# Quantum Mechanics (I): Homework 2
# Due: October 20, 9AM (hand-in box of Room 514 Physics Dept.), 2021

**Ex.1 10** Page 54, ex 1.8.12
**Ex.2 10** Page 30, ex 1.7.1 (1) and (2)
**Ex.3 (a) 10**
The exponential of an operator $\hat{A}$ is defined by

$$\exp(\hat{A}) = I + \frac{\hat{A}}{1!} + \frac{\hat{A}^2}{2!} + \cdots + \frac{\hat{A}^n}{n!} + \cdots$$

Assuming $[\hat{A}, \hat{B}] = c$, where $c$ is a constant, verify, to $O(\lambda^2)$,

$$\exp(\lambda\hat{A})\exp(\lambda\hat{B}) = \exp\lambda(\hat{A}+\hat{B})\exp(\lambda^2 c/2).$$

Therefore, when $\hat{A}$ and $\hat{B}$ are commute, $\exp\hat{A}\exp\hat{B} = \exp(\hat{A}+\hat{B})$.
**(b) 10** In general, when $[\hat{A}, \hat{B}]$ does not vanish and is not a constant, show that the following Feynman's identity is correct

$$\exp(\hat{A}+\hat{B}) = T_\lambda \exp\left[\int_0^1 d\lambda e^{\lambda\hat{B}}\hat{A}e^{-\lambda\hat{B}}\right]\exp(\hat{B}),$$

where similiar to anti time-ordering, $T_\lambda$ orders larger $\lambda$ to the right.
**Ex.4** Suppose that the wavefunction of a free particle at $t=0$ is given by $\psi(x,0) = (\pi\Delta^2)^{-1/4}\exp(-x^2/2\Delta^2)$.
(a) **5** Derive $\psi(x,t)$ for $t>0$.
(b) **5** Verify that the $\psi(x,t)$ you obtain satisfies $\langle\psi|\psi\rangle = 1$ for $t>0$.
(c) **5** Find the momentum uncertainty $\Delta p(t)$ for $t>0$.
**Ex.5 10** Page 163, ex 5.2.1
**Ex.6 10** Page 163, ex 5.2.2
**Ex.7 10** Page 167, ex 5.3.4
**Ex 8 10** Suppose $\{\mathrm{A},\mathrm{B}\} \equiv \mathrm{AB}+\mathrm{BA} = 0$, show that $\mathrm{A}\exp(-\mathrm{B}) = \exp(\mathrm{B})\,\mathrm{A}$.
**Ex.9** The system described by the Hamiltonian $H_0$ has just two orthonormal energy eigenstates, $|1>$ and $|2>$. The two eigenstates have the same eigenvalues, $E_0$. Now suppose the Hamiltonian for the system is changed by the addition of the term $V$, giving $H = H_0 + V$. The matrix elements of $V$ are

$$\langle 1|V|1\rangle = 0, \quad \langle 1|V|2\rangle = V_{12}, \quad \langle 2|V|2\rangle = 0.$$

**(a) 5** Write out $H$ in matrix-product form. **(b) 10** Find the new energy eigenvalues and eigenkets (in terms of $|1>$ and $|2>$).
**(c) 15** Now suppose that instead of being governed by $H$, the system is governed by $\hat{\mathrm{H}}' = a|1\rangle\langle 1| + b|1\rangle\langle 2| + c|2\rangle\langle 1| + d|2\rangle\langle 2|$. Here $\langle 1|1\rangle = \langle 2|2\rangle = 1$ and $\langle 1|2\rangle = \chi$. Show that the eigenvalues $\lambda$ of $\hat{\mathrm{H}}'$ satisfy

$$\begin{vmatrix} a + b\chi^* + c\chi + d|\chi|^2 - \lambda & \sqrt{1-|\chi|^2}(b+d\chi) \\ \sqrt{1-|\chi|^2}(c+d\chi^*) & d(1-|\chi|^2) - \lambda \end{vmatrix} = 0.$$

Why is the equation for $\lambda$ different from the one that determines for the eigenvalues for (b)?
**(d) 5** For problem (c), if one denotes $\hat{\mathrm{H}}'|1\rangle = H'_{11}|1\rangle + H'_{21}|2\rangle$ and $\hat{\mathrm{H}}'|2\rangle = H'_{12}|1\rangle + H'_{22}|2\rangle$, show that the computation of eigenvalues to $\hat{\mathrm{H}}'$ follows the usual way of solving eigenvalues for matrices. That is, if one denotes the eigenket by $\alpha|1\rangle + \beta|2\rangle$, show that

$$\begin{pmatrix} H'_{11} & H'_{12} \\ H'_{21} & H'_{22} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda\begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

hence $\lambda$ satisfies

$$\begin{vmatrix} H'_{11} - \lambda & H'_{12} \\ H'_{21} & H'_{22} - \lambda \end{vmatrix} = 0.$$

