Lecture Notes on Stellar Astrophysics

Hsiang-Kuang Chang

Institute of Astronomy
National Tsing Hua University

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Preface

Chapter 1

Introduction

This lecture is to discuss physics of stars, that is, the physics involved in understanding stellar structure and evolution. At first, we shall discuss some observational properties of stars in Section 1.1. It can serve as a brief review of stellar astronomy at an introductory level. Since stellar mass is a key property of a star, in Section 1.2 we will discuss an important approach for stellar mass determination, that is, using the mass function of a binary system. This approach is also extensively employed in the search of exoplanets. In Section 1.3 we will discuss some examples of order-of-magnitude estimates, which may be useful for a rough understanding of certain phenomena.

1.1 Observation of stars

1.1.1 Distance

To convert the observed flux into the luminosity of a star, which is an intrinsic property of the star and independent of the observers, the distance to the star from the observer needs to be known. Distance determination is a difficult issue in astronomy. Readers are referred to textbooks used for courses of introductory astronomy and astrophysics for various methods employed by astronomers. The most commonly used unit for distance is **parsec**, abbreviated as **pc**, and its relatives, kpc, Mpc and Gpc. The definition of a parsec is based on the parallax measurement:

$$\tan\theta_{\rm p} = \frac{1\rm AU}{d}\ , \tag{1.1}$$

where θ_p is the **parallax angle**, d is the distance, and 1 AU $= 1.496 \times 10^{13}$ cm is the mean distance between the Sun and the Earth, one **astronomical unit**.

Since θ_p is usually extremely small, the distance d can be expressed in terms of pc as

$$d \approx \frac{1\mathrm{AU}}{\theta_\mathrm{p}/\mathrm{radian}} = \frac{206264.8\mathrm{AU}}{\theta_\mathrm{p}/\mathrm{arcsec}} \equiv \frac{1\,\mathrm{pc}}{\theta_\mathrm{p}/\mathrm{arcsec}} \ . \tag{1.2}$$

In terms of Gaussian units, we have 1 pc $\approx 10^{18.5}$ cm.

1.1.2 Magnitude

The **apparent magnitude** m is defined as

$$m - m_0 = -2.5 \log(f/f_0) \ , \tag{1.3}$$

where f_0 is the reference flux corresponding to the magnitude m_0. The **absolute magnitude** M is defined as the apparent magnitude would-be if the star is placed at a distance of 10 pc. Equivalently, we have

$$m - M = 5 \log(d/\mathrm{pc}) - 5 \ . \tag{1.4}$$

In practice the magnitude of a star is measured in different wave bands, mainly due to the different responses of detectors to light at different wavelenghths. So the apparent magnitude in a certain band X is actually

$$m_\mathrm{X} = -2.5 \log \frac{\int_0^\infty R_\mathrm{X} f_\nu \,\mathrm{d}\nu / \int_0^\infty R_\mathrm{X} \mathrm{d}\nu}{f_{0,\mathrm{X}}} \ , \tag{1.5}$$

where R_X is the detector response (including filter transmission) and $f_{0,\mathrm{X}}$ is the flux of magnitude zero in this particular band.

There are various photometry systems with different conventions, notations, and calibration standards. A simple, commonly used one is the UBV system, in which m_U (or simply U) denotes the apparent magnitude in the ultraviolet band, covering wavelength of about 3000Å– 4000Å, m_B (or B) the blue band, covering wavelength of about 3500Å– 5500Å, and m_V (or V) the visual band, covering wavelength of about 4800Å– 6500Å. For further details in photometry, readers are referred to Allen (1973), Jaschek & Jaschek (1990), Budding (1993), and Léna et al (2012).

The difference between magnitudes in two bands is often used to define the **color index**, for example, the U-B color index and the B-V color index. The color index is closely related to the stellar spectral types and the surface temperature.

The quantity which is most directly associated with the intrinsic luminosity of a star is the **absolute bolometric magnitude**. The bolometric magnitude is the magnitude so defined that the electromagnetic radiation from a star at all wavelengths is taken into account. It is often represented by its difference from the V-band magnitude, that is, the **bolometric correction, BC**:

$$m_{\rm bol} = m_{\rm V} + {\rm BC}\ . \tag{1.6}$$

The bolometric magnitude is obviously not measurable. The *absolute* bolometric magnitude and BC may be determined by model computations. They depends on the spectral type of a star, but not on the distance.

Star lights, travelling through space, also suffer from the effect of **interstellar extinction** and **reddening**. The degree of extinction varies with wavelengths and the star light is reddened. Typical values are $\triangle U \approx 1.2 \triangle B$ and $\triangle B \approx 1.3 \triangle V$.

1.1.3 Temperature

The concept of temperature is related to the concept of thermal equilibrium. When thermal equlibrium is reached in a system, the distribution of particles in the system obeys a certain kind of statistics – Fermi-Dirac statistics or Bose-Einstein statistics, depending on whether the particles are fermions or bosons –, which is characterized by the temperature. Strictly speaking, the concept of temperature is only valid when the system is in thermal equilibrium. In nature, however, perfect thermal equilibrium does not exist. Particularly, inside stars, the subject of this lecture, or at stellar surfaces, thermal equilibrium is obviously absent: the thermodynamic states change with time and space. Nonetheless, one can still define, based on some observed properties, various surface temperatures to characterize the system that we want to understand:

- **Effective temperature**
  The effective temperature of a star with a certain luminosity is the

temperature of that star if its surface is a perfect blackbody that would shine with the same lumionosity. That is,

$$L = 4\pi R^2 \sigma T_{\rm e}^4 \ , \tag{1.7}$$

where σ is the Stefan-Boltzman constant. The effective temperature can be determined only when the luminosity L and the stellar radius R are known, which are usually not directly measurable.

- **Color temperature**
  The color temperature is the temperature of a blackbody spectrum which fits best the observed spectrum.

- **Excitation temperature**
  The excitation temperature of a stellar surface for a certain species is determined by the population ratio of different excited states of that species, which can be obtained from the strength of spectral lines due to transitions between different excited states of that species.

- **Ionization temperature**
  The ionization temperature of a stellar surface for a certain species is determined by the population ratio of different ionization states of that species.

These surface temperatures are generally not equal to one another, because the observable 'surface' layer, or the so-called 'atmosphere', of a star is not in thermodynamic equilibrium. They can, on the other hand, be understood with a detailed model atmosphere.

1.1.4 The Hertzsprung-Russel (H-R) diagram

Rich information of stellar structure and evolution can be learned from the H-R diagram, which plots absolute V-band magnitude versus stellar spectral types (or color indices approximately). This diagram may be converted, based on stellar atmosphere models, into a plot of the luminosity versus effective temperatures. Readers should review the concepts of spectral types, luminosity classes, and the general picture of stellar evolution in the H-R diagram from introductory textbooks.

