Conservation laws

In classical physics, conservations of energy and momentum are considered as important principles that go beyond Newton's laws. These principles allow one to define momentum & energy for fields.

This is the reason why action at a distance is abandoned while it is generalized to become the concept of fields.

One example, as we shall see, is the example we have seen: the Newton's 3rd law appears not to hold for particles interacting with magnetostatic fields. However, if one includes fields and assigns momentum to fields, the total momentum of the whole system is conserved.

The Newton's 3rd law, responsible for conservation of momentum of two particles, is then generalized to conservation of total momentum including that of fields.

Continuity equation

From $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$, one gets

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{B} = 0 = \mu_0\vec{\nabla}\cdot\vec{J} + \epsilon_0\epsilon_0\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E}$$

$\because \epsilon_0\vec{\nabla}\cdot\vec{E} = \rho$, one gets

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \quad \text{---} ①$$

which is the continuity equation obtained from the Maxwell's equation.

Eq. ① implies that charges are locally conserved. If one integrates ① in all space, one gets

$$\frac{d}{dt}\int_V \rho\, d\tau + \int_V \vec{\nabla}\cdot\vec{J}\, d\tau = 0$$

$$\because \int_V \vec{\nabla}\cdot\vec{J}\, d\tau = \int_{S=\infty} \vec{J}\cdot d\vec{a} = 0$$

there is no current flowing out at ∞

$\therefore \frac{d}{dt}Q = 0$ ---② $\quad Q = \int_{V=\infty}\rho\, d\tau$ = total charge is conserved.

This is the global conservation of charge.

Clearly, local conservation + $J \to 0$ on S implies global conservation of charges.

The converse is not true!

Energy conservation & the Poynting Theorem

As we have learned, the work necessary to assemble a static charge distribution goes to the field energy

$$W_E = \frac{\epsilon_0}{2} \int E^2 d\tau.$$

The work necessary to get currents goes to the magnetic field energy

$$W_M = \frac{1}{2\mu_0} \int B^2 d\tau$$

$\therefore$ It suggests that total energy stored in electromagnetic fields per volume is

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) \quad \cdots (3)$$

To check the validity of eq. ③, we consider the work done on a charge q by $\vec{E}$ & $\vec{B}$ during dt

$$= \vec{F}\cdot d\vec{\ell} = q(\vec{E}+\vec{v}\times\vec{B})\cdot\vec{v}\,dt$$

$$= q\vec{E}\cdot\vec{v}\,dt \quad \cdots ④$$

$\because$ $q \to \rho\, d\tau$, $\rho\vec{v} \to \vec{J}$ in continuous charge distribution

Total work on all charges in a volume V is

$$\frac{dW}{dt} = \int_V (\vec{E}\cdot\vec{J})\,d\tau \quad \cdots ⑤$$

$\therefore$ $\vec{E}\cdot\vec{J}$ = work done per volume per unit time.

Now, $\because$ $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$, we can rewrite

$$\vec{E}\cdot\vec{J} = \frac{1}{\mu_0}\vec{E}\cdot\vec{\nabla}\times\vec{B} - \epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$$

$$= \frac{1}{\mu_0}\vec{E}\cdot\vec{\nabla}\times\vec{B} - \frac{\epsilon_0}{2}\frac{\partial E^2}{\partial t} \quad \cdots ⑥$$

Using $\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot\vec{\nabla}\times\vec{E} - \vec{E}\cdot\vec{\nabla}\times\vec{B}$

$$= -\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{E}\cdot\vec{\nabla}\times\vec{B}$$

We have

$$\vec{E}\cdot\vec{\nabla}\times\vec{B} = -\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{\nabla}(\vec{E}\times\vec{B})$$

$$= -\frac{1}{2}\frac{\partial B^2}{\partial t} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

∴ Eq. ⑥ becomes

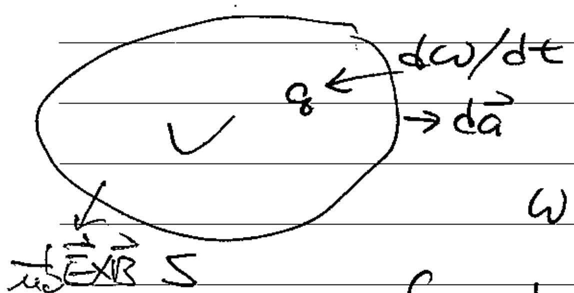
$$\vec{E}\cdot\vec{J} = -\frac{1}{2\mu_0}\frac{\partial B^2}{\partial t} - \frac{\epsilon_0}{2}\frac{\partial E^2}{\partial t} - \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

Hence

$$\frac{dW}{dt} = -\frac{d}{dt}\int_V \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau$$

$$-\frac{1}{\mu_0}\int_V \vec{\nabla}\cdot(\vec{E}\times\vec{B})\,d\tau$$

$$\underset{\text{Stoke's theorem}}{=} -\frac{d}{dt}\int_V \underbrace{\frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)}_{u}d\tau - \frac{1}{\mu_0}\oint_S \vec{E}\times\vec{B}\cdot d\vec{a} \quad \text{--- ⑦}$$



Eq. ⑦ is the Poynting theorem which is the work-energy theorem for electrodynamics. It says

Work done on all charges in V

= decrease of energy remaining in fields in V
− energy flow out of V.

From the Poynting theorem, one identifies that the total energy stored in E & B is

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

and energy transferred across surface per unit per area

$$= \vec{S} \equiv \frac{1}{\mu_0}(\vec{E}\times\vec{B}) \quad \text{--- (8)}$$

which is called the Poynting vector.

