Quantum Mechanics (II): Homework 4
Due: May 6, 2022

Ex 1
(a) 10 Find all related Clebsch-Gordan coefficients for adding $j_1 = 1/2$ and $j_2 = 3/2$.
(b) Consider the addition of two angular momenta: $\vec{S}_1$ and $\vec{S}_2$. The eigenvalues of the resulting total angular momentum $\vec{S} = \vec{S}_1 + \vec{S}_2$ consist of $s = |s_1 - s_2|, \cdots, s_1 + s_2$. It is often needed to confine oneself to the subspace in the Hilbert space of the total angular momenutm with fixed s. This can be done by using the projection operator $\hat{P}_{s_0}$, which projects the states to the subspace with $s = s_0$.
Let us rewrite $\vec{S} = \hbar\vec{s}_1$ and $\vec{S}_2 = \hbar\vec{s}_2$. Find the projection operators in terms of $\vec{s}_1 \cdot \vec{s}_2$ for the following cases:
(i) 8 $s_1 = s_2 = 1/2$, find $\hat{P}_0$ and $\hat{P}_1$.
(ii)17 $s_1 = s_2 = 1$, find $\hat{P}_0$, $\hat{P}_1$, and $\hat{P}_2$.
Ex. 2 10 Exercise 15.2.2.(2)
Ex. 3 10 Exercise 15.3.1.(1)
Ex. 4 10 Exercise 15.3.4.(1).
Ex. 5 5 Exercise 15.3.5.
Ex. 6 5 Exercise 15.2.3.
Ex. 7 (a) 10 Write xy, xz, and $(x^2 - y^2)$ as components of a spherical (irreducible) tensor of rank 2.
(b) 10 The expectation value

$$Q \equiv e\langle \alpha, j, m = j|(3z^2 - r^2)|\alpha, j, m = j\rangle$$

is known as the quadrupole moment. Evaluate

$$e\langle \alpha, j, m'|(x^2 - y^2)|\alpha, j, m = j\rangle,$$

(where $m' = j$, $j - 1$, $j - 2$, ...) in terms of Q and appropriate Clebsch-Gordan coefficients.
(c) 10 As shown in class, the Wigner-Echart theorem implies that when performing averages with respect to eigenstates of total angular momentum, $|jm\rangle$, one may replace the quadrupole operator $\hat{Q}_{ik} = 3\hat{x}_i\hat{x}_k - 2/3\delta_{ik}\hat{r}^2$ by

$$\hat{Q}_{ik} = \frac{Q_J}{2J(2J-1)\hbar^2}\left(\hat{J}_i\hat{J}_k + \hat{J}_k\hat{J}_i - \frac{2}{3}\delta_{ik}\hat{J}^2\right).$$

Here $\hat{J}_i$ are the ith component of the total angular momentum and $Q_J = \langle JJ|\hat{Q}_{zz}|JJ\rangle$. According to the Wigner-Echart theorem, it is also possible to replace $\hat{Q}_{ik}$ by $\hat{Q}_{ik} = q_L(\hat{L}_i\hat{L}_k + \hat{L}_k\hat{L}_i - \frac{2}{3}\delta_{ik}\hat{L}^2)$ with $\hat{L}_i$ being the orbital angular moment operators. Find q_L in terms of Q_J.
Ex. 8 10 Suppose two spin-1/2 particles are known to be in the spin-singlet state. Let $S_a^{(1)}$ be the component of spin for one of the particles along $\hat{\mathbf{a}}$ direction and $S_b^{(2)}$ be the component of spin for the remaining particle along $\hat{\mathbf{b}}$ direction. Show that

$$\langle\, S_a^{(1)} S_b^{(2)}\,\rangle = -\frac{\hbar^2}{4}\,\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}\ .$$

Ex. 9
(a) 10 Consider two electrons (denoted by 1 and 2) interact with each other via the Coulumb interaction $U = \frac{e^2}{|\mathbf{r_1} - \mathbf{r_2}}|$, where $\mathbf{r}_i$ is the position operator for the ith electron. Suppose that the orbital states of electrons can be either $\phi_a(\mathbf{r})$ or $\phi_b(\mathbf{r})$. Find the difference of the average Coulumb energy (ΔU))between spin singlet and spin triplet states. If one tries to attribute ΔU as the difference of spin-spin interactin $-J\sigma_1 \cdot \sigma_2$, find the expression of J and show that it is always positive. Here σ_i is the Pauli spin matrix operator for the ith electron.
(b) 10 A carbon atom has two valence electrons, whose orbital wavefunctions are in one of the $l = 1$ states. The orbital part of the total wavefunction for two electrons, $|LM\rangle$ can be written as summation of $|1m\rangle|1m'\rangle$, where $|LM\rangle$ is the eigenstate to the total orbital angular moment L^2 and L_z. Consider the Coulumb interaction between electrons, what would be the total angular moment l for the two valence electrons?
Ex. 10
(a) 10 In classical physics, to find, say, $(\mathbf{S}_1 - \mathbf{S}_2)^2$ is equivalent to find $[\mathbf{S}_1 + (-\mathbf{S}_2)]^2$. In other words, both $(\mathbf{S}_1 - \mathbf{S}_2)^2$ and $(\mathbf{S}_1 + \mathbf{S}_2)^2$ fall into the range between $||\mathbf{S}_{\ 1}| - |\mathbf{S}_2||^2$ and $||\mathbf{S}_{\ 1}| + |\mathbf{S}_2||^2$. Therefore, if this is also true for Quantum Mechanics, then if both $\mathbf{S}_1$ and $\mathbf{S}_2$ are spin 1/2, the eigenvalues obtained for $(\mathbf{S}_1 - \mathbf{S}_2)^2$ from this argument should be 0 or $2\hbar^2$. Show that this is not correct by finding the correct eigenvalues.

