Quantum Mechanics (II): Homework 3
Due: April 13, 2022

Ex. 1 10 Consider an electron in an atom or molecule such that it can be described by

$$\mathrm{H} = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + V(\mathbf{r}).$$

where $V(\mathbf{r})$ is the effective potential due to nucleus and other electrons. In a uniform magnetic field $\mathbf{B} = (0, 0, B)$, we have seen that by using the symmetry gauge, i.e., $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$,

$$\begin{aligned} \mathrm{H} &= \frac{1}{2m}\mathbf{p}^2 + V(\mathbf{r}) - \mu_L \cdot \mathbf{B} + O(B^2 r^2) \\ &\equiv \mathrm{H}_S + O(B^2 r^2), \end{aligned}$$

where μ_L is the orbital magnetic moment. However, if we adopt the Landau gauge, i.e., $\mathbf{A} = (0, Bx, 0)$, we get

$$\begin{aligned} \mathrm{H} &= \frac{1}{2m}\mathbf{p}^2 + V(\mathbf{r}) + \frac{eB}{mc} x p_y + O(B^2 r^2) \\ &\equiv \mathrm{H}_L + O(B^2 r^2). \end{aligned}$$

H_L does not even look like H_S. Show that to $O(B^2 r^2)$, for the same eigenvalue, the eigenstates ϕ_L to H_L and ϕ_S to H_S differ by a phase

$$\phi_L = \phi_S \exp\left(-i\varphi\right).$$

Therefore, they are equivalent. Find the phase φ.

Ex.2 Equations of motions

(a) 5 Suppose that a particle is governed by the Hamiltonian : $H = \mathbf{p}^2/2m + V(\mathbf{r})$. Show that

$$\frac{d < \mathbf{L} >}{dt} = < \tau >$$

where τ is the torque $\mathbf{r} \times \mathbf{F} = -\mathbf{r} \times \nabla \mathbf{V}$, and $\mathbf{L}$ is the orbital angular momentum.

(b) 10 Suppose that the particle has a magnetic dipole moment $\mu = g\mu_b \mathbf{J}$, where g is the g-factor, $\mu_b = e\hbar/2m$ is the Bohr magneton, and $\mathbf{J}$ is the total angular momentum. If the particle is placed in a constant magnetic field $\mathbf{B}$, $H = -\mu \cdot \mathbf{B}$. Show that

$$\frac{d < \mathbf{J} >}{dt} = g\mu_b < \mathbf{J} > \times \mathbf{B},$$

i.e., $\langle \mathbf{J} \rangle$ is doing Larmor precessing with angular frequency $\omega = g\mu_b B$.

Ex.3 10 Exercise 12.4.4. (Note that $U[R]$ is $\mathcal{D}[R]$ in our notations).

Ex.4 10 A rigid rotator is in a uniform magnetic field along z direction so that the Hamiltonian is

$$H = \frac{L^2}{2I} + \omega_0 L_z,$$

where ω_0 is a constant. If the wavefunction of the rotator at $t = 0$ is

$$\langle \theta, \phi | \psi(0) \rangle = \sqrt{\frac{3}{4\pi}} \sin\theta \sin\phi,$$

find $\langle \theta, \phi | \psi(t) \rangle$.

Ex.5 30 (a) Exercise 12.5.13 **(b)** Exercise 12.6.9. **(c)** Derive Eq.(12.6.37)

Ex.6 10 A fundamental theorem in Linear Algebra is the theorem of Hamilton-Cayley, which says that given a $n \times n$ matrix A, there exists a polynomial $f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0$ such that

$$f(A) = 0.$$

This polynomial is given by

$$f(t) = Det\ (tI - A),$$

of which roots are the eigenvalues of A. Use this theorem to find $\exp(A)$, where A is given by

$$\begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}.$$

Ex.7 10 Exercise 14.5.4.
Ex.8 (a) 10 As we have shown in Ex 3, the definition of a vector operator $\mathbf{V}$ in Quantum Mechanics is defined by $[V_\alpha, J_\beta] = i\hbar\epsilon_{\alpha\beta\gamma}V_\gamma$. Under a rotation, this vector operator transforms according to

$$\exp\left(\frac{i\mathbf{J}\cdot\mathbf{n}\phi}{\hbar}\right) V_i \exp\left(-\frac{i\mathbf{J}\cdot\mathbf{n}\phi}{\hbar}\right).$$

