# Quantum Mechanics (II): Homework 6
# Due: June 8, 9 AM (hand-in box of Room 514 Physics Dept.), 2022

**Ex 1 (a) 15** In this problem, we verify the adiabatic theorem in the spin 1/2 system. Consider an electron in a time-dependent magnetic field

$$\mathbf{B} = B\,(\sin\theta\cos\omega t\,\hat{x} + \sin\theta\sin\omega t\,\hat{y} + \cos\theta\,\hat{z}\,).$$

Let us consider only the spin degree of freedom. Suppose that at $t = 0$ the spin state $|\psi(t=0)\,\rangle$ of the electron is spin-up along the $\mathbf{B}$ direction. By finding the exact solution $|\psi(t)\,\rangle$ to the time-dependent Schrödinger equation, show that $|\,\langle -, t|\psi(t)\,\rangle\,|^2 \to 0$ in the adiabatic limit when $T \to \infty$, where $T = 2\pi/\omega$ and $|\pm, t\rangle$ are instantaneous spin up/down states at time $t$. From here, find $|\psi(T)\,\rangle$ in the adiabatic limit and compare it with the prediction made by the adiabatic theorem.

**(b) 15** A particle of mass $m$ is in an infinite potential well with

$$\begin{aligned} V(x) &= 0 \text{ for } 0 \le x \le a + \upsilon t, \\ &= \infty \text{ otherwise.} \end{aligned}$$

Show that the following wavefunction is a solution

$$\Psi_n(x,t) = \sqrt{\frac{2}{\omega}}\sin\left(\frac{n\pi x}{\omega}\right)\; e^{i(m\upsilon x^2 - 2E_n a t)/(2\hbar\omega)}.$$

where $\omega = a + \upsilon t$ and $E_n = n^2\pi^2\hbar^2/2ma^2$. Find the dynamical phase and the Berry phase for this solution. From here, check how good the adiabatic limit is.

**Ex 2 10** As we have shown in the class that the solution to $(\mathrm{H}_0 + \mathrm{V})\,|\psi\rangle = E\,|\psi\rangle$ is formally given by the Lippmann-Schwinger equation

$$|\psi^+\rangle = |\phi\rangle + \frac{\mathrm{V}}{E - \mathrm{H}_0 + i\varepsilon}\,|\psi^+\rangle,$$

where $|\phi\rangle$ is a solution to $\mathrm{H}_0\,|\phi\rangle = E\,|\phi\rangle$. Derive this equation by the method of adiabatic switch-on. Hint: Turn on V from $t = -\infty$ to 0 by replacing V by $\mathrm{V}\exp(\varepsilon t/\hbar)$ with $\varepsilon \to 0^+$. Write the Schrödinger equation in the interaction picture and convert it to an integral equation.

**Ex 3 10** Consider a two-state system with the Hamiltonian given by $H = E_1|\,1\rangle\,\langle 1\,| + E_2|\,2\rangle\,\langle 2\,| + \gamma e^{i\omega t}\,|\,1\rangle\,\langle 2\,| + \gamma e^{-i\omega t}\,|\,2\rangle\,\langle 1\,|$. Suppose that initially, the system is in $|\,1\rangle$, find the probability of finding it in the state of $|\,2\rangle$ at a later time $t$.

**Ex 4 10** Exercise 18.2.1

**Ex 5 10** Exercise 18.2.6

**Ex 6 (a) 20** Find the one dimensional Lippmann-Schwinger equation. Suppose that the Hamiltonian for scattering is given by

$$H = \frac{p^2}{2m} + V(x),$$

where $V(x)$ vanishes for $|x| > a$. Consider the case when the particle incidents from $-\infty$ to $\infty$ with momentum $\hbar k$, show that the reflection $(R)$ and transmission coefficients $(T)$ are given by

$$R = \left|\frac{\sqrt{2\pi}}{2ik}\int_{-\infty}^{\infty} e^{ikx}U(x)\psi(x)\,dx\right|^2,$$

$$T = \left|1 + \frac{\sqrt{2\pi}}{2ik}\int_{-\infty}^{\infty} e^{-ikx}U(x)\psi(x)\,dx\right|^2,$$

where $U(x) = 2mV(x)/\hbar^2$ and $\psi(x)$ is the exact solution to $H$ with eignevalue $E = \hbar^2k^2/2m$.

**(b) 20** If $U(x) = -\alpha\delta(x)$, use the above result to find $R$ and $T$. Verify that $R + T = 1$. Find the exact Green's function.

