$.

$$\vec{E}+\vec{P}/\varepsilon_0 \neq 0 \quad (\because \vec{E} \neq \text{const.})$$

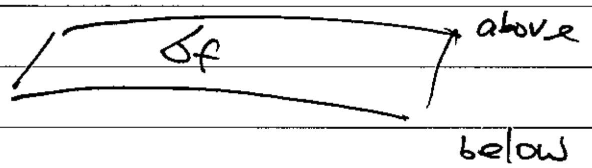
Boundary Condition

As we have shown, the displacement field $\vec{D}$ satisfies

$$\vec{\nabla}\cdot\vec{D} = \rho_f$$

$$\vec{\nabla}\times\vec{D} = \vec{\nabla}\times\vec{P}$$

Therefore, the boundary conditions across a surface with σ_f are (follow the same derivation for $\vec{E}$ in presence of σ)



$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f \quad \text{--- (40)}$$

$$\vec{D}^{\parallel}_{above} - \vec{P}^{\parallel}_{above} = \vec{D}^{\parallel}_{below} - \vec{P}^{\parallel}_{below}$$

$$(\text{i.e. } \vec{E}^{\parallel}_{above} = \vec{E}^{\parallel}_{below})$$

$\to$ i.e

$$\vec{D}^{\parallel}_{above} - \vec{D}^{\parallel}_{below} = \vec{P}^{\parallel}_{above} - \vec{P}^{\parallel}_{below} \quad \text{--- (41)}$$

Linear dielectrics

In the above, the origin of $\vec{P}$ is not included in the discussion of finding $\vec{D}$.

As a result, $\vec{D}$ can't be determined solely by ρ_f as in the form of eq. 38.

One needs to know $\vec{P}$ so that $\nabla\times\vec{D} = -\nabla\times\vec{P}$ is known.

However, as we showed in the beginning, for dielectric materials, $\vec{P}$ is induced by electric fields. This is different from ferroelectrics where $\vec{P}$ is permanent even where there is no electric field.

Furthermore, for many substances, $\vec{P}$ is proportional to the field $\vec{E}$ when $\vec{E}$ is not too strong. $\therefore$ One may write

$$\vec{P}(\vec{r}) = \epsilon_0 \chi_e \vec{E}(\vec{r}) \quad \cdots (42)$$

Where $\vec{E}$ is the total macroscopic field and χ_e is dimensionless and is called the electric susceptibility of the substance.

Materials that obey eq. (42) are called linear dielectrics.

For such materials, we have

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E}$$

$$= \epsilon_0 (1+\chi_e) \vec{E}$$

$\therefore$ $\vec{D}$ is also proportional to $\vec{E}$

$$\vec{D} = \epsilon \vec{E} \quad \text{--- (43)}$$

Where $\epsilon = \varepsilon_0 (1+\chi_e)$ = permittivity of the material. --- (44)

Note that ε_0 is the permittivity of free space.

$$\epsilon / \varepsilon_0 = 1 + \chi_e \equiv \varepsilon_r$$

is called the relative permittivity or the dielectric constant of the material.

In general, χ_e is a matrix $\overleftrightarrow{\chi_e}$; susceptibility tensor so that $\vec{P} = \epsilon_0 \overleftrightarrow{\chi_e} \cdot \vec{E} = \epsilon_0 \begin{pmatrix} \chi_{exx} & \chi_{exy} & \chi_{exz} \\ \chi_{eyx} & \chi_{eyy} & \chi_{eyz} \\ \chi_{ezx} & \chi_{ezy} & \chi_{ezz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$

Examples of linear dielectrics.

	ε_r (dielectric constant)
Vacuum	1
He	1.000065
Ne	1.00013
Air (dry)	1.00536
C_6H_6 (Benzene)	2.28
Silicon	11.7
Water	80.1
Ice (−30°C)	104

Eq. (43) is a relation between $\vec{D}$ & $\vec{E}$, which is called constitutive relation

The field $\vec{E}$ is the total field, which is the sum of external field $\vec{E_0}$ and fields generated by $\vec{P}$.

At first, it may seem that because $\vec{P} = \epsilon_0 \chi_e \vec{E}$ and $\vec{E}$ contains fields generated by $\vec{P}$, there is a cycling relation so that $\vec{E}$ & $\vec{P}$ can't be determined.

This cycling can be broken by noting that $\vec{D}$ is related to ρ_f by $\vec{\nabla} \cdot \vec{D} = \rho_f$. Once $\vec{D}$ is known, $\vec{E}$ & $\vec{P}$ can be determined.

Clausius - Mossotti formula

In linear dielectrics, χ_e is related to atomic polarizability α. To find the relation, one first note that for a given atom, the field that acts on it is the local field $\vec{E}_{loc}$.