$\vec{S}$ = energy flux density

$\vec{S}\cdot d\vec{a}$ = energy acrossing $d\vec{a}$ per unit time

Energy conservation

In the presence of charges, from eq. (4)

$$\vec{F}\cdot d\vec{\ell} = m\frac{d\vec{v}}{dt}\cdot\frac{d\vec{\ell}}{dt}\,dt$$

$$= m\vec{v}\cdot\frac{d\vec{v}}{dt}\,dt = d\left(\frac{1}{2}mv^2\right)$$

$$\therefore \frac{dW}{dt} = \frac{d}{dt}\sum_i \frac{1}{2}m_i v_i^2 = \frac{d}{dt}E_K$$

$E_K = \sum_i \frac{1}{2}m_i v_i^2$ = Kinetic energy of charges.

Hence

$$\frac{dE_K}{dt} = -\frac{d}{dt}\int_V u\, d\tau - \oint_S \vec{S}\cdot d\vec{a} \quad \text{--- (9)}$$

which expresses that the energy is conserved

if one includes both fields & particles

In the absence of charges, say, in

empty space, there is no charge,

$E_K = 0$

$$\therefore \frac{d}{dt}\int_V u\, d\tau + \oint_S \vec{S}\cdot d\vec{a} = 0$$

$$\because \oint \vec{S}\cdot d\vec{a} = \int \vec{\nabla}\cdot\vec{S}\, d\tau$$

$$\therefore \int_V \left(\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S}\right) d\tau = 0$$

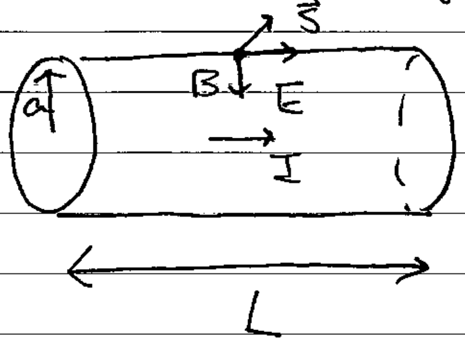
take $V \to 0$, one gets $\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0$ --- (10)

This is the continuity equation of

electromagnetic energy. $u \Leftrightarrow \rho$

$\vec{S} \Leftrightarrow \vec{J}$

Example Poynting vector for conducting wire



$$E = \frac{V}{L}$$

$$B = \frac{\mu_0 I}{2\pi a} \quad \text{at surface}$$

$$E \perp B$$

$$\therefore S = \frac{1}{\mu_0} EB = \frac{1}{\mu_0}\frac{V}{L}\frac{\mu_0 I}{2\pi a}$$

$$= \frac{VI}{2\pi a L}$$

$\vec{S} \parallel \hat{r}$ $\therefore$ Energy per unit time passing through the surface

$$= \int \vec{S}\cdot d\vec{a} = S\cdot(2\pi a L) = IV$$

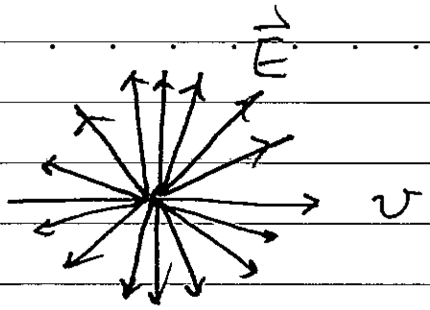
$=$ work done on I

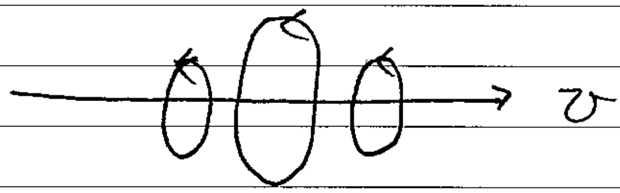
Consistent with our expectation.

Momentum in EM fields

Newton's 3rd law violation in charged particles

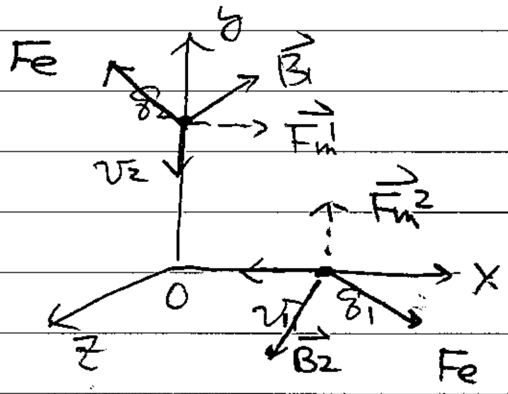
When charges move, the electric fields they generate can't be described Coulomb's law. However, as we shall see later, $\vec{E}$ still points radially for the case when the velocity of the particle v is a constant.





Similarly, $\vec{B}$ fields are circulating around $\vec{v}$ as the case for $\vec{B}$ generated by currents.

Consider two charges q_1 & q_2 that move with $\vec{v_1} = v_1(-\hat{x})$, $\vec{v_2} = -v_2\hat{y}$ as shown in the below.



Clearly, the electric forces that q_1 & q_2 experience are opposite and are in agreement with the Newton's 3rd law.

However, the magnetic field that acts on q_2 due to q_1 points in negative z direction

$$\vec{B_1} = -B_1\hat{z} \quad \cdots (11)$$

While the magnetic fields that acts on q_1 due to q_2 is in positive z direction

$$\vec{B_2} = B_2\hat{z} \quad \cdots (12)$$

$\frac{B_1}{q_1 v_1} = \frac{B_2}{q_2 v_2}$ if distances of q_1 & q_2 to O are the same (qv = current)

As a result, the magnetic force that acts on q_1 is

$$\vec{F_m}^1 = q_2 \vec{v_2} \times \vec{B_1}$$
$$= q_2 v_2 B_1 \hat{x}$$

Similarly, the magnetic force that acts on q_2 is

$$\vec{F_m}^2 = q_1 \vec{v_1} \times \vec{B_2}$$
$$= q_1 v_1 B_2 \hat{y}$$

$\because \frac{B_1}{q_1 v_1} = \frac{B_2}{q_2 v_2} \quad \therefore \quad |\vec{F_m}^1| = |\vec{F_m}^2|$

Clearly, the magnitudes of the magnetic force are the same but their directions are not.