**Ex 7 10** Using the Born approximation, calculate the total cross sections as functions of the wavenumber $k$ of the incident particles for the potentials

$$V(r) = -V_0 \text{ for } r < a \text{ and } V(r) = 0 \text{ for } r > a,$$
$$V(r) = V_0\exp(-r^2/2a^2).$$

**Ex 8 Galilean invariance of the Schrödinger equation**
Consider the Galilean transformation, $x' = x - vt$, $t' = t$, on a system with the Hamiltonian $H = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)$. In the unprimed system, we denote the wavefunction as $\psi(x,t)$, while in the primed system, the wavefunction is $\psi'(x',t')$.
**(a) 10** Verify that

$$\psi'(x',t') = \exp\left[\frac{-i}{\hbar}\left(mvx - \frac{mv^2}{2}t\right)\right]\psi(x,t)$$

**(b) 5** Give an argument why the above is correct without direct verification.
**(c) 5** What is the relation between the probability currents $J'(x',t')$ and $J(x,t)$ ?
**(d) 10** What would the electromagnetic potentials ( $A$, $\phi$) change under Galilean transformation?
**Ex 9 10** Verify the matrices in the Dirac equation, $\alpha_i$ and $\beta$ satisfy the relations

$$\begin{aligned} \alpha_i\alpha_j + \alpha_j\alpha_i &= 2\delta_{i\,j} \\ \alpha_i\beta + \beta\alpha_i &= 0 \\ \beta^2 &= 1. \end{aligned}$$

**Ex 10 10** Deduce the following integral formula

$$e^{i\delta}\sin\delta_l(k) = -\frac{2m}{\hbar^2}k\int_0^\infty j_l(kr)V(r)R_{l,k}(r)r^2dr,$$

where $\delta_l(k)$ is the phase shift for the spherical symmetrical potential $V(r)$ (with angular momentum $l$ and energy $E = \hbar^2k^2/2m$), $j_l(x)$ is the spherical Bessel function and $R_{k,l}$ is the radial part of the wavefunction.
**Ex 11 10** Ex.19.5.5.
**Ex 12** Consider a relativistic spin-1/2 particle in a central potential $V(r)$ governed by the Dirac Hamiltonian

$$H = c\vec{\alpha}\cdot\frac{\hbar}{i}\vec{\nabla} + \beta mc^2 + V(r),$$

where $\vec{\alpha} = \alpha_1, \alpha_2, \alpha_3$.
**(a) 5** Let the orbital angular moment be $\vec{L} = \vec{r}\times\frac{\hbar}{i}\vec{\nabla}$. Find $[H, L_i]$ with $i = 1,2,3$. From here, it is seen that unlike the non-relativisitc case, the Hamiltonian does not commute with the orbital angular moment.
**(b) 10** The total angular momentum in the Dirac equation is found to be $\vec{J} = \vec{L} + \frac{\hbar}{2}\vec{\Sigma}$ with $\Sigma_i = \frac{i}{2}\sum_{jk}\epsilon_{ijk}\beta\alpha_j\beta\alpha_k$. Find the matrix form of $\vec{\Sigma}$ and calculate the expression for $[H, \vec{J}]$.
**Ex 13 10** The tritium atom, $^3$H, is radioactive isotope of hydrogen and decays with a half-life of 12.3 years to $^3He^+$ by the emission of an electron from its nucleus ($\beta$ decay). The electron departs with 16 keV of kinetic energy. Find the probability that the newly-formed $^3He^+$ atom is in an excited state. Justify the approximation regarding dynamics involved in finding the probability.
**Ex 14** Consider the scattering of a particle by a distribution of charges due to Coulomb interaction. The charge of the particle is $e$ ($e > 0$) while the charge of each scatterer is also $e$. Suppose the momentum of incident particle is $\hbar k$ (along $z$ direction), find the differential cross section in the Born approximation for the following configurations.
**(a) 10** The scatterers are placed at eight vertices of cube with length of side being $a$ and one side being parallel to $z$ direction.
**(b) 5** Do the same for an infinite cubic lattice of lattice spacing $b$.

**Bonus**
**Problem 1. +1** Two objects, A and B, have irregular forms and are identical in size and shape. Object A is filled with a positively infinite (impenetrable) potential and scatters particles of mass $m$. Object B is made of metal and has a certain electrostatic capacity $C$ when isolated from other objects. Derive expressions for the differential and total cross sections for elastic scattering of particles of mass $m$ by object A in the limit $k \to 0$, in terms of the capacity C of object B.

**Problem 2. +1** The Hamiltonian for a two-level system is time-dependent and is given by

$$H = \frac{vt}{2}\sigma_z + \Delta\sigma_x,$$

where $t$ represents the time and $v$ and $\Delta$ are positive real numbers. If at $t = -\infty$, the state of the system is $\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, find the exact probability of finding the system to be $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ at $t = \infty$.

**Problem 3.** **+1** Consider a particle of mass in the following potentials: (a) $V(r) = \frac{-\alpha}{r}$ (b) $V(r) = \frac{1}{2}m\omega^2 r^2$. If the particle is in the nth energy eigenstate, find the recursion relation among $\langle r^m \rangle$, $\langle r^{m-1} \rangle$, and $\langle r^{m-2} \rangle$ for (a) and the recursion relation among $\langle r^m \rangle$, $\langle r^{m+2} \rangle$, and $\langle r^{m-2} \rangle$ for (b). Here $m$ is an non-negative integer.