If we enclose the atom by a small sphere of radius R. (shown in the above), $\vec{E}_{loc}$ $= \vec{E}_{out}$ (field generated by outside sphere)

$\therefore \ \vec{p} = \alpha \vec{E}_{out}$

Now, according to Eqs. (31) & (34), the macroscopic total field $\vec{E} = \vec{E}_{in} + \vec{E}_{out}$

where $\vec{E}_{in}$ = average of fields inside sphere due to atom itself! $= -\frac{1}{4\pi\epsilon_0}\frac{\vec{p}}{R^3}$ (eq. (31))

$\therefore \ \vec{E} = -\frac{1}{4\pi\epsilon_0}\frac{\alpha}{R^3}\vec{E}_{out} + \vec{E}_{out} = \left(1 - \frac{\alpha}{4\pi\epsilon_0 R^3}\right)\vec{E}_{out}$

Let the density of atoms $= N = 1/\frac{4\pi}{3}R^3$

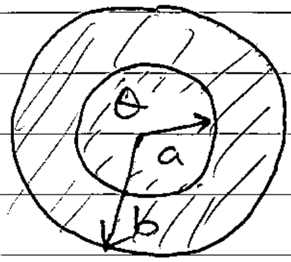
$\therefore \ \vec{E} = \left(1 - \frac{N\alpha}{3\epsilon_0}\right)\vec{E}_{out}$

$\vec{P}(\vec{r}) = N\vec{p} = N\alpha\vec{E}_{out} = \frac{N\alpha}{1 - N\alpha/3\epsilon_0}\vec{E} \equiv \epsilon_0 \chi_e \vec{E}$

$\therefore \ \chi_e = \frac{N\alpha/\epsilon_0}{1 - N\alpha/3\epsilon_0}$, setting $\chi_e = \epsilon_r - 1$,

One can solve $\alpha = \frac{\epsilon_0}{N}\frac{\chi_e}{1 + \chi_e/3} = \frac{3\epsilon_0}{N}\cdot\frac{\epsilon_r - 1}{\epsilon_r + 2}$ which is known as the Clausius - Mossotti formula.

Example. A metal sphere of radius a carries a charge Q and is surrounded by linear dielectric material of permittivity ϵ out to radius b.



Find the potential at the center by setting $V(r=\infty)=0$.

Solution

To compute V, one needs to know $\vec{E}$ ($\because \vec{P}=\epsilon_0\chi_e\vec{E}$)

However, we don't know $\vec{P}$ yet and hence $\rho_b=-\vec{\nabla}\cdot\vec{P}$ is not known.

But the free charge Q is known.

Since the set up is spherically symmetric, from $\oint_{S_r}\vec{D}\cdot d\vec{a}=Q$ for S_r = spherical surface of $r>a$, we get

$$\vec{D}=\frac{Q}{4\pi r^2}\hat{r} \quad \text{for} \quad r>a \quad \text{(44)}$$

Inside the metal sphere, $\vec{D}=0$, $\vec{E}=0$

$\therefore \vec{P}=0$. for $r\leq a$

From eq. (43), we get :

$$\vec{E} = \frac{Q}{4\pi\epsilon r^2}\hat{r} \quad \text{for } a<r<b$$

$$= \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} \quad r>a \ (\epsilon=\epsilon_0)$$

$$\therefore V(r=0) - \underbrace{V(r=\infty)}_{=0} = -\int_{\infty}^{0} \vec{E}\cdot d\vec{l} \quad \leftarrow (45)$$

$$= -\int_{\infty}^{b} \frac{Q}{4\pi\epsilon_0 r^2}dr - \int_{b}^{a} \frac{Q}{4\pi\epsilon r^2}dr$$

$$= \frac{Q}{4\pi}\left[\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right] \leftarrow (46)$$

$\therefore$ It's not necessary to compute $\vec{P}$ or ρ_b first.

From $\vec{E}$, one gets

$$\vec{P} = \epsilon_0\chi_e\vec{E} = \frac{\epsilon_0\chi_e Q}{4\pi\epsilon r^2}\hat{r} \quad \text{for } a<r<b$$

$\leftarrow$ (47)

$$\therefore \rho_b = -\vec{\nabla}\cdot\vec{P} = 0$$

while $\sigma_b = \vec{P}\cdot\hat{n} = \dfrac{\epsilon_0\chi_e Q}{4\pi\epsilon b^2}$ for $r=b$ $(\hat{n}=\hat{r})$

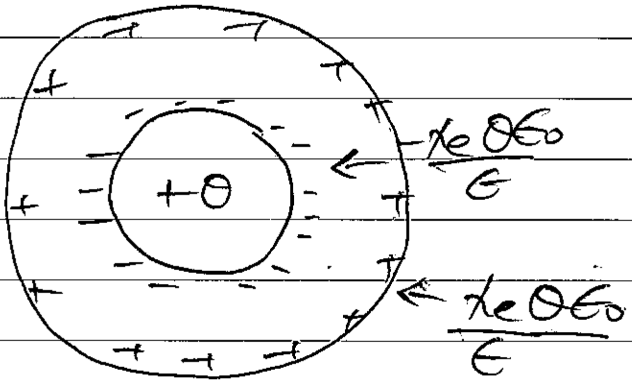
$$= -\frac{\epsilon_0\chi_e Q}{4\pi\epsilon a^2} \quad \text{for } r=a \ (\hat{n}=-\hat{r})$$

∴ The charge Q induces a negative charge

$$-\frac{\epsilon_0 \chi_e Q}{4\pi\epsilon a^2} \times 4\pi a^2 = -\frac{\epsilon_0 \chi_e Q}{\epsilon}$$ on the surface

at $r=a$ & a positive charge

$\frac{\epsilon_0 \chi_e Q}{\epsilon}$ on the surface at $r=b$.