As a result, $\vec{F_m}^1 + \vec{F_m}^2 \neq 0$ --- (13)

The Newton's 3rd law is violated in magnetic forces of charged particles.

Equivalently, since $\vec{F} = \frac{d\vec{p}}{dt} = 0$, when

$\vec{F_1} + \vec{F_2} = \frac{d(\vec{p_1} + \vec{p_2})}{dt} \neq 0 \qquad \vec{p_1} + \vec{p_2} \neq \text{constant}$

$\therefore$ Momentum is not conserved!

i.e. internal forces are not cancelled

$\therefore$ There is no conservation of momentum among particles !

Momentum conservation is generally believed to be very general. Then what happens?

It turns that momentum conservation is correct by realizing that fields themselves also carry momentum.

Maxwell's Stress Tensor

To find where the momentum goes to, we calculate the electromagnetic force on charges in volume V:

$$\vec{F} = \int \rho(\vec{E} + \vec{v}\times\vec{B})\,d\tau$$

$$= \int (\rho\vec{E} + \vec{J}\times\vec{B})\,d\tau \quad \cdots \quad (14)$$

$\therefore$ Force per volume

$$\vec{f} = \rho\vec{E} + \vec{J}\times\vec{B}$$

Replacing $\rho = \varepsilon_0 \vec{\nabla}\cdot\vec{E}$, $\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \varepsilon_0\frac{\partial\vec{E}}{\partial t}$, one gets

$$\vec{f} = \varepsilon_0 (\vec{\nabla}\cdot\vec{E})\vec{E} + \left(\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)\times\vec{B}$$

$$\because \quad \frac{\partial \vec{E}}{\partial t}\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \vec{E}\times\frac{\partial \vec{B}}{\partial t}$$

$$= \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times(\vec{\nabla}\times\vec{E})$$

$$\uparrow$$

$$-\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times\vec{E}$$

$$\therefore \vec{f} = \varepsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E})\right] - \frac{1}{\mu_0}\vec{B}\times(\vec{\nabla}\times\vec{B}) - \varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) \quad \text{---(14)---}$$

Now, $\vec{E}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\underbrace{\left(\frac{\vec{E}\cdot\vec{E}}{2}\right)}_{} - (\vec{E}\cdot\vec{\nabla})\vec{E}$ $\leftarrow$ ∇ only acts on $\vec{C}$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C})$$

$$\therefore \quad \vec{E}\times(\vec{\nabla}\times\vec{E}) = \frac{1}{2}\nabla E^2 - (\vec{E}\cdot\vec{\nabla})\vec{E}$$

Similarly, $\vec{B}\times(\vec{\nabla}\times\vec{B}) = \frac{1}{2}\nabla \vec{B}^2 - (\vec{B}\cdot\vec{\nabla})\vec{B}$

$\because$ $\vec{\nabla}\cdot\vec{B} = 0$, one may include a term $(\vec{\nabla}\cdot\vec{B})\vec{B}$ to make $\vec{f}$ more symmetric:

$$\vec{f} = \varepsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E}\right] + \frac{1}{\mu_0}\left[(\vec{\nabla}\cdot\vec{B})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B}\right]$$

$$-\frac{1}{2}\nabla\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) \quad \text{--- (15)}$$

Eq. (15) can be written in a more concise way by introducing the idea of 2nd-rank tensor t_{ij} $i, j = x, y, z$

$\therefore \{t_{ij}\}$ has 9 components which can ~~be arranged in a 3×3 matrix~~

denoted by $\overleftrightarrow{t}$:

$$\overleftrightarrow{t} = \begin{pmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{pmatrix} \quad \text{---} (16)$$

which is similar to moment of inertia $\overleftrightarrow{I}$ in describing dynamics of rotation or quadrupole moment

$$\overleftrightarrow{Q} = \begin{pmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{yx} & Q_{yy} & Q_{yz} \\ Q_{zx} & Q_{zy} & Q_{zz} \end{pmatrix}$$

Just as matrices, if $\vec{v}$ is a vector (v_x, v_y, v_z)

$$\vec{v}\cdot\overleftrightarrow{t} = (v_x \; v_y \; v_z) \begin{pmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{pmatrix}$$

$$= (v_x t_{xx} + v_y t_{yx} + v_z t_{zx}, \; '' \;, \; '' \;)$$

i.e. $(\vec{v}\cdot\overleftrightarrow{t})_{\underline{\underline{j}}} = \sum_i v_i t_{i\underline{\underline{j}}}$ --- (17)

while $$(\overleftrightarrow{t}\cdot\vec{v}) = \begin{pmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} t_{x\underline{x}} v_{\underline{x}} + t_{x\underline{y}} v_{\underline{y}} + t_{x\underline{z}} v_{\underline{z}} \\ '' \\ '' \end{pmatrix}$$

$$(\overleftrightarrow{t}\cdot\vec{v})_i = \sum_j t_{i\underline{\underline{j}}} v_j \quad \text{---} (18)$$

With the tensor notation, one can rewrite Eq (15) in a more compact form.