1.1.5 Stellar populations

Stars are also classified into different populations according to their metal abundance:

- Population I – metal-rich (Z $\sim$ 0.02), young, mainly in the galactic disk, e.g. OB associations and open clusters
- Population II – metal-poor (Z $\sim$ 0.001), old, in the galactic halo and central bulge, e.g. globular clusters

There could also be extremely metal-poor stars, which are thought to form very early in the univserse's history. These are called population III stars. Definite evidence of their existence is still lacking. They may be progenitors of high redshift Gamma Ray Bursts (GRBs).

1.1.6 The mass-luminosity relation

The mass and luminosity of *main sequence* stars show a power-law like relation, with $L \propto M^4$ for $M < 2M_\odot$ and $L \propto M^3$ for $M > 2M_\odot$ (Popper 1980; Carroll & Ostlie 1996, page 212). The data points to infer this relation have a noticeable spread, and at both higher ($M > 10M_\odot$) and lower ($M < 0.5M_\odot$) mass end the power indices 3 and 4 are no longer good. We will try to derive, or understand, this mass-luminosity function in Section 4.3.

1.2 Mass functions of spectroscopic binaries

The mass of a star is a very crucial parameter in determining various properties of a star. It is also quite intriguing that there exists a mass-luminosity relation for main sequence stars, whose establishment relies on estimates of stellar masses. It is usually difficult to accurately determine the mass of a star observationally. The most reliable information is obtained by observing the orbital motion of binary systems. It is, however, very often that binaries are not spatially resolved. Spectroscopic binaries are therefore the major systems that can provide mass information. The **mass function** of a binary system with orbital eccentricity $\varepsilon = 0$ reads

$$\frac{m_2^3 \sin^3 \theta_\mathrm{i}}{(m_1 + m_2)^2} = \frac{P}{2\pi G}\, v_{1r}^3 \,, \tag{1.8}$$

where $v_{1r} = v_1 \sin\theta_\mathrm{i}$ is the observed maximal velocity along the line of sight, with the center-of-mass velocity removed, and θ_i is the inclination angle between the line of sight and the orbital plane normal. In general, the binary orbit is elliptical, instead of circular. In the following we will derive the mass function of a binary system with an arbitrary eccentricity. To do that, Kepler's laws will be derived first. We will then achieve our goal with Kepler's third law, which links the orbital period (observable) with stellar masses and the orbital semi-major axis, and a relation describing stellar line-of-sight velocity (observable) in terms of stellar masses and orbital parameters.

1.2.1 Kepler's laws

The Lagrangian in the center-of-mass frame for a two-body system is

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r) \ , \tag{1.9}$$

where $m = m_1m_2/(m_1+m_2)$ is the reduced mass, r is the distance between the two bodies, and ϕ is the phase angle with respect to a chosen direction of $\phi = 0$. The equation of motion is then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}) - \frac{\partial\mathcal{L}}{\partial q_i} = 0 \ , \tag{1.10}$$

with $q_1 = r, q_2 = \phi$. We can see that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}) = 0 \ ,$$

that is,

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mr^2\dot{\phi} \equiv \ell \tag{1.11}$$

is a constant. This is just a statement of angular momentum conservation, from which we can have **Kepler's second law**: consider $\mathrm{d}A = \frac{1}{2} \times r \times r\mathrm{d}\phi$ and

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{r^2\dot{\phi}}{2} = \frac{\ell}{2m} \ ,$$

which is obviously a constant.

Now let's turn to Kepler's first law. The energy as formulated in the center-of-mass frame is

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + V(r) \ , \tag{1.12}$$

and then we have

$$\dot{r} = \sqrt{\frac{2}{m}(E - V(r)) - \frac{\ell^2}{m^2 r^2}} \tag{1.13}$$

from

$$\dot{\phi} = \frac{\ell}{mr^2} \ . \tag{1.14}$$

Taking the ratio of the above two equations, we may find the orbital phase, ϕ, as a function of the distance, r, which is a description of the orbit and reads as the following:

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{\ell}{r^2\sqrt{2m(E - V(r)) - \frac{\ell^2}{r^2}}} \ . \tag{1.15}$$

With the potential form of

$$V(r) = -\frac{Gm_1 m_2}{r} \ ,$$

we can integrate over r to get (e.g. Gradshteyn et al. 1994, page 101, item 2.266)

$$\phi(r) = \cos^{-1}\left(\frac{\frac{\ell}{r} - \frac{Gmm_1m_2}{\ell}}{\sqrt{2mE + (\frac{Gmm_1m_2}{\ell})^2}}\right) + \phi_0$$

and find that

$$\frac{1}{r} = \frac{\sqrt{2mE + (\frac{Gmm_1m_2}{\ell})^2}}{\ell}\cos(\phi - \phi_0) + \frac{Gmm_1m_2}{\ell^2} \ ,$$

that is,

$$\frac{1}{r} = \frac{Gmm_1m_2}{\ell^2}(1 + \varepsilon\cos(\phi - \phi_0)) \ ,$$

where

$$\varepsilon = \sqrt{1 + \frac{2E\ell^2}{m(Gm_1m_2)^2}} \tag{1.16}$$

is the eccentricity expressed in terms of energy, angular momentum and mass. One can show that the eccentricity in the above form is always real, that is, it is always true that

$$E \geq -\frac{G^2mm_1^2m_2^2}{2\ell^2}$$

(see Exercises of this chapter).

The trajectory can now be written as

$$r = \frac{\ell^2}{Gmm_1m_2}\frac{1}{1+\varepsilon\cos(\phi-\phi_0)} \ . \tag{1.17}$$

This is the trajectory of a conic curve *with the coordinate origin defined at one focus.* For a bound system, $E < 0$, so $\varepsilon < 1$. The trajectory is an ellipse. This is **Kepler's first law**.

The orbit certainly can be described with the semi-major axis and eccentricity. The semi-major axis a of an ellipse is

$$\begin{aligned} a &= \frac{r_\mathrm{p}+r_\mathrm{a}}{2} \\ &= \frac{1}{2}\frac{\ell^2}{Gmm_1m_2}(\frac{1}{1+\varepsilon}+\frac{1}{1-\varepsilon}) \\ &= \frac{\ell^2}{Gmm_1m_2}\frac{1}{1-\varepsilon^2} && (1.18) \\ &= -\frac{Gm_1m_2}{2E} \ , && (1.19) \end{aligned}$$

where r_p is the distance at periastron (c.f. perigee, perihelion) and r_a is that at apastron (c.f. apogee, aphelion). In terms of the semi-major axis and eccentricity, the orbit has the function form

$$r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos(\phi-\phi_0)} \ . \tag{1.20}$$

Now come to the 3rd law. Recall that $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{r^2\dot{\phi}}{2} = \frac{\ell}{2m}$, and then we have

$$\int \frac{\mathrm{d}A}{\mathrm{d}t}\mathrm{d}t = A = \frac{\ell}{2m}P \ ,$$

where P is the orbital period. With $A = \pi ab$, where $b = a\sqrt{1-\varepsilon^2}$ is the semi-minor axis, the period can be written as, with the help of Eq.(1.18),

$$P = \frac{2m\pi ab}{\ell} = \frac{2m\pi a^2\sqrt{1-\varepsilon^2}}{\sqrt{aGmm_1m_2(1-\varepsilon^2)}} \ ,$$

that is,

$$P^2 = \frac{4m\pi^2 a^3}{Gm_1m_2} = \frac{4\pi^2}{G(m_1+m_2)}a^3 \ . \tag{1.21}$$

This is **Kepler's third law**. We note that the semi-major axis a here is that of the relative orbit, not of each individual orbits, which we denote as a_1 and a_2.