Consider a rotation about z-axis through the angle ϕ, show that

$$\begin{aligned}
\exp\left(\frac{i\mathbf{J}\cdot\mathbf{n}\phi}{\hbar}\right) V_x \exp\left(-\frac{i\mathbf{J}\cdot\mathbf{n}\phi}{\hbar}\right) &= V_x\cos\phi - V_y\sin\phi \\
\exp\left(\frac{i\mathbf{J}\cdot\mathbf{n}\phi}{\hbar}\right) V_y \exp\left(-\frac{i\mathbf{J}\cdot\mathbf{n}\phi}{\hbar}\right) &= V_x\sin\phi + V_y\cos\phi \\
\exp\left(\frac{i\mathbf{J}\cdot\mathbf{n}\phi}{\hbar}\right) V_z \exp\left(-\frac{i\mathbf{J}\cdot\mathbf{n}\phi}{\hbar}\right) &= V_z
\end{aligned}$$

(b) 10 Consider a rotation $R(\phi,\theta,\psi)$, where ϕ,θ,ψ are three Euler angles in Sakurai's notations. The angle θ describes the rotation with respect to the intermediate y-axis: η, and is given by $R_\eta(\theta) = R_z(\phi)R_y(\theta)R_z^{-1}(\phi)$. Using the result of (a), express $D_\eta(\theta)$ in the form $\exp(-iA/\hbar)$. Find A in terms of J_x, J_y, and J_z. From here, find J_η in terms of J_x, J_y, and J_z, explain the result.
Ex.9 (a) 20 Exercise 14.4.3.
(b) 15 Neutrinos are neutral particles that come in three kinds, namely ν_e, ν_μ and ν_τ . Until recently they were considered to be massless. In this problem, we shall show that the observation of neutrino oscillations, i.e. transformation between ν_μ and ν_e, is a proof of their massiveness.
(i) Consider for simplicity two neutrino species only, namely ν_e and ν_μ. We shall adopt a similar view as isospin and try to view them as two states of the same particle. Ignoring the spatial degrees of freedom and treating the momentum as a parameter p (a number), then similar to the spin-1/2 system, the Hamiltonian is a 2×2 matrix with eigenvetor ν_1 and ν_2 with the corresponding energies being $E_{1,2} = \sqrt{p^2c^2 + m_{1,2}^2c^4} \approx pc + m_{1,2}^2c^3/2p$. In general, there is no reason to identify ν_e and ν_μ as ν_1 and ν_2. Instead, let us assume

$$\begin{aligned}
\nu_e &= \cos\theta\nu_1 + \sin\theta\nu_2 \\
\nu_\mu &= -\sin\theta\nu_1 + \cos\theta\nu_2
\end{aligned}$$

where θ is a parameter,termed as mixing angle. Suppose that at $t = 0$, one produces a neutrino of momentum p of the kind ν_e. Find the probability of for this neutrino to be detected as ν_μ at a later time t (in terms of E_1, E_2 and θ).
(ii) Suppose that the speed of neutrinos can be approximated by the speed of light c. If $pc = 4$ MeV, one finds that the neutrino oscillates back and forth between ν_e and ν_μ in the length 100km. Estimate the mass difference $(\Delta m^2)c^4$ with $\Delta m^2 \equiv m_1^2 - m_2^2$.
Ex 10 10 Consider $j = 1$. Find matrices (3×3) for $\exp(-iJ_x\alpha/\hbar)$ and $\exp(-iJ_y\alpha/\hbar)$.
Ex.11 10 Exercise 14.3.5.
Ex.12 10 Exercise 14.3.8.
Ex.13 5 The normalized wave function of an electron in a central potential is given by

$$\psi_{nlm}(\mathbf{r},t) = R_{nl}(r)\,Y_l^m(\theta,\phi)\exp(-iE_nt/\hbar).$$

Show that the probability current for this electron is given by

$$\mathbf{j} = \hat{\phi}\,\frac{|\psi_{nlm}|^2 m\hbar}{m_e\,r\sin\theta}$$

where m_e is the mass of the electron.
Ex.14 One of the classic problems in classical mechanics is to solve the motion for a rotating top. The rotation energy can be rewritten as

$$\mathrm{H} = \frac{1}{2}\left(\frac{\mathrm{J}_a^2}{\mathrm{I}_a} + \frac{\mathrm{J}_b^2}{\mathrm{I}_b} + \frac{\mathrm{J}_c^2}{\mathrm{I}_c}\right),$$