For instance

$$(\vec{\nabla}\cdot\vec{E})E_j = \sum_i \left(\frac{\partial}{\partial x_i}E_i\right)E_j$$

$$= \sum_i \frac{\partial}{\partial x_i}(E_i E_j) - E_i \frac{\partial}{\partial x_i} E_j$$

Let $D_i = \frac{\partial}{\partial x_i}$ $\left(= \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \; i=1,2,3\right)$

$$\therefore (\vec{\nabla}\cdot\vec{E})E_j = \sum_i D_i(E_i E_j) - (\vec{E}\cdot\vec{D})\vec{E_j}$$

$$(\vec{\nabla}\cdot\vec{E})E_j + (\vec{E}\cdot\vec{D})\vec{E_j} = (\vec{D}\cdot\overleftrightarrow{T_1})_j = \sum_i D_i(\overleftrightarrow{T_1})_{ij}$$

$$(\overleftrightarrow{T_1})_{ij} = E_i E_j \quad \cdots (19)$$

Similarly, $(\nabla E^2)_j = \sum_i D_i(\delta_{ij}E^2) = D_j E^2$

$\delta_{ij} = 1$ if $i=j$, otherwise $\delta_{ij}=0$

$\therefore \vec{f}$ can be rewritten as (Kronecker delta)

$$f_j = (\vec{D}\cdot\overleftrightarrow{T})_j - \left[\epsilon_0 \frac{\partial}{\partial t}(\vec{E}\times\vec{B})\right]_j$$

where $T_{ij} = \epsilon_0(E_i E_j - \frac{1}{2}\delta_{ij}E^2) + \frac{1}{\mu_0}(B_i B_j - \frac{1}{2}\delta_{ij}B^2)$

L. (20)

$\because \vec{E}\times\vec{B} = \mu_0\vec{S}$ ($\vec{S}$ = Poynting vector)

$$\therefore \vec{f} = \vec{\nabla}\cdot\overleftrightarrow{T} - \mu_0\epsilon_0 \frac{\partial \vec{S}}{\partial t} \quad \cdots (21)$$

$\overleftrightarrow{T}$ is known as the Maxwell's stress tensor.

From eq. (14), one gets that the total force on charges in V is

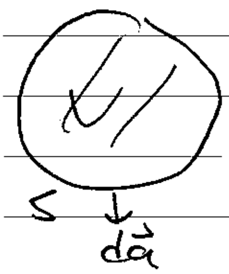
$$\vec{F} = \int_V \vec{\nabla}\cdot\overleftrightarrow{T}\, d\tau - \mu_0\epsilon_0 \frac{d}{dt}\int_V \vec{S}\, d\tau$$

$\cdots (22)$

For j component,

$$\int_V (\vec{\nabla}\cdot\overleftrightarrow{T})_j\, d\tau$$

$$= \int_V \sum_i \partial_i T_{ij}\, d\tau = \int_V \vec{\nabla}\cdot\vec{t_j}\, d\tau$$



$$\vec{t_j} = (T_{xj}, T_{yj}, T_{zj})$$

$$= \oint_S \vec{t_j}\cdot d\vec{a} = \oint_S \sum_i T_{ij}\, da_i$$

$$= \oint_S (d\vec{a}\cdot\overleftrightarrow{T})_j$$

$$\therefore \vec{F} = \oint_S d\vec{a}\cdot\overleftrightarrow{T} - \epsilon_0\mu_0 \frac{d}{dt}\int_V \vec{S}\, d\tau \quad \cdots (23)$$

Meaning: Eq. 23 implies that the total force

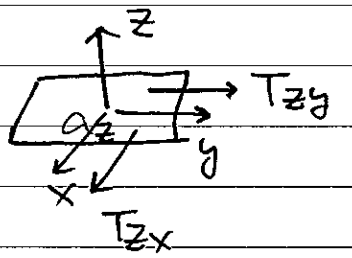
$= $ force acts on surface of V

$+ \frac{d}{dt}\left(-\varepsilon_0 \mu_0 \int \vec{S}\, d\tau\right)$

$\therefore \overleftrightarrow{T}$ contributes force in the following way:

$\because (F_T)_j = \oint_S da_i\, T_{ij}$

$\therefore$ For a given area in z direction



$$\frac{F_j}{a_z} = T_{zj}$$

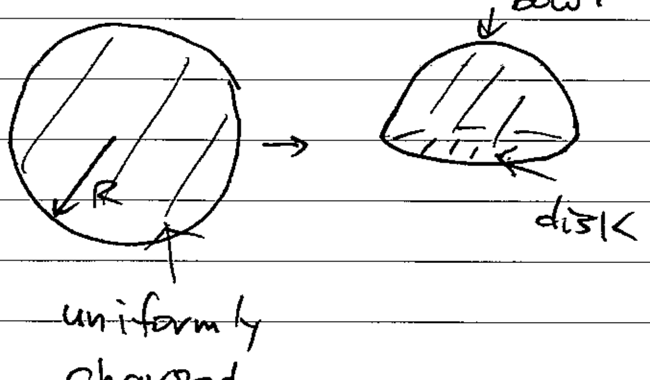
$T_{zj} =$ force per area in j direction that acts on $a\hat{z}$

$\therefore$ Generally, $T_{ij} =$ force per area in j direction that acts on $\vec{a}$ in i direction

If $i=j$, $T_{ij} =$ pressure that acts on a!

Generally, $i \neq j$ T_{ij} is the force per area in parallel to the surface; we call them shears. Overall, $\overleftrightarrow{T}$ is the stress tensor due to electromagnetic fields.

Example

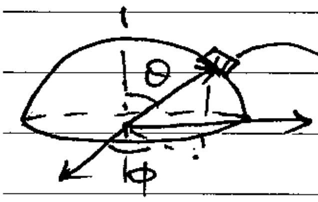


force on northern hemisphere

→ take S = surface of bowl + disk

Solution: $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\hat{r}$ on surfaces of sphere

bowl surface



$$\because d\vec{a} = R^2\sin\theta\, d\theta\, d\phi\, \hat{r}$$

$$= R^2\sin\theta\, d\theta\, d\phi \times (\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z})$$

$$= da_x\hat{x} + da_y\hat{y} + da_z\hat{z} \quad \text{— ①}$$

By symmetry, net force should be along z direction $\therefore$ We will focus on z component.

$$\because (d\vec{a}\cdot\overleftrightarrow{T})_z = da_x T_{xz} + da_y T_{yz} + da_z T_{zz} \quad \text{— ②}$$