(b) 5 Following (a), find the eigenvalues of $(a\mathbf{S}_1 + b\mathbf{S}_2)^2$, where a and b are real numbers.
Ex. 11 10 A system of two particles with spins $s_1 = 3/2$ and $s_2 = 1/2$ is described by the approximated Hamiltonian $H = \alpha \vec{S}_1 \cdot \vec{S}_2$, with α being a given constant. At $t = 0$, the system is in the simutaneous eignestate of S_1^2, S_2^2, S_{1z}, and S_{2z}: $|\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\rangle$. Evaluate the probability of finding the system in the state $|\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}\rangle$ at $t > 0$.
Ex. 12 10 Consider the addition of angular momenta $\vec{J}_1$ and $\vec{J}_2$. Let $|j_1 j_2 jm\rangle$ be the eigenstate to the total angular moment J^2 and J_z. Calculate the matrix elements $\langle j_1 j_2 jm|\vec{J}_1|j_1 j_2 jm\rangle$ and $\langle j_1 j_2 jm|\vec{J}_2|j_1 j_2 jm\rangle$.
Ex. 13 10 Consider the dipole-dipole interaction between two magnetic moments, $\vec{m}_1$ and $\vec{m}_2$,

$$V(r) = \frac{\vec{m}_1 \cdot \vec{m}_2}{r^3} - \frac{3(\vec{m}_1 \cdot \vec{r})(\vec{m}_2 \cdot \vec{r})}{r^5},$$

where r is the relative distance between two magnetic moments. Since the magnetic moment is proportinal to the spin of the particle, the interaction can be expressed in terms of Pauli matrices ($\vec{\sigma}_1$ and $\vec{\sigma}_2$) as $V(r) = -\frac{g^2 \mu_b{}^2}{4} \frac{v(r)}{r^3}$ with $v(r)$ being given by

$$v(r) = 3\frac{(\vec{\sigma}_1 \cdot \vec{r})(\vec{\sigma}_2 \cdot \vec{r}))}{r^2} - \vec{\sigma}_1 \cdot \vec{\sigma}_2.$$

Find the expression of $v(r)$ in terms of the irreducible 2nd-rank tensor operators constructed by the total spin $\vec{S} = (\vec{\sigma}_1 + \vec{\sigma}_2)/2$ and the position operator $\vec{r}$.
Ex. 14 10 Consider the addition of two angular moment of same magnitude, $j_1 = j_2 = j$. Show that the state with zero angular momentum can be put in the following form

$$|0,0\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} (-1)^{m-1/2} |jm; j-m\rangle \quad \text{j = half − integer},$$

$$|0,0\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} (-1)^{m} |jm; j-m\rangle \quad \text{j = integer}.$$

Ex. 15 8 Consider two particles governed by the Hamiltonian $H = \frac{\mathbf{J}^2}{2I}$, where $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ is the total angular moment of two particles and I is the moment of inertia. Show that the angle between $\mathbf{J}$ and $\mathbf{J}_1$ or $\mathbf{J}$ and $\mathbf{J}_2$ is a constant. Demostrate that $\mathbf{J}_1$ and $\mathbf{J}_2$ precess about $\mathbf{J}$.
Ex. 16 Consider the coupling of three spin-1/2 particles. Let $|\alpha\beta\gamma\rangle$ denote the state when the first particle in ithe state $|\alpha\rangle$, the 2nd in $|\beta\rangle$ and the 3rd in $|\gamma\rangle$, where α, β, and γ are either + (spin up) or − (spin down).
(a) 7 Construct all states with total angular momentum $J = 3/2$.
(b) 8 Construct all states with $J = 1/2$. (Hint: add two spin-1/2 partciles first, and then include the 3rd particle.)

Bonus problems (+1): Exact formula for Clebsch-Gordan coefficients
We have learned the Schwinger boson model for the angular momentum. One of the advantage for this model is that it allows one to find the exact formula for Clebsch-Gordan coefficients.
Consider the addition of two angular momenta, $\vec{J}_1$ and $\vec{J}_2$. Let the boson annhilation operators for $\vec{J}_1$ be a_+ and a_- and those for $\vec{J}_2$ be b_+ and b_-. The operator that is crucial in derving the Clebsch-Gordan coefficeints is the K-operator defined by

$$K^\dagger = a_+^\dagger b_-^\dagger - a_-^\dagger b_+^\dagger.$$

By computing relevant commutators of K with operators formed by $\vec{J}_1$ and $\vec{J}_2$, find the Clebsch-Gordan coefficients $\langle j_1, m_1; j_2, m_2|jm; j_1 j_2\rangle$. You may need to use the following identity:

$$\sum_m C_{n_1}^{n_1+r-s} C_{n_2}^{n_2+s} = C_{n-1+n_2+1}^{n_1+n_2+r+1}.$$