So that the effect charge seen in $a<r<b$

can be identified as $\vec{E} = \frac{1}{4\pi\epsilon_0}\Big[\underbrace{Q - \frac{\epsilon_0 \chi_e}{\epsilon} Q}_{Q_{eff}}\Big]\frac{1}{r^2}\hat{r}$

$$\Rightarrow Q_{eff} = \frac{\epsilon_0}{\epsilon} Q = Q/\epsilon_r \quad \cdots (48)$$

$$\vec{E} = \frac{1}{\epsilon_r}\frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r} = \frac{\vec{E}_{vac}}{\epsilon_r} \quad \cdots (49)$$

The picture is clear: dielectrics reduces strength of Q and thus strength of $\vec{E}$ fields.

Reduction of fields ∵

Eqs. (48) & (49) are general features of fields in linear dielectrics.

For the space is filled with a homogeneous linear dielectrics, one has

$$\vec{\nabla}\cdot\vec{D} = \rho_f \qquad \vec{\nabla}\times\vec{D} = \vec{\nabla}\times(\epsilon\vec{E})$$

$$= \epsilon\vec{\nabla}\times\vec{E} = 0 \quad \text{--- (50)}$$

Hence for the displacement $\vec{D}$, it is the same problem as it is in the vacuum (without linear dielectrics) with ρ_f :

$$\vec{\nabla}\cdot(\epsilon_0\vec{E}) = \rho_f$$

$$\vec{\nabla}\times(\epsilon_0\vec{E}) = 0$$

Hence $\vec{D} = \epsilon_0 \vec{E}_{vac}$ --- (51)

$$\therefore \quad \vec{E} = \frac{\vec{D}}{\epsilon} \quad \text{(fields in the dielectrics)}$$

$$= \frac{\epsilon_0}{\epsilon}\vec{E}_{vac} \quad \text{(fields in the presence of } \rho_f \text{ without dielectrics)}$$

$$= \frac{1}{\epsilon_r}\vec{E}_{vac}$$

--- (52)

Note that $\vec{\nabla}\times\vec{D} = \epsilon\vec{\nabla}\times\vec{E} = 0$ is only true inside the dielectrics. In the interface between vacuum and the dielectrics, ϵ becomes position dependant.

Vacuum $\epsilon = \epsilon_0$

dielectrics $\epsilon = \epsilon$

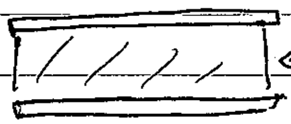
Hence $\vec{\nabla}\times(\epsilon\vec{E})$

$$= \nabla\epsilon\times\vec{E} + \underbrace{\cancel{\epsilon\vec{\nabla}\times\vec{E}}}_{0}$$

$$\neq 0$$

$\therefore\ \vec{\nabla}\times\vec{D} = 0$

Example Parallel plate capacitor filled with dielectrics with dielectric constant ϵ_r.



$\because$ relative to vacuum

$$E = \frac{1}{\epsilon_r}E_{vac}$$

$\therefore\ V = \frac{1}{\epsilon_r}V_{vac}$ with Q fixed

$$\therefore\ C = Q/V = \frac{Q}{V_{vac}}\epsilon_r = \epsilon_r \underbrace{C_{vac}}_{\text{Capacitance when there is no dielectrics filled.}}$$

Boundary value problems with linear dielectrics

For linear dielectrics,

$$\because \rho_b = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot\left(\epsilon_0 \chi_e \vec{E}\right)$$

$$= -\vec{\nabla}\cdot\left(\frac{\epsilon_0\chi_e}{\epsilon}\vec{D}\right)$$

$$= -\frac{\epsilon_0\chi_e}{\epsilon}\vec{\nabla}\cdot\vec{D} = -\frac{\epsilon_0\chi_e}{\epsilon_0}\rho_f \quad \text{-- (53)}$$

$\therefore$ ρ_b is proportional to free charge density ρ_f.

Hence unless ρ_f is embedded in dielectrics, $\rho_b = 0$, $\rho = \rho_f + \rho_b = 0$ are generally true inside dielectrics.

$$\therefore \quad \vec{\nabla}\cdot\vec{D} = 0$$
$$\vec{\nabla}\times\vec{D} = 0$$

$$\vec{D} = -\epsilon\vec{\nabla}V \Rightarrow \nabla^2 V = 0 \quad \text{--- (54)}$$

The potential obeys the Laplace equation inside the dielectrics

However, it does not mean that the solution is the same as if there is no dielectrics.

The crucial difference lies in the boundaries and the boundary conditions.

As mentioned, on the boundary of dielectrics,

$$\vec{\nabla} \times \vec{D} \neq 0$$

Across a free surface charge density σ_f,

$\vec{D}$ obeys eq. (40)

ϵ_{above} σ_f

ϵ_{below}

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$$

$$\because D = \epsilon E$$

$$\therefore \epsilon_{above} E^{\perp}_{above} - \epsilon_{below} E^{\perp}_{below} = \sigma_f \quad \text{--- (55)}$$

In terms of V, one gets

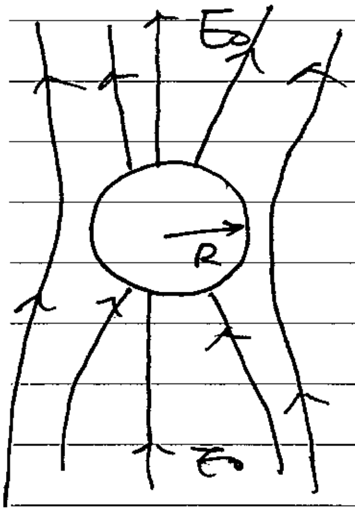
$$\epsilon_{above} \frac{dV_{above}}{dn} - \epsilon_{below} \frac{dV_{below}}{dn} = -\sigma_f \quad \text{--- (56)}$$

together with

$$V_{above} = V_{below} \quad \text{--- (57)}$$

Eq. (54) is solved together with eqs. (56) and (57).