We need to find T_{xz}, T_{yz} & T_{zz}:

$$T_{xz} = \epsilon_0 E_x E_z = \epsilon_0\left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \sin\theta\cos\phi\cos\theta$$

$$T_{yz} = \epsilon_0 E_y E_z = \epsilon_0\left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \sin\theta\sin\phi\cos\theta$$

$$T_{zz} = \epsilon_0\left(E_z^2 - \frac{E^2}{2}\right) = \epsilon_0\left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2\left[\cos^2\theta - \frac{1}{2}\right] \quad \text{— ③}$$

$$= \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2(\cos^2\theta - \sin^2\theta)$$

Combining eqs ①-③, one gets

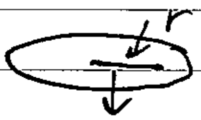
$$(d\vec{a}\cdot\overleftrightarrow{T})_z$$

$$= \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 \sin\theta\cos\theta\, d\theta\, d\phi \quad \text{for bowl's surface}$$

$$\therefore F_{bowl} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin\theta\cos\theta$$

$$= \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 \times 2\pi \times \frac{1}{2} = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{8R^2}$$

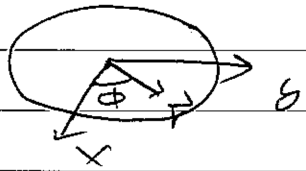
For disk, $d\vec{a} = -r\,dr\,d\phi\,\hat{z}$ — ④



$d\vec{a} = -\hat{z}\cdot r\,dr\,d\phi$

Inside sphere at radius r

$$E\cdot 4\pi r^2 = \frac{1}{\epsilon_0}Q(r) = \frac{1}{\epsilon_0}\frac{Q}{\frac{4}{3}\pi R^3}\times\frac{4\pi}{3}r^3$$



$$\therefore \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\vec{r} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}(r\cos\phi\,\hat{x} + r\sin\phi\,\hat{y})$$

∴

$$\therefore T_{xz} = \epsilon_0 E_x E_z = 0 \quad (E_z = 0)$$

$$T_{yz} = \epsilon_0 E_y E_z = 0$$

$$T_{zz} = \frac{\epsilon_0}{2}(E_z^2 - E_x^2 - E_y^2) = -\frac{\epsilon_0}{2}E^2$$

$$= -\frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 r^2 \quad \text{— ⑤}$$

$$\therefore (d\vec{a}\cdot\overleftrightarrow{T})_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 r^3\,dr\,d\phi$$

↑ ④ & ⑤

$$\therefore F_{disk} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 \int_0^{2\pi}\int_0^R r^3\,dr\,d\phi$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\theta^2}{16R^2} \quad \text{----} \quad (16)$$

$\therefore \ F = F_{bowl} + F_{disk}$

$$= \frac{1}{4\pi\epsilon_0}\frac{3\theta^2}{16R^2}$$

<u>Alternatively</u>: $\because \ \vec{E} \parallel \hat{r}, \quad \vec{\nabla}\cdot\vec{E} = 0$

$$\therefore \ \vec{\nabla}\cdot\overleftrightarrow{T} = \epsilon_0\left[\underbrace{(\vec{\nabla}\cdot\vec{E})}_{0}\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E} - \frac{1}{2}\nabla E^2\right]$$

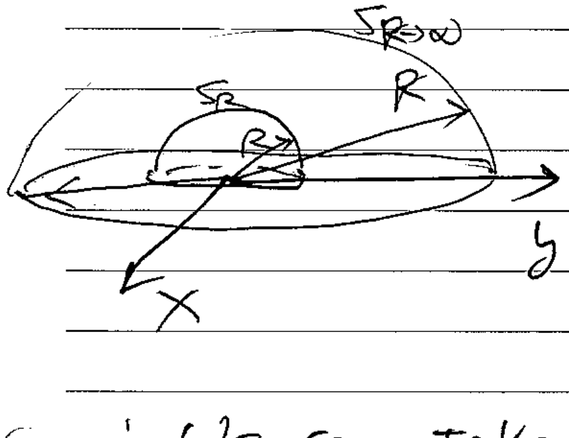
$$(\vec{E}\cdot\vec{\nabla})\vec{E} = \left(E\frac{\partial}{\partial r}\right)E\hat{r} = \frac{1}{2}\hat{r}\frac{\partial}{\partial r}E^2$$

$$= \frac{1}{2}\vec{\nabla}_r E^2$$

$$\therefore \ (\vec{\nabla}\cdot\overleftrightarrow{T})_r = \epsilon_0\left[\vec{\nabla}_r\frac{E^2}{2} - \frac{1}{2}\nabla_r E^2\right] = 0$$

$$(\vec{\nabla}\cdot\overleftrightarrow{T})_{\theta,\phi} = 0 \quad \therefore \ \vec{\nabla}\cdot\overleftrightarrow{T} = 0$$

$r \to \infty$



$$\therefore \int_{V_r - V_R} \vec{\nabla}\cdot\overleftrightarrow{T}\, d\tau = 0$$

$$\therefore \oint_{disk+bowl} d\vec{a}\cdot\overleftrightarrow{T} = \int_{S_{R\to\infty}+S_{r=\infty}} d\vec{a}\cdot\overleftrightarrow{T}$$

$\therefore$ We can take bowl (with $R\to\infty$) + the whole xy plane

$$\left.\begin{array}{l} \because \ E \to 0 \text{ at } S_{R\to\infty} \ \& \ T \sim \frac{1}{R^4} \\ \quad \sim \frac{1}{R^2} \qquad\qquad\qquad da \sim R^2 \end{array}\right\} \int_{S_{R=\infty}} d\vec{a}\cdot\overleftrightarrow{T} \to 0$$