1.2.2 Velocity curves

In Eq.(1.8) we have the mass function of a binary system with a circular orbit. Now we would like to find the mass function for binaries with arbitrary eccentricities. For spectroscopic binaries, the observables are its period P and the line-of-sight velocity v_r, inferred from the measured Doppler shift of spectral lines, of one or both members in the binary system. At first we note that the trajectory keeps the same form for the case of describing the relative distance r as a function of position angle ϕ and the case of r_1 , or r_2, versus ϕ, where r_1 and r_2 are the distance of star 1 and 2 to the center of mass. In the former case the coordinate origin is set at one of the two stars, while in the latter it is at the center of mass.

For one of the two stars, say, star 1, we have always $r_1 = \frac{m_2}{m_1+m_2}r$, that is,

$$r_1 = \frac{a_1(1-\varepsilon^2)}{1+\varepsilon\cos\phi} \ , \tag{1.22}$$

where $a_1 = \frac{m_2}{m_1+m_2}a$, $a_1 + a_2 = a$, and *the position angle is set zero at periastron.* Consider the geometry depicted in Figure 1.1. The line-of-sight

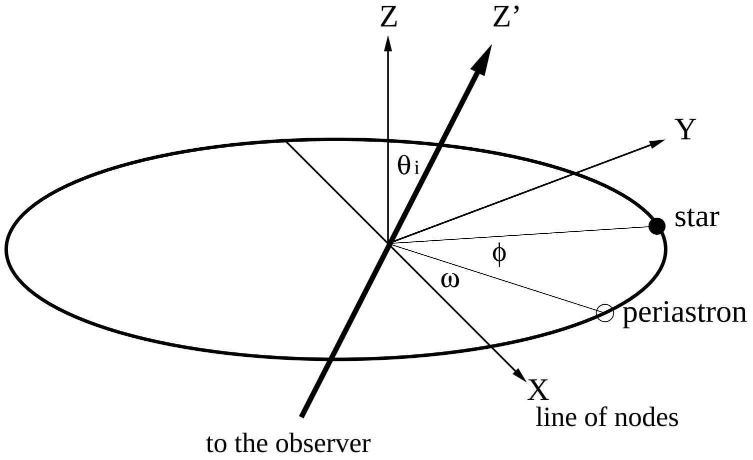

Figure 1.1: The geometry of a star in a binary system. The line of nodes, defined as the X-axis direction, is the intersection of the orbital plane and the plane of sky. The line of sight is in the Z' direction, which is parallel to the Y-Z plane.

velocity can be found as the following. At first, the projection onto the line of sight of the distance from star 1 to the center of mass of the system is

$$z'_1 = r_1 \sin(\phi + \omega) \sin\theta_{\rm i} \ ,$$

where $\theta_{\rm i}$ is the inclination angle, the angle between the line of sight and the orbital normal, and ω is the angle between line of nodes and the direction from the center of mass to the periastron. To find the line-of-sight velocity $\dot{z}'_1$, noting that

$$\dot{\phi} = \frac{\ell}{mr^2} = \frac{2mA}{mr^2 P} = \frac{2\pi}{P}\frac{(1+\varepsilon\cos\phi)^2}{(1-\varepsilon^2)^{3/2}}$$

and

$$\dot{r}_1 = \frac{a_1(1-\varepsilon^2)}{(1+\varepsilon\cos\phi)^2}\varepsilon\sin\phi\,\dot{\phi} = \frac{2\pi}{P}\frac{a_1\varepsilon\sin\phi}{\sqrt{1-\varepsilon^2}} \ ,$$

we have

$$\begin{aligned} \dot{z}'_1 &= \frac{2\pi a_1 \sin\theta_{\rm i}}{P\sqrt{1-\varepsilon^2}}(\cos(\phi+\omega) + \varepsilon\cos\omega) \\ &= K_1(\cos(\phi+\omega)+\varepsilon\cos\omega) \ , \end{aligned} \tag{1.23}$$

where

$$K_1 \equiv \frac{2\pi a_1 \sin\theta_{\rm i}}{P\sqrt{1-\varepsilon^2}} \ . \tag{1.24}$$

In the special case of a circular orbit, K_1 reduces to the v_{1r} in Eq.(1.8), the maximal line-of-sight velocity.

From Kepler's third law, $P^2 = \frac{4\pi^2}{G(m_1+m_2)}\,a^3$, and Eq.(1.24) one can see that if K_1 and the eccentricity ε can be deduced from observations, with the measured period P, the value of the mass function, a combination of m_1, m_2, and $\sin\theta_{\rm i}$, can be obtained.

The line-of-sight velocity as a function of time is

$$V_{\rm r}(t) = V_{\rm c} + K(\cos(\phi(t)+\omega) + \varepsilon\cos\omega) \ , \tag{1.25}$$

where we have dropped the subscript of K since these formalisms apply to both stars and $V_{\rm c}$ is the line-of-sight component of the center-of-mass velocity (relative to the observer). With well measured velocity curves, one may hope to pin down those parameters appearing in Eq.(1.25). However, $\phi(t)$ does not have a simple form for an elliptical orbit (see, e.g., Goldstein, Poole & Safko (2000), Section 3-8). On the other hand, to determine the values of K and ε, we may pick out some particular values from the observed velocity curve, for example, the largest and smallest velocities, which occur when the star is passing the nodes, and that at the periastron. They are

$$V_{\rm r,l} = V_{\rm c} + K(1+\varepsilon\cos\omega) \ , \tag{1.26}$$

$$V_{\rm r,s} = V_{\rm c} + K(-1+\varepsilon\cos\omega) \ , \tag{1.27}$$

and

$$V_{\rm r,p} = V_{\rm c} + K\cos\omega(1+\varepsilon) \ , \tag{1.28}$$

$$V_{\rm r,a} = V_{\rm c} + K\cos\omega(-1+\varepsilon) \ . \tag{1.29}$$

$V_{\rm r,l}$ and $V_{\rm r,s}$ are directly measurable. By noting that

$$\frac{V_{\rm r,p}+V_{\rm r,a}}{2} = V_{\rm c} + K\varepsilon\cos\omega = \frac{V_{\rm r,l}+V_{\rm r,s}}{2} \ , \tag{1.30}$$