**Ex.10**
**(a) 10** Verify that the operator

$$\mathrm{T}\,[e^{-i/\hbar \int_0^t dt\,\hat{\mathrm{H}}(t)}]$$

is a solution to the operator differential equation

$$i\hbar\frac{d}{dt}\hat{\mathrm{U}}(t,0) = \hat{\mathrm{H}}(t)\hat{\mathrm{U}}(t,0),$$

where $\mathrm{T}[\cdots]$ is the time-ordered product.
**(b) 10** Under certain electromagnetic interaction, the Hamiltonian of an electron is given by $\hat{H}(t) = \hat{p}^2/2m + \lambda(\hat{x}\cos\omega t + \hat{p}\sin\omega t)$. Using the time-ordered product expression of $\hat{U}(t,0)$, find $\hat{U}(T,0)$ accurately to $O(\hat{H}^3)$ (i.e., neglect $O(\hat{H}^3)$), where $T = 2\pi/\omega$.
**Ex.11** Consider an operator $\hat{\mathrm{A}}$ of which $\langle x|\hat{\mathrm{A}}|x'\rangle = \hat{\mathbf{A}}(x)\delta(x-x')$, where $\hat{\mathbf{A}}(x)$ is the corresponding operator acting on $\delta(x-x')$. For instance, if $\hat{\mathrm{A}} = \hat{\mathrm{p}}$, $\hat{\mathbf{A}}(x) = -i\hbar\partial/\partial x$. Similarly, for $\hat{\mathrm{A}}^\dagger$, we can also write $\langle x|\hat{\mathrm{A}}^\dagger|x'\rangle = \hat{\mathbf{A}}^\dagger(x)\delta(x-x')$.
**(a) 5** Show that the relation $\langle\phi|\hat{\mathrm{A}}^\dagger|\psi\rangle = \langle\psi|\hat{\mathrm{A}}|\varphi\rangle^*$ reduces to $\int dx\phi^*(x)[\hat{\mathbf{A}}^\dagger\psi(x)] = \int dx\psi(x)[\hat{\mathbf{A}}^*\phi^*(x)]$.
**(b) 10 Nonlocal potential** In the usual Schrödinger equation, the potential $\hat{\mathrm{V}}$ is *local* in the sense that $\langle x|\hat{\mathrm{V}}|x'\rangle = V(x)\delta(x-x')$, i.e., $\hat{\mathrm{V}}$ is diagonal in "x-basis". As a result, $\psi(x,t)$ does not depend *explicitly* on $\psi(x',t)$ at other position $x'$. Suppose now that the potential is nonlocal, i.e., $\langle x|\hat{\mathrm{V}}|x'\rangle = V(x,x')$, where $V(x,x') \neq 0$ when $x \neq x'$. What is the corresponding Schrödinger equation in "x-basis" ?
**Ex.12 5** $\hat{\mathrm{P}}=|i\rangle\langle i|$ is a projection operator that projects any state to the space spanned by $|i\rangle$. Show that $g^{\hat{\mathrm{P}}} = 1-(1-g)\hat{\mathrm{P}}$ where $g$ is a real number.
**Ex.13** Consider a quatum system whose Hilbert space is two-dimensional. The system is usually termed as a two-level system.
**(a) 10** Show that in general, the density matrix $\hat{\rho}$ is characterized by three real numbers ($b_x$, $b_y$ and $b_z$) and can be written in the form $\hat{\rho} = \frac{1}{2}\left(I + \vec{b}\cdot\vec{\sigma}\right)$ with $\vec{b}\cdot\vec{\sigma} = b_x\sigma_x + b_y\sigma_y + b_z\sigma_z$. Here $I$, $\sigma_x$, $\sigma_y$ and $\sigma_z$ are given by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

**(b) 5** Find the condition for $\vec{b}$ so that the $2\times 2$ matrix, $\frac{1}{2}\left(I + \vec{b}\cdot\vec{\sigma}\right)$, is indeed a density matrix. What is the condition in terms of $\vec{b}$ when $\hat{\rho}$ describes a pure state?
**Ex.14 10** As we know, the probability current density in the Schrödinger equation is given by

$$J(x) = \frac{i\hbar}{2m}\left[\Psi(x)\frac{\partial}{\partial x}\Psi^*(x) - \Psi^*(x)\frac{\partial}{\partial x}\Psi(x)\right].$$

Construct the current density operator $\hat{\mathrm{J}}(x)$ in terms of $\hat{\mathrm{x}}$ and $\hat{\mathrm{p}}$ such that $\langle\Psi|\hat{\mathrm{J}}(x)|\Psi\rangle = J(x)$. Note that the $x$ in $\hat{\mathrm{J}}(x)$ is a parameter, not an operator.