Example.



A dielectric sphere of radius R is placed in a uniform external field $\vec{E_0}$.

The dielectric constant of the sphere is ε_r. Find $\vec{E}$ inside the sphere.

Solution

One needs to solve V that satisfies

$\nabla^2 V = 0$ for $V_{in}(r,\theta)$ $r \leq R$

$V_{out}(r,\theta)$ $r > R$

With Boundary conditions

(i) $V_{in}(R,\theta) = V_{out}(R,\theta)$

(ii) $\epsilon \frac{\partial V_{in}}{\partial r}(R,\theta) = \epsilon_0 \frac{\partial V_{out}}{\partial r}(R,\theta)$

(iii) $V_{out} \rightarrow -E_0 r\cos\theta$ $r \gg R$

Following the same steps for solving $\vec{E}$ of a conducting sphere in $\vec{E_0}$, we first note the general solution with azimuthal symmetry has the form

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta) \quad \text{--- (52)}$$

For $r<R$, in order U to be finite at $r=0$,

one sets $B_\ell = 0$

$$\therefore U_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \quad \cdots (59)$$

For $r>R$, however, in order $U \to -E_0 r\cos\theta$,

One sets $A_\ell = 0$ $(\ell \geq 2)$, and $A_1 = -E_0$

$$\therefore U_{out}(r,\theta) = -E_0 r\cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$\cdots (60)$$

Now (iii) is satisfied,

To satisfy (i), one requires

$$\sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = -E_0 R\underbrace{\cos\theta}_{P_1(\cos\theta)} + \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

$$\cdots (61)$$

$\{P_\ell(\cos\theta)\}$ are orthogonal $\therefore$ Coefficients of

$P_\ell(\cos\theta)$ in eq. (61) must be equal for

LHS (Left hand side) & RHS :

$$\ell = 1 \quad A_1 R = -E_0 R + \frac{B_1}{R^2}$$

$$\ell \neq 1 \quad A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}} \quad \cdots (62)$$

No.
Date.

Similarly, to satisfy (iii), one requires ($\epsilon = \epsilon_r \epsilon_0$). using 4-44

$$\epsilon_r \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta)$$

$$= -E_0 \cos\theta - \sum_{l=0}^{\infty} \frac{(l+1) B_l}{R^{l+2}} P_l(\cos\theta) \quad \text{--- } (63)$$

We find

$l=1$ $\quad \epsilon_r A_1 = -E_0 - \dfrac{2B_1}{R^3}$

$l \neq 1$ $\quad \epsilon_r l A_l R^{l-1} = -\dfrac{(l+1) B_l}{R^{l+2}}$ --- (64)

Combing eqs. (63) & (64) for $l \neq 1$, one finds

$$A_l = B_l = 0.$$

For $l=1$, $\quad A_1 = -\dfrac{3}{\epsilon_r+2} E_0$, $\quad B_1 = \dfrac{\epsilon_r - 1}{\epsilon_r + 2} R^3 E_0$

--- (65)

Hence $\quad V_{in}(r,\theta) = -\dfrac{3E_0}{\epsilon_r+2} r\cos\theta = -\dfrac{3E_0}{\epsilon_r+2} z$

$$\vec{E}_{in} = \frac{3}{\epsilon_r+2} \vec{E}_0 \quad \text{for } r < R$$

is uniform.

$$V_{out}(r,\theta) = -E_0 r\cos\theta + \frac{\epsilon_r - 1}{\epsilon_r+2} \frac{R^3 E_0}{r^2} \cos\theta$$

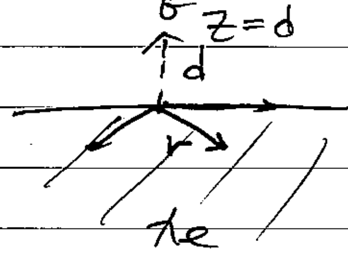
$$= -E_0 z + \frac{1}{r^2} \frac{\epsilon_r - 1}{\epsilon_r + 2} R^3 \hat{r} \cdot \vec{E}$$

$$\equiv -E_0 z + \frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_0 r^2} \quad \text{--- } (66)$$

∴ For $r>R$, the dielectric sphere acts a single dipole with dipole moment

$$\vec{p} = 4\pi R^3 \frac{\epsilon_r - 1}{\epsilon_r + 2}\hat{z} \quad --- \quad (67)$$

<u>Example</u> A charge q is placed at a distance d above a dielectric material occupying $z<0$. Find the force that acts on the charge.



<u>Solution</u>: To find the force, one needs to find $\vec{E}$ outside the dielectrics.