one can identify $V_{\rm r,p}$ and $V_{\rm r,a}$, since these two velocities occur at times separated by one half of the period. This condition ($\phi(t_2) = \phi(t_1) + \pi$ and $t_2 - t_1 = \frac{P}{2}$) is met *only* at periastron and apastron, but due to observational uncertainty, the determination may not be unique. Only three among Eq.(1.26)–(1.29) are independent, as one can see from Eq.(1.30). In fact all the unknowns appear in three combinations: K, $K\cos\omega$, and $V_{\rm c} + K\varepsilon\cos\omega$. These equations are then solvable. Unfortunately, ε does not stand alone in these combinations. In order to determine ε – remember that we want to know K and ε, the center-of-mass velocity $V_{\rm c}$ must be determined independently from the observed velocity curve. $V_{\rm c}$ is obviously the average velocity in the velocity curve. Then, with Eq.(1.26)–(1.28), the three unknowns – K, ε, and $\cos\omega$ – can be easily solved.

The ambiguity of properly identifying $V_{\rm r,p}$ versus $V_{\rm r,a}$ can be resolved by requiring the eccentricity ε to be positive. Without loss of generality, we may consider the case of $V_{\rm c} = 0$ (therefore $V_{\rm r,s} < 0$). From Eq.(1.26)–(1.29), we have

$$K = \frac{1}{2}(V_{\rm r,l} - V_{\rm r,s}) \ , \tag{1.31}$$

$$K\varepsilon\cos\omega = \frac{1}{2}(V_{\rm r,l} + V_{\rm r,s}) \ , \tag{1.32}$$

$$\varepsilon = \frac{\frac{1}{2}(V_{\rm r,l} + V_{\rm r,s})}{V_{\rm r,p} - \frac{1}{2}(V_{\rm r,l} + V_{\rm r,s})} \tag{1.33}$$

and

$$\varepsilon = \frac{\frac{1}{2}(V_{\rm r,l} + V_{\rm r,s})}{\frac{1}{2}(V_{\rm r,l} + V_{\rm r,s}) - V_{\rm r,a}} \ . \tag{1.34}$$

$V_{\rm r,p}$ and $V_{\rm r,a}$ should, therefore, be so identified that $V_{\rm r,p} > \frac{1}{2}(V_{\rm r,l} + V_{\rm r,s})$ and $V_{\rm r,a} < \frac{1}{2}(V_{\rm r,l} + V_{\rm r,s})$ if $(V_{\rm r,l} + V_{\rm r,s}) > 0$, and $V_{\rm r,p} < \frac{1}{2}(V_{\rm r,l} + V_{\rm r,s})$ and $V_{\rm r,a} > \frac{1}{2}(V_{\rm r,l} + V_{\rm r,s})$ if $(V_{\rm r,l} + V_{\rm r,s}) < 0$. There are two possibilities if $(V_{\rm r,l} + V_{\rm r,s}) = 0$. One is that $\varepsilon = 0$, i.e., a circular orbit. In such a case, the velocity curve is purely sinusoidal and periastron and apastron are not defined. The other is $\cos\omega = 0$, for which $V_{\rm r,p} = V_{\rm r,a} = 0$. The latter case should be very rare, since the orbital major axis needs to be in the plane spanned by the orbital plane normal and the line of sight. Alternatively, one may also easily identify

$V_{\mathrm{r,p}}$ and $V_{\mathrm{r,a}}$ with the condition that $|V_{\mathrm{r,p}}| > |V_{\mathrm{r,a}}|$ (with V_{c} removed). This can be understood since the orbital speed at periastron is larger than that at apastron (Kepler's 2nd law), or, from Eq.(1.28) and Eq.(1.29).

1.2.3 Mass functions

We are now ready to write down the mass functions. At first we consider a double-line spectroscopic binary. Noting that

$$K_1 = \frac{2\pi a_1 \sin\theta_{\mathrm{i}}}{P\sqrt{1-\varepsilon^2}} \ ,$$

$$K_2 = \frac{2\pi a_2 \sin\theta_{\mathrm{i}}}{P\sqrt{1-\varepsilon^2}}$$

and

$$K_1 + K_2 = \frac{2\pi a \sin\theta_{\mathrm{i}}}{P\sqrt{1-\varepsilon^2}} \ ,$$

and from Kepler's third law, we have

$$P^2 = \frac{4\pi^2 \left(\frac{(K_1+K_2)P\sqrt{1-\varepsilon^2}}{2\pi\sin\theta_{\mathrm{i}}}\right)^3}{G(m_1+m_2)} \ ,$$

that is,

$$(m_1+m_2)\sin^3\theta_{\mathrm{i}} = \frac{P}{2\pi G}(1-\varepsilon^2)^{\frac{3}{2}}(K_1+K_2)^3 \ . \tag{1.35}$$

Since $m_1/m_2 = K_2/K_1$, we can separate m_1 and m_2, and write explicitly

$$m_1\sin^3\theta_{\mathrm{i}} = \frac{P}{2\pi G}(1-\varepsilon^2)^{\frac{3}{2}}K_2(K_1+K_2)^2 \tag{1.36}$$

and

$$m_2\sin^3\theta_{\mathrm{i}} = \frac{P}{2\pi G}(1-\varepsilon^2)^{\frac{3}{2}}K_1(K_1+K_2)^2 \ . \tag{1.37}$$

For a single-line spectroscopic binary, the mass function can be derived in a similar way, which gives

$$\frac{m_2^3\sin^3\theta_{\mathrm{i}}}{(m_1+m_2)^2} = \frac{P}{2\pi G}(1-\varepsilon^2)^{\frac{3}{2}}K_1^3 \ . \tag{1.38}$$

Eqs.(1.36)–(1.38) are **the mass functions of binaries with elliptical orbits**. The orbital inclination, θ_i, is usually not known. It may be inferred from other observations, e.g., light curves, with some assumptions. When it is unknown, one can only obtain lower limit estimates from the mass function. For single-line spectroscopic binaries, one very often can estimate the mass of the observable star, i.e., m_1, from its spectral information. Therefore, the lower limit to m_2, the mass of the unobservable star, can be derived.

1.3 Understanding stars: estimate, analytic models, numerical computations

To understand a phenomenon, or activities and structures of celestial bodies, some order-of-magnitude estimates are very often useful and instructive. These estimates may help in identifying the nature of a phenomenon and point out a direction for further analysis – to construct analytic models. From these analytic models one can verify, with observations, whether their predictions are correct or not. Various approximations are often adopted in constructing analytic models. It is very common, particularly with the rapid development of computing power, that a more realistic model, or a complicated model prediction, can be realized by numerical computations. Results of numerical computations are compared with data of observations to improve our understanding of the universe. In this section we will discuss some examples of order-of-magnitude estimates.