where $\mathbf{J}$ is the angular momentum in the body frame and (I_a, I_b, I_c) are the principal moments of inertia. But how do we generalize the angular momentum in the body frame in quantum mechanics? An obvious way is to define $\hat{\mathrm{J}}_a \equiv \hat{\mathbf{J}}\cdot\hat{a}$, $\hat{\mathrm{J}}_b \equiv \hat{\mathbf{J}}\cdot\hat{b}$, and $\hat{\mathrm{J}}_c \equiv \hat{\mathbf{J}}\cdot\hat{c}$, where $\hat{\mathbf{J}}$ is the angular momentum operator in the fixed frame, while $\hat{a}$, $\hat{b}$, and $\hat{c}$ are three unit vectors along the principal axes. The tricky point, however, is that $\hat{a}$, $\hat{b}$, and $\hat{c}$ also depend on how $\mathbf{J}$ changes in classical mechanics, i.e., they are not independent from $\mathbf{J}$. How do we deal with this in quantum mechanics? This is where we can apply the concept used in Ex. 8. In quantum mechanics, all the above consideration says is that we can not treat $\hat{a}$, $\hat{b}$, and $\hat{c}$ as constant vectors but have to treat them as vector operators.
(a) 10 Show that

$$(\hat{\mathbf{J}}\cdot\hat{\mathbf{u}})(\hat{\mathbf{J}}\cdot\hat{\mathbf{v}}) - (\hat{\mathbf{J}}\cdot\hat{\mathbf{v}})(\hat{\mathbf{J}}\cdot\hat{\mathbf{u}}) = -i\hbar\,\hat{\mathbf{J}}\cdot\hat{\mathbf{u}}\times\hat{\mathbf{v}}$$

where $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are two commuting vector operators.
(b) 10 From (a), show that $[\hat{\mathrm{J}}_\alpha,\, \hat{\mathrm{J}}_\beta] = -i\hbar\,\epsilon_{\alpha\beta\gamma}\hat{\mathrm{J}}_\gamma$, where α, β, and γ represent a, b, or c. Note that it differs from the usual commutation relation by a minus sign. Give a physical argument for the minus sign.
(c) 5 Find the equation of motion for $\hat{\mathrm{J}}_\alpha$ in the Heisenberg picture.
Ex.15 SU(2) matrices in the Euler description As mentioned in the class, any 2×2 SU(2) matrix U can be considered as a rotation in the spin space. In particular, we can consider U as a rotational matrix in the Euler description so that

$$U = \exp(-\frac{i}{\hbar}\phi S_z)\exp(-\frac{i}{\hbar}\theta S_y)\exp(-\frac{i}{\hbar}\alpha S_z).$$

(a) 5 Show that U rotates the spin states along the z direction ($|\pm\rangle$) to the direction $\mathbf{n} = (\theta, \phi)$ ($|n, \pm\rangle$). If we define $z_1 = \exp(-i\alpha/2)\exp(-i\phi/2)\cos(\theta/2)$ and $z_2 = \exp(-i\alpha/2)\exp(i\phi/2)\sin(\theta/2)$, then $|n, +\rangle = (z_1, z_2)$ and $|n, -\rangle = (-z_2^*, z_1^*)$.
(b) 10 Show that $z_\alpha^*\vec{\sigma}_{\alpha\beta}z_\beta = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, where $\vec{\sigma}$ is the Pauli matrix vector and α and β are either 1 or 2.
Ex.16 10 Use the method of Schwinger bosons to find the rotation matrix $d(\beta) = \exp(-\frac{i}{\hbar}J_y\beta)$ for $j = 1$. Compare your result with the one obtained previously using the matrix representation of J_y.
Ex.17 10 Consider an electron incident from $x = -\infty$ to the region $x \geq 0$ with a unform magnetic field $\vec{B} = (B\sin\theta, 0, B\cos\theta)$. Here θ is the angle between $\vec{B}$ and z axisand B is the magnitude of $\vec{B}$. Except for the presence of $\vec{B}$ in $x \geq 0$, the electron is free. Suppose that the incident electron is in the spin up (toward $+z$) state with incident momentum being along x-axis and energy E. If the incident energy is in the range $0 < E < \mu_b B$ with μ_b being the Bohr magneton, find the transition probability that the electron can go into the region $x \geq 0$. (Assuming that the gyromagnetic ratio for the spin of the electron is 2.)
Ex. 18 15 Let $d^j_{mm'}(\beta) = \langle jm|e^{-\frac{i}{\hbar}J_y\beta}|jm'\rangle$ be the rotation matrix. Calculate the following quantities: (i) $\sum_{m=-j}^{j} m|d^j_{mm'}(\beta)|^2$ and (ii) $\sum_{m=-j}^{j} m^2|d^j_{mm'}(\beta)|^2$ in terms of β, j, and m'.