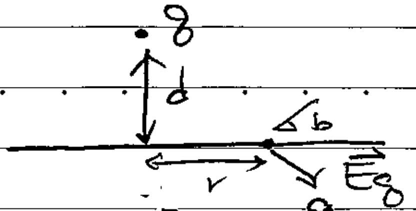
For this purpose, one needs to know ρ_b & σ_b

∵ $\rho_f = 0$ for $z<0$, from eq (53), $\rho_b = -\frac{\chi_e}{1+\chi_e}\rho_f = 0$

∴ Only $\sigma_b = \vec{P}\cdot\hat{n} = \epsilon_0 \chi_e \vec{E}\cdot\hat{z} \neq 0$ at $z=0$ contributes $\vec{E}$. --- (68)

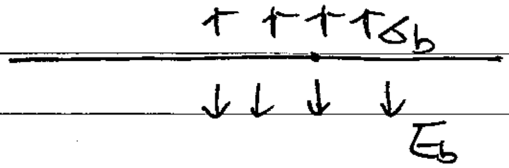
Now, to find $\sigma_b = \epsilon_0 \chi_e E_z$, one needs to find E_z. There are two contributions to E_z at $z=0^-$ (just below σ_b):

$$E_q = -\frac{1}{4\pi\epsilon_0}\frac{q\,d}{(r^2+d^2)^{3/2}}$$



$$E_q^z = -\frac{q}{4\pi\epsilon_0}\frac{d}{(r^2+d^2)^{3/2}}$$ due to q. $r^2 = x^2+y^2$

the other is due to σ_b



$$E_b = \frac{-\sigma_b}{2\epsilon_0}$$

$$\therefore \quad E_z = -\frac{1}{4\pi\epsilon_0}\frac{qd}{(r^2+d^2)^{3/2}} - \frac{\sigma_b}{2\epsilon_0} \quad \cdots (69)$$

Eq. (68) becomes

$$\sigma_b = \epsilon_0\chi_e\left[-\frac{1}{4\pi\epsilon_0}\frac{qd}{(r^2+d^2)^{3/2}} - \frac{\sigma_b}{2\epsilon_0}\right] \quad \cdots (70)$$

which enables us to solve for σ_b

$$\sigma_b = -\frac{1}{2\pi}\frac{\chi_e}{\chi_e+2}\frac{qd}{(r^2+d^2)^{3/2}} \quad \cdots (71)$$

Once σ_b is known, one can find $\vec{E}$

by performing the integration

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\sigma_b(r')\,da' \quad \cdots (72)$$

However, just as in the case of the conducting plane, one can try if the method of image can be applied here.

For this purpose, we examine E_z at $z=0^-$.

using eq. 71, eq. 68 becomes

$$E_z(z=0^-) = -\left[1-\frac{\chi_e}{\chi_e+2}\right]\frac{1}{4\pi\epsilon_0}\frac{qd}{(r^2+d^2)^{3/2}} \quad --- (72)$$
(E_z^{below})

Similarly, for $z=0^+$ (above σ_b)

$$E_z(z=0^+) = -\frac{1}{4\pi\epsilon_0}\frac{qd}{(r^2+d^2)^{3/2}} + \frac{\sigma_b}{2\epsilon_0}$$
(E_z^{above})

$$= -\left[1+\frac{\chi_e}{\chi_e+2}\right]\frac{1}{4\pi\epsilon_0}\frac{qd}{(r^2+d^2)^{3/2}} \quad --- (73)$$

For the method of image, one needs to consider the region $z>0$.

From eq. 73, it is clear that E_z right above the dielectrics consists of two parts

$$E_z(\text{above}) = E_q + E_{q'}$$

$E_{q'}$ is due to an image charge

$$q' = -\frac{\chi_e}{\chi_e+2}q = -\frac{\epsilon_r-1}{\epsilon_r+1}q \text{ at } z=-d \quad -- (74)$$

From the uniqueness theorem, placing q' at $z=-d$

generates exactly the same E_z on the surface just above the dielectrics at $z=0$.

Therefore, $\vec{E}(z>0)$ is the sum of fields due to q and q'!

Note that eqs (12) & (13) satisfy the boundary condition:

$$E_z^{below} = -\frac{2}{\epsilon_r+1}\frac{1}{4\pi\epsilon_0}\frac{qd}{(d^2+r^2)^{3/2}}$$

$$E_z^{above} = -\frac{2\epsilon_r}{\epsilon_r+1}\frac{1}{4\pi\epsilon_0}\frac{qd}{(d^2+r^2)^{3/2}} \quad \text{-- (75)}$$

$$= \epsilon_r E_z^{below}$$

Using the image charge, one can immediately find

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(2d)^2}\hat{z} = -\frac{1}{4\pi\epsilon_0}\frac{\chi_e}{\chi_e+2}\frac{q^2}{4d^2}\hat{z}$$

Finally, if one wants to find $\vec{E}(z<0)$ in the dielectric material, Eq. (75) implies that one can replace vacuum for $z>0$ by the same dielectric material and remove q/replace q by $q'' = \frac{2\epsilon_r}{\epsilon_r+1}q$ at $z=d$ -- (76)

thus $E(z<0)$ is generated by the image charge q''.

Energy of charges in the presence of dielectrics

In the presence of dielectric materials, to assemble charges (free charges), one needs to do additional work to polarize dielectric materials.

To find the work, consider the assymbling of $\Delta\rho_f$ when there is ρ_f already.