1.3.1 Time scales

The free-fall time scale

The time scale of gravitational free-fall of a star can be estimated by dimentional analysis:

$$t_\mathrm{ff} = \sqrt{\frac{R^3}{GM}} \ . \tag{1.39}$$

This expression can also be obtained by considering

$$\ddot{r} = -\frac{GM}{r^2}$$

$$\langle \ddot{r} \rangle \approx \frac{R}{t^2} \approx \frac{MG}{R^2}$$

$$\Rightarrow \; t \approx \sqrt{\frac{R^3}{GM}} \; .$$

For a star like the Sun, the free-fall time scale is

$$t_{\rm ff} = 1.6 \times 10^3 \, (M/M_\odot)^{-\frac{1}{2}} (R/R_\odot)^{\frac{3}{2}} \text{ sec} \; . \tag{1.40}$$

This time scale is much less than the evolution time scale of the Sun. It implies that the Sun possesses other mechanisms to maintain a hydrostatic equilibrium, so that its physical configuration is not significantly changed in such a free-fall time scale. The orbital time scale of planets around the Sun is of the order of the free-fall time scale of such a Sun-planet system, as is shown in Kepler's third law, Eq.(1.21).

The time scale of stellar pulsation due to structural instability is also of the order of the free-fall time scale, which, when expressed in terms of the average density $\bar{\rho} \approx M/R^3$, is

$$t_{\rm pulsation} \sim \frac{1}{\sqrt{G\bar{\rho}}} \; . \tag{1.41}$$

Pulsation periods are shorter for denser stars.

The Kelvin-Helmholtz time scale

The time scale of sustaining a certain luminosity with gravitational potential energy is called the Kelvin-Helmholtz time scale. For the Sun, we have

$$U_\odot \approx \frac{GM_\odot^2}{R_\odot} \approx 4 \times 10^{48} \, \text{erg} \; ,$$

and the corresponding time scale is

$$t_K = U_\odot / L_\odot \approx 10^{15} \, \text{sec} \approx 3 \times 10^7 \, \text{yr} \; .$$

This time scale is much shorter than Sun's life time. It indicates that the energy source of the solar luminosity is not gravitational.

The Einstein time scale

Another time scale similar to the previous one is the Einstein time scale, which involves the total energy available from mass conversion, instead of the gravitational potential energy. For the Sun, we have

$$E_\odot = M_\odot c^2 = 1.8 \times 10^{54}\,\mathrm{erg} \ ,$$

and its Einstein time scale is

$$t_\mathrm{E} = E_\odot / L_\odot = 4.5 \times 10^{20}\,\mathrm{sec} = 1.4 \times 10^{13}\,\mathrm{yr} \ .$$

Considering a typical conversion efficiency of 0.007 (see Chapter 6), this is consistent with the understanding of nuclear energy being the major energy source of the Sun.

1.3.2 The Jeans radius and the Jeans mass

Consider a proton and a cloud of mass M and radius R. For the proton to be bound, it requires

$$\frac{1}{2}v^2 < \frac{GM}{R} \ .$$

Assuming the cloud has a constant density $\rho \approx M/R^3$ and temperature $T \approx m_\mathrm{p} v^2/k$, the above requirement can be written as

$$R^2 > \frac{k}{Gm_\mathrm{p}}\frac{T}{\rho} \ .$$

So, for given T and ρ, there is a required minimum radius for the system to be gravitationally bound, that is, possible for gravitational collapse to occur to form stars. This minimum radius is called the **Jeans radius**, R_J. A more careful analysis shows

$$R_\mathrm{J} = \sqrt{\frac{\pi k}{Gm_\mathrm{p}}\frac{T}{\rho}} \approx 0.6 \times 10^8 T^{\frac{1}{2}} \rho^{-\frac{1}{2}}\,\mathrm{cm} \qquad (1.42)$$

(c.f. Rose 1998, page 16). Correspondingly, the **Jeans mass** is

$$M_\mathrm{J} = \frac{4\pi}{3} R_\mathrm{J}^3 \rho \approx 5 \times 10^{-10} T^{\frac{3}{2}} \rho^{-\frac{1}{2}}\, M_\odot \ . \qquad (1.43)$$

For a typical interstellar medium with $T = 100$ K and $\rho = 10^{-22}$ g/cm^3, we have $R_\mathrm{J} \approx 0.6 \times 10^{20}$ cm ≈ 20 pc, and $M_\mathrm{J} \approx 5 \times 10^4\, M_\odot$. Stars are therefore formed in a large group with a complicated process involving fragmentation, momentum and mass transportation, and so on.

1.3.3 Pressure of gas and radiation in stars

Typical pressure inside a star of mass M and radius R can be estimated as roughly

$$P \approx \frac{GM^2}{R^4} \ . \tag{1.44}$$

For the Sun, the above estimate gives 1.1×10^{16} dyne cm^{-2}. In usual solar models, the pressure is about 2.5×10^{17} dyne cm^{-2} at the solar center, 7.5×10^{14} dyne cm^{-2} at half radius, and 1.7×10^{13} dyne cm^{-2} at 0.8 times the radius. The pressure estimated above occurs at about 0.3 radius from the center (e.g. Cox, Livinston & Matthews 1991, page 1242–1247).

Pressure usually comes from two components: gas and radiation. Treating the gas as an ideal gas and the radiation field as characterized by the same temperature, the pressure is

$$P = \frac{\rho k T}{\mu m_\mathrm{p}} + \frac{1}{3} a T^4 \ , \tag{1.45}$$

where μ is the mean molecular weight (Section 2.2) and $a = 4\sigma/c$ with $\sigma = 5.67 \times 10^{-5}$ erg cm^{-2} sec^{-1} K^{-4} being the Stefan-Boltzman constant (Section 2.3.1). Roughly speaking, temperature in more massive stars tends to be higher, due to stronger graviational binding. From the temperature dependence in the above equation, one may infer that radiation pressure will dominate in massive stars. As for how massive a star should be in order to have radiation pressure dominating, detailed modelling is needed. In the following we try to estimate in an approximate way the stellar mass beyond which radiation pressure dominates. At first, in the case of gas pressure being the dominant one, taking $\rho \approx M/R^3$, the temperature behaves like (cf. Eq.(1.44)

$$T_\mathrm{g} \approx \frac{\mu m_\mathrm{p} G}{k} \frac{M}{R} \ , \tag{1.46}$$

while for the case of radiation pressure dominating, the temperature is

$$T_{\mathrm{r}} \approx \left(\frac{3G}{a}\right)^{\frac{1}{4}} \frac{M^{\frac{1}{2}}}{R} \ . \tag{1.47}$$

The above expressions of T_{g} and T_{r} are the typical interior temperature of the star for the two different cases. The wanted mass can be estimated by considering these two temperatures being comparable. This is left to readers in the Exercises.