$\therefore F = \int_{xy\ plane} d\vec{a} \cdot \overleftrightarrow{T}$

$= \int_{disk} d\vec{a} \cdot \overleftrightarrow{T} + \int_{r>R} d\vec{a} \cdot \overleftrightarrow{T}$

$= F_{disk} + \int_{r>R} d\vec{a} \cdot \overleftrightarrow{T}$

For $r>R$ on equator,

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} (\cos\phi\,\hat{x} + \sin\phi\,\hat{y})$$

i.e. set $R=r$ in $E(r<R)$

$\therefore T_{xz}=0,\ T_{yz}=0$ are still correct

$$T_{zz} = -\frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 \frac{1}{r^4}$$

$$\therefore \int_{r>R} d\vec{a} \cdot \overleftrightarrow{T} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 \int_R^\infty \frac{1}{r^4}\, r \cdot dr \int_0^{2\pi} d\phi$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Q^2}{8R^2}$$

$$\therefore F = F_{disk} + \frac{1}{4\pi\epsilon_0}\frac{Q^2}{8R^2}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{3Q^2}{16R^2}$$ is the same.

Conservation of momentum

From eq. (23), $\vec{F} = \oint_S d\vec{a} \cdot \overleftrightarrow{T} - \epsilon_0 \mu_0 \frac{d}{dt} \int_V \vec{S} \, d\tau$.

Since $\vec{F}$ = force on charge particles,

$\vec{F} = \frac{d \vec{P}_{mech}}{dt}$ $\vec{P}_{mech}$ denotes momentum of particle. "Mechanical" indicates it's mechanical origin.

$$\therefore \quad \frac{d \vec{P}_{mech}}{dt} = -\epsilon_0 \mu_0 \frac{d}{dt} \int_V \vec{S} \, d\tau + \oint_S d\vec{a} \cdot \overleftrightarrow{T}$$

i.e $$\oint_S d\vec{a} \cdot \overleftrightarrow{T} = \frac{d \vec{P}_{mech}}{dt} + \epsilon_0 \mu_0 \frac{d}{dt} \int_V \vec{S} \, d\tau \quad \cdots (24)$$

Eq. (24) has a clear implication:

RHS = $\underset{\text{total}}{\wedge}$ force on volume V

LHS = momentum change rate of particles

$+ \epsilon_0 \mu_0 \frac{d}{dt} \int_V \vec{S} \, d\tau$

Therefore, one identifies the momentum stored in field as $\epsilon_0 \mu_0 \int_V \vec{S} \, d\tau$.

$$\therefore \vec{P}_{field} = \mu_0 \epsilon_0 \int_V \vec{S}\, d\tau \quad \cdots (25)$$

The momentum density in fields (momentum per volume)

$$\vec{g} \equiv \mu_0 \epsilon_0 S = \epsilon_0 (\vec{E} \times \vec{B}) \quad \cdots (26)$$

Furthermore, one can re-interpretate

$\oint_S d\vec{a} \cdot \overleftrightarrow{T}$ as momentum flowing through S into V per unit time.

$\therefore -d\vec{a} \cdot \overleftrightarrow{T}$ = electromagnetic momentum passing $d\vec{a}$ (into direction of $d\vec{a} > 0$) per unit time.

$\therefore -\overleftrightarrow{T}$ = electromagnetic momentum per area per unit time
= momentum flux

Note that current = charge flux
= $\rho \cdot \vec{v}$ ρ = charge density

this is current of scalar.

For flux of momentum, it is "$\vec{g}\,\vec{v}$" so it carries two indices. One can view it as $g_x \vec{v}$, $g_y \vec{v}$ & $g_z \vec{v}$

More precisely, $-\sum_i da_i T_{ij}$ = electromagnetic momentum $\vec{j}$-component that passes $d\vec{a}$ per unit time

$\therefore -(T_{xj}, T_{yj}, T_{zj})$ = momentum j flux

$\simeq$ "g_j" $\vec{v}$

↑ j component of momentum density

In the case of empty space or $\vec{P}_{mech}$ = fixed,

Eq. (24) becomes

$$\oint_S d\vec{a}\cdot\overleftrightarrow{T} = \frac{d}{dt}\int_V \vec{g}\, d\tau = \int_V \frac{\partial \vec{g}}{\partial t} d\tau$$

$$\because \oint_S d\vec{a}\cdot\overleftrightarrow{T} = \int_V \vec{\nabla}\cdot\overleftrightarrow{T} d\tau$$

$$\therefore \int_V \frac{\partial \vec{g}}{\partial t} d\tau = \int_V \vec{\nabla}\cdot\overleftrightarrow{T} d\tau$$

$$\therefore \frac{\partial \vec{g}}{\partial t} = \vec{\nabla}\cdot\overleftrightarrow{T} \quad \cdots \text{(25)}$$

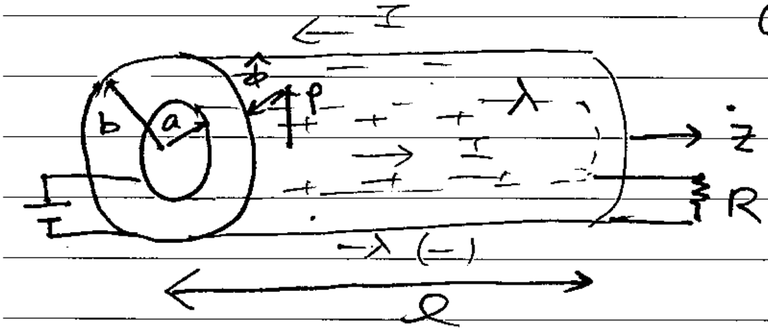
This is continuity equation of electromagnetic momentum. with $-\overleftrightarrow{T}$ playing the role of momentum flux. Eq. (25) implies $\int \vec{g}\, d\tau$ is conserved if $V=\infty$.

No.
Date.