The work needs to be done is to bring in ρ_f

$$\therefore \Delta W = \int (\Delta\rho_f) \cdot V(\vec{r})\, d\tau$$

$$\because \vec{\nabla}\cdot\vec{D} = \rho_f \qquad \therefore \Delta\rho_f = \vec{\nabla}\cdot(\Delta\vec{D})$$

$$\therefore (\Delta\rho_f)\, V(\vec{r}) = \vec{\nabla}\cdot(\Delta\vec{D})\, V(\vec{r})$$

$$= \vec{\nabla}\cdot\left[\Delta\vec{D}\, V(\vec{r})\right] - \Delta\vec{D}\cdot\vec{\nabla}V$$

$$\therefore \Delta W = \int \vec{\nabla}\cdot(\Delta\vec{D}\, V(\vec{r})) + \int (\Delta\vec{D})\cdot\vec{E}\, d\tau$$

Using the Stoke's theorem and assuming $\Delta\vec{D}\, V(\vec{r}) \to 0$ faster than $\frac{1}{r^2}$,

$$\int_{V_r} \vec{\nabla}\cdot(\Delta\vec{D}\, V)\, d\tau = \int_{\partial V_r} \Delta\vec{D}\cdot V(\vec{r})\cdot d\vec{a} \to 0.$$

Hence

$$\Delta W = \int \Delta \vec{D} \cdot \vec{E} \, d\tau \quad --- \quad (77)$$

Eq. (77) is the most general expression for the work that needs to be done.
By integration ΔW, one gets W which generally depends on how free charges are assembled.

Linear dielectrics

For linear dielectrics, eq. (77) can be integrated explicitly.

In this case, $\vec{D} = \epsilon \vec{E}$

$$\therefore \Delta \vec{D} \cdot \vec{E} = \epsilon \Delta \vec{E} \cdot \vec{E} = \frac{1}{2} \epsilon (\Delta \vec{E}^2)$$

$$= \frac{1}{2} \Delta (\vec{D} \cdot \vec{E})$$

$$\therefore \Delta W = \Delta \int \frac{1}{2} \vec{D} \cdot \vec{E} \, d\tau$$

$$\therefore W = \frac{1}{2} \int \vec{D} \cdot \vec{E} \, d\tau \quad --- \quad (78)$$

$$= \frac{1}{2} \int \epsilon E^2 d\tau \quad -- \quad (79)$$

Using $\vec{E} = -\nabla V$, one can write

$$\vec{D}\cdot\vec{E} = -(\vec{\nabla} V)\cdot\vec{D} = -\vec{\nabla}\cdot(V\vec{D}) + V\vec{\nabla}\cdot\vec{D}$$

$$\because \int_{\mathcal{V}_{r=\infty}} \vec{\nabla}\cdot(V\vec{D}) = \int_{\partial\mathcal{V}_{r=\infty}} V\vec{D}\cdot d\vec{a} \to 0 \quad \text{as } r\to\infty$$

$$\therefore \frac{1}{2}\int \vec{D}\cdot\vec{E}\, d\tau = \frac{1}{2}\int V\vec{\nabla}\cdot\vec{D}\, d\tau$$

$$= \frac{1}{2}\int \rho_f V\, d\tau \quad \text{---} \quad (80)$$

Therefore, similar to the assembling charges in vacuum, the work done for linear dielectrics is given in eqs. (79) or (80), with ε_0 replaced by ε!

Note that eqs. (79) & (80) include the work to polarize the material. If one thinks dipoles as $\pm$ charges connected by a spring with spring constant k, eqs (79) & (80) include the work to provide the potential energy for the spring $\frac{1}{2}kx^2$.

On the other hand, if one considers the work for assambling all charges density ρ_f & ρ_b, the work need to be done is

$$\bar{W} = \frac{1}{2}\int(\rho_f+\rho_b)V\,d\tau$$

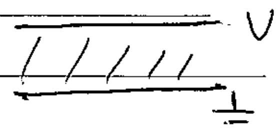
$$= \frac{1}{2}\int(\epsilon_0 \vec{\nabla}\cdot\vec{E})V\,d\tau$$

$$= \frac{1}{2}\epsilon_0\int\left[\vec{\nabla}\cdot(\vec{E}V) - \vec{E}\cdot\nabla V\right]d\tau$$

$$= \int \frac{1}{2}\epsilon_0 E^2\,d\tau \quad \text{--- (\#1)}$$

$\bar{W}$ in eq (#1) is the same expression as before. It does not include the work to polarize $\vec{P}(\vec{r})$!

Example: Parallel-plate capacitor with filled dielectrics. with fixed potential.

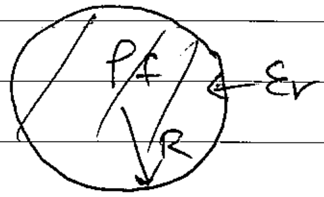


$$W = \frac{\epsilon}{2}\int E^2\,d\tau = \frac{\epsilon_0\epsilon_r}{2}\int E^2\,d\tau$$

$$= \epsilon_r W_{vac} \quad (E \text{ is fixed})$$

i.e. ∵ $Q = CV$, for fixed V, Q increases by ϵ_r. ∴ W increases by ϵ_r.

Example. A sphere of radius R is filled with dielectric material (dielectric constant ϵ_r) & uniform charge density ρ_f.



What is the energy of this configuration?