1.3.4 Mass limits of stars

Now we have the sense that in very massive stars the radiation pressure will dominate. On the other hand, radiation can also blow off the outer-layer material of a star, making a star unstable. The maximum luminosity of a stable star with a given mass M can be estimated as

$$\frac{L}{4\pi R^2}\frac{\sigma_{\mathrm{T}}}{c} \approx \frac{Gm_{\mathrm{p}}M}{R^2} \ ,$$

where $\sigma_{\mathrm{T}} = \frac{8\pi}{3}(\frac{e^2}{mc^2})^2$ is the **Thomson cross section**. For electrons, it is 0.665×10^{-24} cm^2. Then we have

$$L \approx 1.3 \times 10^{38}(M/M_{\odot})\,\mathrm{erg/sec} \ . \tag{1.48}$$

This luminosity is called the **Eddington luminosity**. For a given luminosity, the above expression gives the mass lower limit of that star. For a given mass, it gives the upper limit of the luminosity.

Comparing the mass-luminosity relation of the main sequence stars as described in Section 1.1.6 with the Eddington luminosity, one can see that there is a mass upper limit at about $130M_{\odot}$, beyond which a stable star is impossible. Most model computations, involving different assumptions, set the upper limit at about $100M_{\odot} \sim 200M_{\odot}$. The mass lower limit of a star is constrained by the ignition of nuclear reactions inside stars. Most of model computations give a value of this lower limit at about $0.08M_{\odot}$.

Exercises

1. From the velocity curve of a single-line spectroscopic binary containing a G2V star, one finds $K = 3$ km/s, $\varepsilon = 0.2$ and $P = 100$ days. Assuming this is an edge-on system, find the mass of the unseen companion and the average separation between the two members of this binary.

2. From Kepler's third law verify Eq.(1.38).

3. If the luminosity of a star is about 10^{40} erg/sec, what would you say about its mass?

4. Consider a spherical gas cloud of temperature 50 K and radius 10^{18} cm. Find the density, assuming homogeneous, so that its total mass is just the Jeans mass. How large is this Jeans mass?

5. Estimate how fast the Sun can rotate without breaking itself. How about a white dwarf? A neutron star? Express your answers with rotation periods.

6. As a star evolves and changes radius, its angular momentum should stay constant. Will contraction or expansion of a star cause it to become unstable if it is initially close to rotational instability?

7. Show that the expression within the square root sign in Eq.(1.16) for the eccentricity of a binary orbit is never negative. (Hint: Consider Eq.(1.12) and Eq.(1.17).)

8. From Eqs.(1.46) and (1.47) make a quick estimate of the mass beyond which radiation pressure dominates.

9. Derive the mass upper limit at about $130M_\odot$ mentioned at the end of this chapter.

Chapter 2

Thermal and Statistical Properties of Matter

Before getting into the topic of stellar structures, a brief review on relevant thermal and statistical properties of matter is in order. We will very often assume the matter to be an ideal gas, which is in principle never true. Such an approximation greatly simplifies the problem to deal with and in many cases offers the essential ingredients of a proper model.

2.1 The ideal gas

2.1.1 Pressure and heat capacity

An ideal gas is an idealized non-interacting Boltzmann gas, whose **equation of state** is

$$P = \frac{N}{V}kT = nkT = \frac{\rho}{m}kT \ , \tag{2.1}$$

which can be derived from the Maxwell-Boltzmann distribution (Section 2.4.1). Sometimes the word ‘ideal’ is loosely used to refer to ‘non-interacting’ only. In such a case, the equation of state can be different from the above when the quantum statistical property is taken into account.

The **heat capacity at constant volume** is defined as

$$C_V \equiv \left(\frac{\partial Q}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \ , \tag{2.2}$$

then from the first law of thermodynamics,

$$\mathrm{d}U = \mathrm{d}Q - P\mathrm{d}V \ , \tag{2.3}$$

we have

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V \ . \tag{2.4}$$

For an ideal gas, whose internal energy U depends on temperature T only,

$$U = \int_0^T C_V \, \mathrm{d}T' \ . \tag{2.5}$$

For a monatomic ideal gas, $U = \frac{3}{2}NkT = \frac{3}{2}PV$. So, $C_V = \frac{3}{2}Nk$ is a constant.

The **heat capacity at constant pressure** is defined similarly as

$$C_P \equiv \left(\frac{\partial Q}{\partial T}\right)_P = T\left(\frac{\partial S}{\partial T}\right)_P \ . \tag{2.6}$$

For an ideal gas

$$\mathrm{d}U = C_V \mathrm{d}T = \mathrm{d}Q - P\mathrm{d}V \ ,$$

one can see that

$$C_V = \left(\frac{\partial Q}{\partial T}\right)_P - P\left(\frac{\partial V}{\partial T}\right)_P \ , \tag{2.7}$$

that is,

$$C_P = C_V + Nk \ . \tag{2.8}$$

2.1.2 Polytropic processes

A polytropic process is a thermodynamic process in which

$$\frac{\mathrm{d}Q}{\mathrm{d}T} = c \ , \tag{2.9}$$

where c is a constant. For an ideal gas undergoing such a process, we have $c\,\mathrm{d}T - C_V\mathrm{d}T = \mathrm{d}Q - \mathrm{d}U = P\mathrm{d}V = \frac{NkT}{V}\mathrm{d}V = (C_P - C_V)T\frac{\mathrm{d}V}{V}$, that is,

$$\frac{\mathrm{d}T}{T}(c - C_V) - \frac{\mathrm{d}V}{V}(C_P - C_V) = 0 \ .$$

If we now define a **polytropic exponent** Γ as

$$\Gamma \equiv \frac{C_P - c}{C_V - c} \ , \tag{2.10}$$

then we have

$$\frac{\mathrm{d}T}{T} + (\Gamma - 1)\frac{\mathrm{d}V}{V} = 0 \ . \tag{2.11}$$

This is equivalent to

$$TV^{\Gamma-1} = \text{constant} \ . \tag{2.12}$$

Recalling that $PV = NkT$, we can reformulate Eq.(2.11) as

$$\frac{\mathrm{d}(PV)}{PV} + (\Gamma - 1)\frac{\mathrm{d}V}{V} = 0 \ ,$$

which is just

$$\frac{\mathrm{d}P}{P} + \Gamma\frac{\mathrm{d}V}{V} = 0 \ . \tag{2.13}$$

This is again equivalent to

$$PV^{\Gamma} = \text{constant} \ . \tag{2.14}$$

Similarly, we have

$$\frac{\mathrm{d}P}{P} + \frac{\Gamma}{1-\Gamma}\frac{\mathrm{d}T}{T} = 0 \ , \tag{2.15}$$

and equivalently,

$$P^{1-\Gamma}T^{\Gamma} = \text{constant} \ . \tag{2.16}$$

For adiabatic processes, which are polytropic processes with $c = 0$, we have the polytropic exponent $\Gamma = C_P/C_V = \gamma$, the **adiabatic index**. For isothermal processes, $c = \infty$ and $\Gamma = 1$. Instead of defining the polytropic exponent with Eq.(2.10), one may take Eqs.(2.11), (2.13) and (2.15) as definitions to discuss thermal properties of a system. In such a case, the polytropic exponents in those three equations are different in general, if the gas is not ideal, or the system considered has a mixture of radiation and gas, which is apparently an important case in stellar astrophysics and will be discussed in later sections.