In general, $\frac{d\vec{P}_{mech}}{dt} \neq 0$; eq. (25) is not valid. Hence both the monentum of particles & electromagnetic momentum are not conserved individually. Their sum is conserved, however.

Example

a Coaxial cable



Find the momentum stored in the fields

Solution:

$$\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{\rho}\hat{\rho} \qquad a<\rho<b, \quad V = \int_a^b E\,d\rho$$

$$\vec{B} = \frac{\mu_0}{2\pi}\frac{I}{\rho}\hat{\phi} \qquad = \frac{1}{2\pi\epsilon_0}\lambda \ln b/a$$

$$\therefore \vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{(2\pi)^2\epsilon_0}\frac{\lambda I}{\rho^2}\underbrace{\hat{\rho}\times\hat{\phi}}_{\hat{z}}$$

$$= \frac{1}{4\pi^2\epsilon_0}\frac{\lambda I}{\rho^2}\hat{z}$$

$$\therefore \vec{g} = \mu_0\epsilon_0\vec{S} = \frac{\mu_0}{4\pi^2}\frac{\lambda I}{\rho^2}\hat{z}$$

$\therefore \vec{p} = \mu_0 \epsilon_0 \int \vec{S}\, d\tau$

$$= \frac{\mu_0 \lambda I}{4\pi^2} \hat{z} \int_a^b \frac{1}{\rho^2} 2\pi\rho\, d\rho \times l$$

$$= \frac{\mu_0 \lambda I l}{2\pi} \ln b/a\ \hat{z} = \frac{\lambda \ln b/a}{2\pi\epsilon_0} \mu_0 \epsilon_0 l I \hat{z}$$

$$= \frac{1}{c^2} I V l \hat{z} \neq 0$$

Power transport :

$$P = \int \vec{S} \cdot d\vec{a} = \frac{\lambda I}{4\pi^2 \epsilon_0} \int_a^b \frac{1}{\rho^2} 2\pi\rho\, d\rho$$

$$= \frac{\lambda I}{2\pi\epsilon_0} \ln b/a = IV$$

When I decreases to 0, it induces an electric field.

$$\vec{E} = \left[\frac{\mu_0}{2\pi} \frac{dI}{dt} \ln\rho + K\right] \hat{z}$$

which exerts forces on $\pm\lambda$

$$\vec{F} = \lambda l \left[\frac{\mu_0}{2\pi} \frac{dI}{dt} \ln b + K\right] \hat{z}$$

$$- \lambda l \left[\frac{\mu_0}{2\pi} \frac{dI}{dt} \ln a + K\right] \hat{z} = -\frac{\mu_0 \lambda l}{2\pi} \frac{dI}{dt} (\ln b/a) \hat{z}$$

$$\therefore \vec{P}_{charges} = \vec{P}_{mech} = \int_0^T \vec{F}\, dt = -\frac{\mu_0 \lambda l}{2\pi} \ln b/a\, \hat{z} \int_I^0 dI$$

$= \frac{\mu_0 \lambda l I}{2\pi} \ln b/a\ \hat{z}$ = precisely $\vec{p}$ stored in fields

Hence turning off I restores the momentum back to charges!
from fields

Now there may seem to be strange:
why $\vec{E}$ & $\vec{B}$ are static, there is a non-vanishing field momentum?
The answer is that there is an energy flow from the battery to resistance.
According to the special relativity, the energy flow must accompany a momentum flow. We shall come back to it in Chapter 12. (Ex 12.12).

Angular momentum

Since fields carry energy density

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

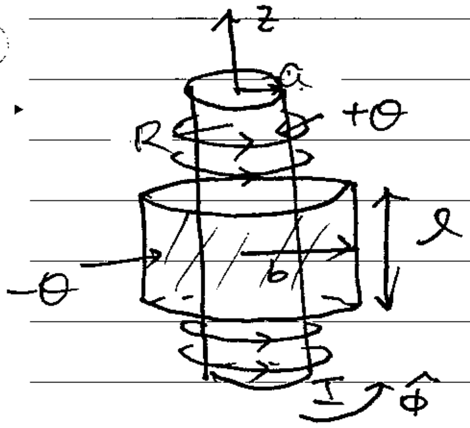
and momentum density $\vec{g} = \epsilon_0 (\vec{E} \times \vec{B})$.

it also carries angular moment

$$\vec{\ell} = \vec{r}\times\vec{g} = \epsilon_0\left[\vec{r}\times(\vec{E}\times\vec{B})\right] \quad \cdots (26)$$

Total angular momentum $= \vec{L}_{field} + \vec{L}_{matter}$ is conserved. Therefore, changes of angular momentum in fields induce rotation of matters as shown in the next example.

Example



Two coaxial conducting cylindrical shells (of radii $a<b$) carry $\pm Q$

Solenoid is also coaxial with radius $R>a$.

I decreases to 0 gradually.
Cylinders start to rotate.
Where does the angular momentum come from?

Solution Before I is turned off,

$$\vec{E} = \frac{Q}{2\pi\epsilon_0 \ell}\frac{1}{\rho}\hat{\rho} \qquad a<\rho<b \quad \text{between cylinders}$$

$$\vec{B} = \mu_0 n I \hat{z} \qquad \rho<R$$

$\therefore \vec{g} = \epsilon_0 \vec{E} \times \vec{B}$

$$= \frac{\mu_0 n I Q}{2\pi \ell \rho} \hat{\rho} \times \hat{z} = -\frac{\mu_0 n I Q}{2\pi \ell \rho} \hat{\phi}$$

for $a < \rho < R$

$$\therefore \ell_z = (\vec{r} \times \vec{g})_z = -\frac{\mu_0 n I Q}{2\pi \ell} = \text{constant}$$

$$\vec{L}_{field} = \ell_z \times \pi (R^2 - a^2) \hat{z}$$

$$= -\frac{1}{2} \mu_0 n I Q (R^2 - a^2) \hat{z}$$

in fields.