Bonus problems (+1): Spin coherent states

Based on what we learned for the coherent states of the harmonic oscillator, we know that the salient point of coherent states is that one can replace the operators by its eigenvalues. Specifically, we have

$$\hat{\mathrm{a}}\,|\lambda\rangle = \lambda\,|\lambda\rangle, \text{ or equivalently, } \langle\lambda|\,\hat{\mathrm{a}}\,|\lambda\rangle = \lambda\,,$$

where $|\lambda\rangle$ is a coherent state. Similarly, one defines the coherent states $|\mathbf{\Omega}\rangle$ for the spin operator by

$$\langle\mathbf{\Omega}|\,\mathbf{S}\,|\mathbf{\Omega}\rangle = \hbar\, s\,\mathbf{\Omega},$$

where $\mathbf{\Omega}$ is a unit vector specifying the direction $\mathbf{\Omega} \equiv (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$ and $\mathbf{S}$ is the spin operator with eigenvalue s (which is not necessary 1/2). In other words, using the coherent states, one can replace the operator $\mathbf{S}$ by $\hbar s\,\mathbf{\Omega}$ just as if the spins are classical vectors.

(a) The state $|\mathbf{\Omega}\rangle$ may be obtained by applying a rotation to the state $|s, s\rangle$. From this fact, show that

$$|\mathbf{\Omega}\rangle = \frac{(u\hat{\mathrm{a}}_+^\dagger + v\hat{\mathrm{a}}_-^\dagger)^{2s}}{\sqrt{(2s)!}}\,|0\rangle,$$

where $\hat{\mathrm{a}}_+^\dagger$ and $\hat{\mathrm{a}}_-^\dagger$ are the creation operators defined in the Schwinger model of the angular momentum, $u = \cos(\theta/2)\exp(-i\phi/2)$ and $v = \sin(\theta/2)\exp(i\phi/2)$. From this result, show that

$$\langle\mathbf{\Omega}_2|\mathbf{\Omega}_1\rangle = e^{is(\phi_2-\phi_1)}\left(\cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + e^{i(\phi_1-\phi_2)}\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\right)^{2s}.$$

(b) If we define $d\mathbf{\Omega} = \sin\theta\, d\theta\, d\phi$, show that

$$\frac{2s+1}{4\pi}\int d\mathbf{\Omega}\,|\mathbf{\Omega}\rangle\langle\mathbf{\Omega}| = \mathbf{1},$$

$$\frac{(s+1)(2s+1)}{4\pi}\int d\mathbf{\Omega}\,\mathbf{\Omega}\,|\mathbf{\Omega}\rangle\langle\mathbf{\Omega}| = \mathbf{S}.$$

(c) Consider the time evolution of a spin characterized by a Hamiltonian $\hat{\mathrm{H}} = \hat{\mathrm{H}}(\mathbf{S})$. For example, $\hat{\mathrm{H}} = \alpha\,\mathbf{S}\cdot\mathbf{B}$. Show that the path integral for the evolution operator of this system can be written as

$$\langle\mathbf{\Omega}_f|\hat{\mathrm{U}}(t)|\mathbf{\Omega}_i\rangle = \int\mathcal{D}\mathbf{\Omega}\exp\left\{i\int_{t_i}^{t_f}\left[s\cos\theta\frac{d\phi}{dt} - \frac{H(\mathbf{\Omega})}{\hbar}\right]dt\right\},$$

where as usual, $\int\mathcal{D}\mathbf{\Omega} \equiv \lim_{N\to\infty}\left(\frac{2s+1}{4\pi}\right)^N \int d\mathbf{\Omega}_N \int d\mathbf{\Omega}_{N-1}\cdots\int d\mathbf{\Omega}_1$, which represents summing over all possible spin orientations.

(d) The first term in (c) is new and known as the Hoft term. To understand its physical meaning, let us rewrite it as

$$\int d\phi\ \cos\theta,$$

and consider a special case when $\phi(t_f) = \phi(t_i)$. On a unit sphere, the spin orientation traces out a closed curve. Show that this term is equal to $2\pi-$ the cap area bounded by this closed curve (See fig. 1). Therefore, this term measures the area the spin traces out on a unit sphere.