Quantum Mechanics (I): Homework 1
Due: October 6, 9AM (hand-in box of Room 514 Physics Dept.), 2021

Ex.1 In early period of development of quantum mechanics, a few physicists (including Schrödinger himself) tried to develop so-called hidden-variable theories in which quantum fluctuations are attributed to some unknown variables in classical mechanics. In the pilot-wave theory, one tries to interpretate that the role of mater wave is to pilot how the classical particle move. Since probabilities are associated with the matter wave, in the pilot-wave theory, the quantum probability gives rise random forces applied to classical particles. In 1952, Bohm succeeded in identifying such kind of random forces in the Schrödinger equation: Suppose ψ satisfies the Schrödinger equation with the the potential $V(\mathbf{r})$. Let $\psi = |\psi| \exp(iS/\hbar)$, show that

(a) **5** S is real

(b) **5** when ψ is a plane wave, $\nabla S = \hbar\,\mathbf{k}$ is the momentum.

In general, ∇S has the meaning of momentum. Let $\mathbf{p} \equiv \nabla S$. Here $\mathbf{p}$ is a function of $\mathbf{r}$ and is the momentum of the particle that arrives at $\mathbf{r}$. This is in the Euler description. In general, we want to follow the same particle in the so-called Lagragian description. In this case, show that

(c) **10** $\frac{d\vec{p}}{dt} = -\nabla(V + V_Q)$, where $V_Q = -\frac{\hbar^2}{2m|\psi|}\nabla^2|\psi|$.

$-\nabla V_Q$ is the so-called quantum mechanical force that was interpreted as the fluctuations due to the unknown hidden-variable. It is claimed that this theory can also explain the two-slit expt. (see Philippidis et al. Nuovo Cimento, **71B**, 75-87, 1982)

Ex.2 10

In the double-slit experiment shown in the class done by Tonomura et al. (American Journal of Physics 57, 117, 1989), the setup of the experiment is shown in Fig. 1. Suppose that the incident velocity of the electron along z direction is $c/2$, the voltage at the metal cylinder with radia $a = 0.6\mu m$ is $10V$, and $b = 5mm$. Find the typical fringe spacing.

Ex.3 10 Page 139, ex 4.2.2

Ex.4 10 Page 139, ex 4.2.3

Ex.5 10 Show that $[p, x^n] = -ni\hbar x^{n-1}$. In general, if $f(x)$ is a differentiable function of x, this implies that $[p, f(x)] = -i\hbar\frac{df(x)}{dx}$.

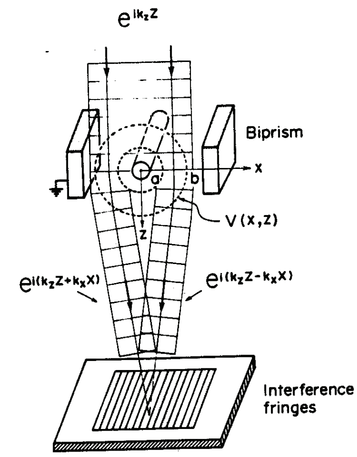

FIG. 1: Schematic plot of experimental setup used by Tonomura et al.

Ex.6 The state space of a certain physical system is three-dimensional. Let $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ be an orthonormal basis of this space. The kets $|\psi_1\rangle$ and $|\psi_2\rangle$ are defined by :

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle$$
$$|\psi_2\rangle = \frac{1}{\sqrt{3}}|u_1\rangle + \frac{1}{\sqrt{3}}|u_2\rangle$$

(a) **5** Are these kets normalized?

(b) **5** What is $\langle\psi_1|\psi_2\rangle$?

Ex.7 Consider the probability amplitude

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{i\omega t}$$

where A, λ, and ω are positive real constants. (a) **5** Normalize Ψ. (b) **5** Find $\langle x\rangle$ and $\langle x^2\rangle$.

Ex.8 10 Show that in the Schrödinger equation, the operator $H = p^2/2m + V(x)$ is hermitian.

Ex.9 10 Consider two *independent* and positive classical random numbers: x_1 and x_2, of which the probability densities are

$$p(x_i) = \lambda e^{-\lambda x_i}, \quad i = 1, 2,$$

where λ is positive and real. Let us form another random number x by defining $x = (x_1 + x_2)/2$. Find the probability density for x, $\mathrm{P}(x)$.

Ex.10 10 Suppose that at $t = 0$, a particle is described by the wavefunction

$$\Psi(x,0) = \frac{1}{\sqrt{2L}} \quad |x| < L$$
$$= 0 \quad \text{otherwise.}$$

If, at the same instant, we measure the momentum of the particle. What are the possible values we will get and what are the corresponding probabilities?

Ex.11 **10** Suppose that we do a measurement of the observable Ô on some particle and get the value α. Using the concept of "collapse of state", argue that after the measurement, the state of the particle has to be an eigenstate of Ô with eigenvalue of α.

Ex.12 **10** Following the same arguement used in the class, show that in the momentum space, the position operator x̂ is represented by $i\hbar\frac{d}{dp}$.

Ex.13 **10** Page 46, ex.1.8.10.

Ex.14 **(a)5** Consider a particle governed by $\hat{\mathrm{H}} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$. Now, if the particle is further confined to be at $x = na$ with n be integers and a being a positive constant, one needs to discretize the system: replacing any function $f(x = na)$ by f_n and $\frac{d^2}{dx^2}\Psi(x)\Big|_{x=na}$ by $\frac{\Psi_{n+1}+\Psi_{n-1}-2\Psi_n}{a^2}$. Let us denote the state when the particle is localized at $x = na$ by $|n\rangle$, and assume that $\langle n|m\rangle = \delta_{nm}$. Express $\hat{\mathrm{H}}$ in terms of the Dirac notations (ket and bra and etc.) **(b)5** In the above discrete system, the momentum operator should be changed accordingly. However, there are many possible ways to do: For example, $\frac{\hbar}{i}\frac{d\Psi(x)}{dx} = \frac{\hbar}{i}\frac{\Psi_{n+1}-\Psi_n}{a}$ or $\frac{\hbar}{i}\frac{d\Psi(x)}{dx} = \frac{\hbar}{i}\frac{\Psi_{n+1}-\Psi_{n-1}}{2a}$, which one is correct? why? **(c)5** Suppose now that the particle is in the state $|\phi\rangle \equiv |1\rangle + i|2\rangle + 2|3\rangle$. Evaluate the averaged value of energy for $|\phi\rangle$. **(d)5** Now suppose that the particle is further confined to be at $n = 1$ and $n = 2$, i.e., $\Psi_n = 0$ when $n = 3, 4, 5, ...$ and $n = 0, -1, -2, ...$ The wavefunction $|\phi(t)\rangle$ of the particle at time t can be thus written as $|\phi(t)\rangle = \alpha(t)|1> +\beta(t)|2>$, Find the differential equations that $\alpha(t)$ and $\beta(t)$ obey (**5**). If at $t = 0$, $|\phi(0)\rangle = |1\rangle$, find $|\phi(t)\rangle$ at $t > 0$. (**5**)

Bonus(+1) Suppose that $V(x \leq 0) = \infty$ and $V(x \geq (N+1)a) = \infty$, $V(x) = 0$ otherwise. Using the above correct expression to find the eigenvalues and eigenvectors of the discrete momentum operator. Find the corresponding expressions when $a \to 0$.