Condensed Matter Physics (I): Homework 3
Due: November 11, 2019

Exercises in Ashcroft/Mermin

22-1,22-5, 22-4, 23-3, 25-3(a)

Ex.1 10% Evaluate number off independent elastic constants for orthorhombic and tetragoal lattice systems.

Ex.2 Discrete rotations and isotropy of tensors

A rank-n tensor in 2-dimension, $T^{\alpha_1\alpha_2\cdots\alpha_n}$, $\alpha_i = 1.2$, is an object which transforms under rotations via the rule

$$(R_\theta T)^{\alpha_1\alpha_2\cdots\alpha_n} = \sum_{\beta_1,\cdots,\beta_n} R(\theta)^{\alpha_1\beta_1}\cdots R(\theta)^{\alpha_n\beta_n} T^{\beta_1\cdots\beta_n} \tag{1}$$

where

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2}$$

is the rotation by angle θ. T is isotropic if

$$T^{\alpha_1\alpha_2\cdots\alpha_n} = \int_0^{2\pi} \frac{d\theta}{2\pi} (R_\theta T)^{\alpha_1\cdots\alpha_n}, \tag{3}$$

while it has discrete p-fold symmetry if

$$T^{\alpha_1\alpha_2\cdots\alpha_n} = \frac{1}{p}\sum_{j=0}^{p-1}\left(R_{\frac{2\pi j}{p}} T\right)^{\alpha_1\cdots\alpha_n}. \tag{4}$$

(a) 10% Show that T is isotroic if and only if $R_\theta T = T$ for any angle θ and it is a p-fold symmetric if and only if $R_{\frac{2\pi j}{p}} T = T$.

(b) 10% Show that if $n < p$, a p-fold symmetric tensor is isotropic. Hint: You may use the identity $\frac{1}{p}\sum_{j=-\infty}^{\infty}\delta(\theta - \frac{2\pi j}{p}) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ikp\theta}$ to show that, for $l + m < p$,

$$\int_0^{2\pi} \cos^l(\theta)\sin^m(\theta)\frac{d\theta}{2\pi} = \frac{1}{p}\sum_{j=0}^{p-1}\cos^l(\frac{2\pi j}{p})\sin^m(\frac{2\pi j}{p}). \tag{5}$$

Ex.3 20% Consider a linear chain of atoms with only nearest neighoboring interaction

$$U = \sum_{n=0}^{n=N}\frac{1}{2}m\omega^2|u_n - u_{n+1}|^2, \tag{6}$$

where m is the mass of the atom, ω is some characteristic frequence, and $u_n = u(na)$ with a being the lattice constant. The linear chain is fixed to a wall at $x = 0$ and $x = (N+1)a$.

(a) Find normal modes and determine all possible frequencies w_k of normal modes.

(b) Express the Hamiltonian in terms of phonon operators.

(c) Evaluate $\langle e^{iku_n} \rangle$ in terms of a summation over number of phonons. Here k is a wavevector.

Ex.4 10% Assuming that the change of the phonon mode of frequency $\omega_{\mathbf{k}}$ due to the change of volume is given by $\delta\omega_{\mathbf{k}}/\omega_{\mathbf{k}} = -\gamma\Delta$ with γ being independent of $\mathbf{k}$ (Grüneisen constant) and Δ being the fractional volume change, find Δ in terms of γ, total thermal energy due to phonons $U(T)$, and the bulk modulus B. Hint: consider the contribution due to the volume change to the free energy up to $O(\Delta^2)$.

Ex.5 10% A simple, monoatomic, two-dimensional square lattice is modelled by balls and springs. The lattice constant is a, the mass of the balls is m, the spring constant of the nearest-neighbor springs is k_1, and the srping constant between the next nearest neighbors (along the diagonal of the squares) is k_2. All other interactions can be neglected. Find the frequencies of phonons at $\mathbf{k} = (\pi/a, 0)$. (Hint: you do not need to find the complete solutions to find the answer.)

Ex.6 10% Wick's theorem for classical Gaussian averages

(a) Show that if Y is a Gaussian variable (i.e., its probability density is given by $P(Y) = \frac{e^{-*^2/2\sigma_y^2}}{\sqrt{2\pi}\sigma_y}$ for some σ_y) then $\langle e^{tY} \rangle = e^{\frac{1}{2}t^2\langle Y^2 \rangle}$ where t is an arbitraty (possibly complex) constant.

(b) Show that if $Y_1, Y_2, \ldots.., Y_n$ are Gaussian (that is, they have a joint probability density $N^{-1}\exp(-\frac{1}{2}\sum_{i,j=1}^{n} a_{ij}Y_iY_j)$, where N is the normalization constant and a_{ij} is a symmetric positive definite matrix), then $Y = t_1Y_1 + \cdots + t_nY_n$ is also Gaussian. Find $\langle Y_iY_j \rangle$.

(c) Using part (a), expanding both sides as a Taylor series in the t's, derive the Wick's theorem:

$$\langle Y_1Y_2Y_3\cdots Y_n \rangle = \sum_{\text{all possible pairing}} \langle Y_{i_1}Y_{j_1} \rangle \langle Y_{i_2}Y_{j_2} \rangle \cdots \langle Y_{i_p}Y_{j_p} \rangle \tag{7}$$