2.2 The mean molecular weight

When the system is composed of different species of particles, the concept of the mean molecular weight is often useful. The **mean molecular weight** μ is the average particle mass in units of a.m.u., i.e., defined as

$$\mu \equiv \frac{\rho}{n m_{\mathrm{u}}} \ , \tag{2.17}$$

where ρ is the mass density, n the number density, and m_{u} is one a.m.u., which is about 1.66×10^{-24} g. The number density n includes all the different species, that is, $n = \sum n_i$, where n_i is the number density for a certain species of particles. The gas pressure is the combination of partial pressures from all the different components. When the equation of state of an ideal gas is written in terms of the mass density instead of the number density, the mean molecular weight comes into play:

$$P = \frac{\rho}{\mu m_{\mathrm{u}}} kT \ . \tag{2.18}$$

The mean molecular weight of a gas depends on its composition and ionization degree. Consider a gas consisting of various elements with atomic number Z and mass fraction X_Z, i.e., $X_Z = \rho_Z/\rho$ and $\sum_{Z=1} X_Z = 1$. The number density n_Z of element Z is

$$n_z = \frac{\rho_Z}{A_Z m_{\mathrm{u}}} = \frac{X_Z}{A_Z} \frac{\rho}{m_{\mathrm{u}}} \ ,$$

where A_Z is the **atomic weight** (e.g. $A_1 \approx 1.008$, $A_2 \approx 4.003$). Denote the average number of free particles contributed by *one atom* of element Z with $\bar{N}_Z$, which depends on the ionization degree of the element. For example, $\bar{N}_Z$ is equal to $1+Z$ for the case of complete ionization. The particle number density n of the gas, *including ions and electrons*, is

$$n = \sum_{Z=1} n_Z \bar{N}_Z = \sum_{Z=1} \frac{\rho}{m_{\mathrm{u}}} \frac{X_Z}{A_Z} \bar{N}_Z \ ,$$

and then we have

$$\frac{1}{\mu} = \sum_{Z=1} \frac{X_Z}{A_Z} \bar{N}_Z \ . \tag{2.19}$$

The most abundant elements in the Universe are hydrogen and helium. All other elements are called 'metal' in astronomical terms. Conventionally the mass fraction of hydrogen is denoted as X, that of helium as Y, and the metallic mass fraction as Z. (The notation Z can be confusing, unfortunately. We will try to avoid its use, with $Z = 1 - X - Y$, when the atomic number Z is around.) The metallic abundance, although very low, about 0.02 for population I, and 0.001 for population II, affects the opacity significantly, and therefore plays an important role in determining the outer-layer structure of a star and its emergent spectrum.

In terms of X and Y, the mean molecular weight can be written as

$$\mu = \left(\frac{X\bar{N}_1}{1.008} + \frac{Y\bar{N}_2}{4.003} + (1 - X - Y)\langle\frac{\bar{N}_Z}{A_Z}\rangle \right)^{-1} \quad . \tag{2.20}$$

For the case of complete ionization, we have $\bar{N}_1 = 2$, $\bar{N}_2 = 3$, and $\bar{N}_Z = 1+Z$. Taking $A_Z \approx 2Z + 2$, the mean molecular weight is then

$$\begin{aligned} \mu &\approx (2X + \frac{3}{4}Y + \frac{1}{2}(1 - X - Y))^{-1} \\ &= \frac{2}{1 + 3X + 0.5Y} \quad . \end{aligned} \tag{2.21}$$

We can see that under the approximations taken here, the mean molecular weight falls in the range of

$$0.5 \leq \mu \leq 2 \ . \tag{2.22}$$

The lower end occurs when $X = 1$ and the higher end at $X = Y = 0$.

Electrons deserve a special treatment, particularly for systems in which electrons dominate in contributing pressure. The **mean molecular weight per electron**, or simply the **electron molecular weight**, of a gas is defined in a similar way:

$$\mu_{\mathrm{e}} \equiv \frac{\rho}{n_{\mathrm{e}} m_{\mathrm{u}}} \ . \tag{2.23}$$

With

$$n_{\mathrm{e}} = \sum_{Z=1} n_Z(\bar{N}_Z - 1) = \frac{\rho}{m_{\mathrm{u}}} \sum_{Z=1} \frac{X_Z}{A_Z}(\bar{N}_Z - 1)$$

the electron molecular weight can be expressed as

$$\mu_{\rm e} = \left(\sum_{Z=1} \frac{X_Z}{A_Z} (\bar{N}_Z - 1) \right)^{-1} \ . \tag{2.24}$$

Consider again the case of complete ionization,

$$\begin{aligned} \mu_{\rm e} &= \left(\sum_{Z=1} \frac{X_Z Z}{A_Z} \right)^{-1} \\ &\approx \frac{1}{X + 0.5Y + 0.5(1 - X - Y)} \\ &= \frac{2}{1+X} \ . \end{aligned} \tag{2.25}$$

Note that

$$1 \leq \mu_{\rm e} \leq 2 \ . \tag{2.26}$$

2.3 Radiation and matter

In previous sections gas pressure was discussed. The gas was assumed to be ideal, which is often a good approximation for stellar matter. Radiation, however, may also play an important role in determining stellar structure, in particular for more massive stars. A system consisting of an ideal gas and a radiation field is no longer 'ideal', as we can see that the ideal gas law does not apply any more. Radiation pressure has a dependence on temperature different from that of an ideal gas. In the following we will discuss radiation pressure first, and then introduce three adiabatic exponents which characterize thermodynamic properties of a system, including the system consisting of an ideal gas and a radiation field that we discuss in this section. Entropy, which is important when considering evolution, is introduced for such a system at the end of this section.

2.3.1 Pressure of radiation

When a system is in thermal equilibrium, the photon field is isotropic and follows the Planck function:

$$B_\nu = \frac{2\nu^2}{c^2} \frac{h\nu}{e^{\frac{h\nu}{kT}} - 1} \ , \tag{2.27}$$

where B_ν is the specific intensity, whose unit is erg cm^{-2} sec^{-1} Hz^{-1} str^{-1} in Gaussian units. $B_\nu \mathrm{d}\nu$ is the intensity in the frequency range of $\mathrm{d}\nu$. The approximation of using such an isotropic Planck function to describe the photon field is usually fine in the deep stellar interior, but not really applicable in stellar atmospheres, in which anisotropy prevails and the photon field is obtained by solving the radiation transfer problem in a model atmosphere.