Solution $\because \oint \vec{D}\cdot d\vec{a} = Q_f$ and $\vec{D} = D\hat{r}$

$$\therefore \quad D(r)\cdot 4\pi r^2 = \rho_f \cdot \frac{4\pi}{3} r^3 \qquad r<R$$

$$= \rho_f \frac{4\pi}{3} R^3 \qquad r>R$$

$$\therefore \quad \vec{D}(\vec{r}) = \begin{cases} \frac{\rho_f}{3}\vec{r} & r<R \\ \frac{\rho_f}{3}\frac{R^3}{r^2}\hat{r} & r>R \end{cases}$$

$$W = \frac{1}{2}\int \vec{D}\cdot\vec{E}\, d\tau$$

$$= \frac{1}{2}\int_0^R \Big(\underbrace{\frac{\rho_f}{3\epsilon}\vec{r}}_{E}\cdot \underbrace{\frac{\rho_f}{3}\vec{r}}_{D}\Big) 4\pi r^2 dr$$

$$+ \frac{1}{2}\int_R^\infty \Big(\underbrace{\frac{\rho_f}{3}\frac{R^3}{r^2}\hat{r}}_{\vec{D}} \cdot \underbrace{\frac{\rho_f R^3}{3\epsilon_0 r^2}\hat{r}}_{\vec{E}}\Big) 4\pi r^2 dr$$

$$= \frac{1}{2}\left[\frac{\rho_f^2}{9\epsilon_0\epsilon_r}\times 4\pi \int_0^R r^4 dr + \frac{\rho_f^2}{9\epsilon_0}4\pi R^6 \int_R^\infty \frac{dr}{r^2}\right]$$

$$= \frac{2\pi}{9\epsilon_0} \rho_f^2 R^5 \left(1+\frac{1}{5\epsilon_r}\right) \quad \cdots (82)$$

One can check eq. (82) is indeed the work done to bring $dq = \rho_f 4\pi r^2 dr$ from $r=\infty$ to r.

On the other hand,

$$\bar{W} = \frac{\epsilon_0}{2}\int E^2 d\tau$$

$$= \frac{\epsilon_0}{2}\left[\left(\frac{\rho_f}{3\epsilon_0\epsilon_r}\right)^2 \int_0^R r^2 4\pi r^2 dr\right.$$

$$\left. + \left(\frac{\rho_f}{3\epsilon_0}\right)^2 R^6 \int_0^R \frac{1}{r^4} 4\pi r^2 dr\right]$$

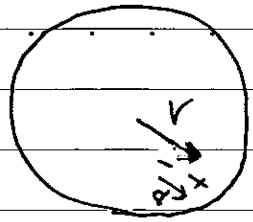
$$= \frac{2\pi}{9\epsilon_0} \rho_f^2 R^5 \left(1+\frac{1}{5\epsilon_r^2}\right) \quad \cdots (83)$$

is the work to assemble ρ_f & ρ_b.

$W - \bar{W}$ = work to polarize the material
= work to establish $\vec{p}(\vec{r})$.

$$= \frac{2\pi}{9\epsilon_0} \rho_f^2 R^5 \cdot \frac{\epsilon_r - 1}{5\epsilon_r^2} = \frac{2\pi}{45\epsilon_0\epsilon_r^2} \rho_f^2 R^5 (\epsilon_r - 1) \quad \cdots (84)$$

To check eq. (84), we first note that at $\vec{r}$, the electric field $E = \frac{\rho_f}{3\epsilon} r$



Therefore, the dipole at r is stretched by E.

If $\pm q$ charge is separated by d and their connection is characterized by a spring with spring constant K,

one gets $$Eq = Kd = \frac{q\rho_f}{3\epsilon} r \quad \text{--- (85)}$$

$$\because\ qd = p = P(r)\,dz \quad \therefore\ q = \frac{P}{d}\,dz \quad \text{--- (86)}$$

$\therefore$ (85) & (86) imply $$Kd^2 = \frac{\rho_f}{3\epsilon} r\, P(\vec{r})\, dz \quad \text{--- (87)}$$

The work on spring for dz

$$dW_{spring} = \frac{1}{2}Kd^2 = \frac{\rho_f}{6\epsilon_0\epsilon_r} r\, P\, dz$$

$$\therefore\ W_{spring} = \frac{\rho_f}{6\epsilon_0\epsilon_r}\int r P(\vec{r})\, dz \quad \text{--- (88)}$$

Now $\vec{P} = \epsilon_0\chi_e\vec{E}$ $\therefore$ $$P = \epsilon_0\chi_e \frac{\rho_f}{3\epsilon_0\epsilon_r} r = \frac{\epsilon_r - 1}{3\epsilon_r}\rho_f r$$

$$\therefore\ W_{spring} = \frac{\rho_f}{6\epsilon_0\epsilon_r}\frac{\epsilon_r-1}{3\epsilon_r}\rho_f 4\pi\int_0^R r^4 dr = \frac{2\pi}{45\epsilon_0\epsilon_r^2}\rho_f^2 R^5(\epsilon_r - 1) = W - \bar{W}$$

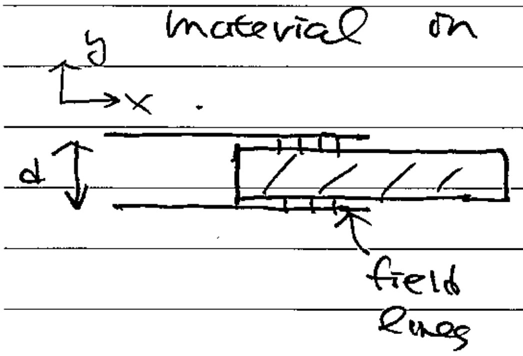
Forces on dielectrics

As we have shown, for a surface density σ on a conductor, there is a force acting on σ. The force per area

$$\vec{f} = \frac{\sigma^2}{2\epsilon_0}\hat{n}.$$

There are also forces acting on dielectric materials. However, the calcution of these forces are tricky.