When I is turning off, one has an induced E_{in} field : $2\pi \rho E_{in} = -\frac{d}{dt} \Phi_B = -\mu_0 n \frac{dI}{dt} \cdot \text{area}$.

$$\therefore \vec{E}_{in} = \begin{cases} -\frac{1}{2} \mu_0 n \frac{dI}{dt} \frac{R^2}{\rho} \hat{\phi} & \rho > R \\ -\frac{1}{2} \mu_0 n \frac{dI}{dt} \rho \hat{\phi} & \rho < R \end{cases}$$

$\therefore$ Torque on outer cylinder

$$\frac{d\vec{L}_b}{dt} = \vec{\tau}_b = \vec{r} \times (-Q \vec{E}_{in}) = \frac{1}{2} \mu_0 n \frac{dI}{dt} R^2 Q \hat{z}$$

$$\therefore \vec{L}_b = \frac{1}{2} \mu_0 n R^2 Q \hat{z} \int_I^0 \frac{dI}{dt} dt = -\frac{1}{2} \mu_0 n R^2 Q I \hat{z}$$

On the other hand, torque on inner cylinder.

$$\frac{d\vec{L_a}}{dt} = \vec{\tau_a} = \vec{r} \times (Q\vec{E_{in}})$$

$$= -\frac{1}{2}\mu_0 n Q a^2 \frac{dI}{dt}\hat{z}$$

$$\therefore \vec{L_a} = \frac{1}{2}\mu_0 n I Q a^2 \hat{z}$$

$$\vec{L_a} + \vec{L_b} = \frac{1}{2}\mu_0 n I Q (a^2 - R^2) = -\vec{L_{field}}$$

$\therefore$ Angular momentum in fields is transformed into angular momenta of cylinders a and b.

<u>Field energy & momentum in Matter.</u>

In the presence of matter, eq. ① becomes

$$\frac{dW}{dt} = \int_V \vec{E}\cdot\vec{J_f}\, d\tau$$ is the work on free charges. --- (27)

$$\because \vec{\nabla}\times\vec{H} = \vec{J_f} + \frac{\partial \vec{D}}{\partial t}$$

$$\therefore \vec{E}\cdot\vec{J_f} = \vec{E}\cdot\vec{\nabla}\times\vec{H} - \vec{E}\cdot\frac{\partial \vec{D}}{\partial t}$$

$$\because \vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot\vec{\nabla}\times\vec{H} = \vec{H}\cdot\left(-\frac{\partial \vec{B}}{\partial t}\right) - \vec{E}\cdot\vec{\nabla}\times\vec{H}$$

$$\therefore\ \vec{E}\cdot\vec{J}_f = -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} - \nabla\cdot(\vec{E}\times\vec{H}) \quad \cdots (28)$$

$\therefore$ Eq (27) becomes

$$\frac{dw}{dt} = -\int_V\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right)d\tau$$

$$-\oint_S \vec{E}\times\vec{H}\cdot d\vec{a} \quad \cdots (29)$$

$\therefore$ The Poynting vector can be identified $\vec{S} = \vec{E}\times\vec{H}$ --- (30)

Furthermore, the energy density u becomes

$$\frac{\partial u}{\partial t} = \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t} \quad \cdots (31)$$

For linear materials, $\vec{D} = \epsilon\vec{E}$, $\vec{B} = \mu\vec{H}$

$$\therefore\ \frac{\partial u}{\partial t} = \frac{1}{2}\epsilon\frac{\partial E^2}{\partial t} + \frac{1}{2\mu}\frac{\partial B^2}{\partial t}$$

$$= \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{E}\cdot\vec{D} + \frac{1}{2}\vec{B}\cdot\vec{H}\right)$$

$$u = \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}) = \frac{1}{2}\epsilon E^2 + \frac{1}{2\mu}B^2 \quad \cdots (32)$$

To find $\vec{g}$, Eq. (14) becomes

$$\vec{f} = \rho_f\vec{E} + \vec{J}_f\times\vec{B} \quad \cdots (33)$$

ρ_f = density of free charges

$\because \rho_f = \vec{\nabla}\cdot\vec{D}, \quad \vec{J}_f = \vec{\nabla}\times\vec{H} - \frac{\partial \vec{D}}{\partial t}$

$\therefore \vec{f} = \vec{E}(\vec{\nabla}\cdot\vec{D}) + (\vec{\nabla}\times\vec{H})\times\vec{B} - \frac{\partial \vec{D}}{\partial t}\times\vec{B}$

$\because \frac{\partial \vec{D}}{\partial t}\times\vec{B} = \frac{\partial}{\partial t}(\vec{D}\times\vec{B}) - \vec{D}\times\frac{\partial \vec{B}}{\partial t}$

$= \frac{\partial}{\partial t}(\vec{D}\times\vec{B}) + \vec{D}\times(\vec{\nabla}\times\vec{E})$

$\therefore \vec{f} = \vec{E}(\vec{\nabla}\cdot\vec{D}) - \vec{D}\times(\vec{\nabla}\times\vec{E}) - \vec{B}\times(\vec{\nabla}\times\vec{H}) - \frac{\partial}{\partial t}(\vec{D}\times\vec{B})$

Adding $\vec{H}(\vec{\nabla}\cdot\vec{B}) = 0$, we get

$$\vec{f} = \left[\vec{E}(\vec{\nabla}\cdot\vec{D}) - \vec{D}\times(\vec{\nabla}\times\vec{E})\right] + \left[\vec{H}(\vec{\nabla}\cdot\vec{B}) - \vec{B}\times(\vec{\nabla}\times\vec{H})\right] - \frac{\partial}{\partial t}(\vec{D}\times\vec{B})$$

which generalizes eqs. (14)-1 & (15).

$\therefore$ One identifies the field momentum density

as $\vec{g} = \vec{D}\times\vec{B}$ --- (34)