For example, consider a slab of dielectric material on half way inside a capacitor.



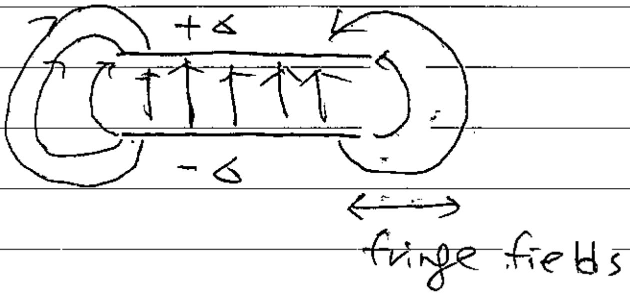
According to the field lines we have discussed, all field lines are perpendicular to the slab. There appears to be no tranverse force along x direction.

This, however, is not correct.

In real materials, there are fringe fields.

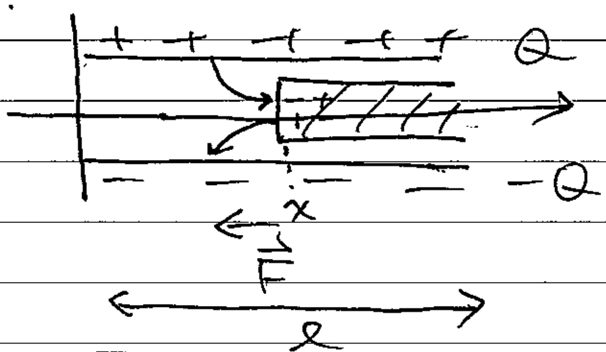
The field lines we obtain are based on symmetry using the Gauss's law. These field lines are only correct only near the center of plate.

Near edges, field lines are no longer perpendicular to slab or plates of capacitor as shown in the below.



Fields that deviate from fields at center are called fringe fields.

It is this fring fields that can drive the slab of dielectrics into the capacitor.



It would seem that we need to $\vec{E}_{fring}$ to find the force acting on dielectrics. This, however, turns out not to be necessary as shown below.

· The energy change of the whole system do not have to be concentrated in the fringing region.

Suppose we apply a force $\vec{F_a}$ to counter $\vec{F}$ and move the slab by dx, the work we need to do is

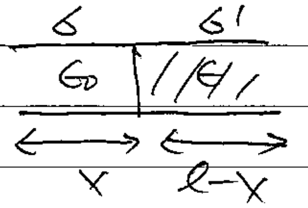
$$dW = F_a\,dx = -F\,dx$$

$$\therefore \quad F = -\frac{dW}{dx} \quad \cdots \text{(89)}$$

Now, for fixed Q, $W = \frac{1}{2}\frac{Q^2}{C} \quad \cdots \text{(90)}$

C depends on x and can be found:

computing V:



$V = Ed = E'd \Rightarrow \frac{\sigma}{\epsilon_0} = \frac{\sigma'}{\epsilon}$

total charge fixed

$$\sigma x + \sigma'(l-x) = \sigma_0 l = \frac{Q}{A} l$$

area of plate

$A = l w$

width

$$\therefore \quad \sigma = \frac{l}{(1-\frac{\epsilon}{\epsilon_0})x + \frac{\epsilon}{\epsilon_0} l}\,\frac{Q}{A}$$

$$C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{Q}{\frac{\sigma}{\epsilon_0} d} = \frac{\epsilon_0 w}{d}\left[\frac{\epsilon}{\epsilon_0} l + \left(1-\frac{\epsilon}{\epsilon_0}\right)x\right]$$

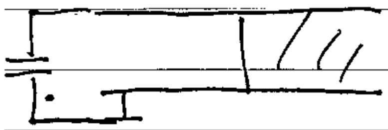
$$= \frac{\epsilon_0 w}{d}\left[\epsilon_r l - \chi_e x\right] \quad \cdots \text{(91)}$$

Combining eq. (89) & (90), (91), we get

$$F = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx} = \frac{1}{2}V^2\left(-\frac{\epsilon_0 w}{d}\chi_e\right) = -\frac{\epsilon_0 \chi_e w}{2d} V^2 \quad \cdots \text{(92)}$$

$\therefore F < 0$ the slab is attractive into the capacitor.!

For fixed potential, one can connect the capacitor to a battery,



In this case, in addition to do work on dielectrics, to maintain the potential, the battery also needs to do work to move in/out charges to keep V fixed.

$$\therefore dW = -F dx + V dQ$$

$$\therefore F = -\left.\frac{dW}{dx}\right|_{V\text{ fixed}} + V \left.\frac{dQ}{dx}\right|_{\text{fixed } V}$$

For fixed V, $W = \frac{1}{2} C V^2$, $\therefore \frac{dW}{dx} = \frac{V^2}{2} \frac{dC}{dx}$

On the other hand, $Q = CV$

$$\therefore V \left.\frac{dQ}{dx}\right|_{\text{fixed } V} = V^2 \frac{dC}{dx}$$

$$\therefore F = -\frac{V^2}{2} \frac{dC}{dx} + V^2 \frac{dC}{dx} = \frac{V^2}{2} \frac{dC}{dx}$$

which is in consisten with eq.(90)!

Note that if the work by battery is not included, F becomes repulsive and is not correct!