The specific energy density u_ν is the specific intensity integrated over all solid angles (called 'specific flux' or 'flux density') divided by the speed of light:

$$u_\nu = \frac{4\pi}{c} B_\nu \tag{2.28}$$

and the energy density is its integration over all frequencies:

$$u = \int_0^\infty u_\nu \mathrm{d}\nu = aT^4 = \frac{4\sigma}{c} T^4 \ , \tag{2.29}$$

where $\sigma = 5.67 \times 10^{-5}$ erg cm^{-2} sec^{-1} K^{-4} is the **Stefan-Boltzman constant**. It is also useful to note that the number density of photons is

$$n_\gamma = \int_0^\infty \frac{u_\nu}{h\nu} \mathrm{d}\nu \approx 20 (T/\mathrm{K})^3 \, \mathrm{cm}^{-3} \ . \tag{2.30}$$

The radiation pressure in such an isotropic field is simply one third of the energy density,

$$P = \frac{1}{3} u = \frac{1}{3} aT^4 \ . \tag{2.31}$$

This can be understood by considering that the energy flux divided by the speed of light is the momentum flux ($E = pc$ for photons; we also note that energy density, momentum flux and pressure have the same dimension). To get the pressure, because of isotropy, a factor of one third should be included when taking all directions into account and considering a total-reflecting imaginary plane, that is,

$$P_\nu = \frac{1}{3} \int \frac{B_\nu}{c} \mathrm{d}\Omega = \frac{4\pi}{3} \frac{B_\nu}{c} = \frac{1}{3} u_\nu \ ,$$

and $P = \int P_\nu \mathrm{d}\nu = \frac{1}{3} u$. Alternatively, P_ν can also be understood as

$$P_\nu = 2 \times \int \frac{B_\nu}{c} \cos^2 \theta \mathrm{d}\Omega = \frac{4\pi}{3} \frac{B_\nu}{c} \ ,$$

by considering a unit area in an imaginary plane bouncing back photons, with θ being the angle from the surface normal and integration over half a hemisphere.

The pressure in a system can be attributed to contributions from gas and radiation, respectively, that is,

$$P = P_{\mathrm{g}} + P_{\mathrm{r}} = nkT + \frac{1}{3}aT^4 \ , \tag{2.32}$$

where P_{g} is the gas pressure and P_{r} is the radiation pressure. An ideal gas is assumed when we use $P_{\mathrm{g}} = nkT$. Defining the gas pressure fraction β as

$$\beta \equiv \frac{P_{\mathrm{g}}}{P} \ , \tag{2.33}$$

the total pressure can be written as

$$P = \frac{P_{\mathrm{g}}}{\beta} = \frac{\rho kT}{\beta \mu m_{\mathrm{u}}} \ , \tag{2.34}$$

which expresses the total pressure of a system in an ideal-gas formalism with a modified mean molecular weight $\beta\mu$.

2.3.2 Adiabatic exponents

In Section 2.1.2 the polytropic exponent of an ideal gas was introduced and shown to link the variation of two thermodynamic variables in a certain way for any polytropic processes; see Eqs.(2.11)–(2.16). Since the system we study may not always be ideal, Eqs.(2.11)–(2.16) may not hold in general. On the other hand, adiabatic processes, depending on the time scales involved, are often good approximation to describe thermodynamical change in a system which is not necessarily ideal. Under the adiabatic assumption, variation of any two thermodynamic variables can still be related to each other in a form similar to Eqs.(2.11)–(2.16), but with different 'exponents', called adiabatic exponents now, for each set of two variables. These three adiabatic exponents, of which only two are independent, describe thermodynamic properties of the system.

Considering the three variables, P, ρ and T, and their variation expressed in terms of variation in entropy and in one of the other two variables, that is,

$$\mathrm{d}P(\rho, S) = \left(\frac{\partial P}{\partial \rho}\right)_S \mathrm{d}\rho + \left(\frac{\partial P}{\partial S}\right)_\rho \mathrm{d}S \ , \tag{2.35}$$

$$\mathrm{d}P(T,S) = \left(\frac{\partial P}{\partial T}\right)_S \mathrm{d}T + \left(\frac{\partial P}{\partial S}\right)_T \mathrm{d}S \tag{2.36}$$

and

$$\mathrm{d}T(\rho,S) = \left(\frac{\partial T}{\partial \rho}\right)_S \mathrm{d}\rho + \left(\frac{\partial T}{\partial S}\right)_\rho \mathrm{d}S \ , \tag{2.37}$$

one may see that by defining

$$\Gamma_1 \equiv \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_S \tag{2.38}$$

$$\frac{\Gamma_2}{\Gamma_2 - 1} \equiv \left(\frac{\partial \ln P}{\partial \ln T}\right)_S \tag{2.39}$$

$$\Gamma_3 - 1 \equiv \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_S \ , \tag{2.40}$$

we have, for adiabatic processes ($\mathrm{d}S = 0$),

$$\frac{\mathrm{d}P}{P} + \Gamma_1 \frac{\mathrm{d}V}{V} = 0 \ , \tag{2.41}$$

$$\frac{\mathrm{d}P}{P} + \frac{\Gamma_2}{1-\Gamma_2}\frac{\mathrm{d}T}{T} = 0 \ , \tag{2.42}$$

and

$$\frac{\mathrm{d}T}{T} + (\Gamma_3 - 1)\frac{\mathrm{d}V}{V} = 0 \ , \tag{2.43}$$

or equivalently,

$$PV^{\Gamma_1} = C_1 \ , \tag{2.44}$$

$$P^{1-\Gamma_2}T^{\Gamma_2} = C_2 \ , \tag{2.45}$$

and

$$TV^{\Gamma_3 - 1} = C_3 \ . \tag{2.46}$$

These adiabatic exponents are in general different from one another, unless the system is simply an ideal gas. Recall in Section 2.1.2 these exponents are all the same for an ideal gas and in such a case these relations hold not only for adiabatic processes but also for all the polytropic ones : they are called 'polytropic exponents' there. Note that we have used terms like 'polytropic processes', 'polytropic exponents', 'adiabatic indices', and 'adiabatic exponents'. Among these adiabatic exponents, we note that only two of them are independent:

$$\Gamma_3 - 1 = \frac{\Gamma_2 - 1}{\Gamma_2} \Gamma_1 \ . \tag{2.47}$$

The values of these three exponents, Γ_1, Γ_2, and Γ_3, are related to dynamic, convective, and pulsational instabilities respectively.

Adiabatic exponents in terms of β and γ of the ideal gas in a gas-radiation system

The adiabatic process is often a good approximation for describing processes subject to a finite time scale. We shall now consider a system consisting of an ideal gas and radiation (photons) and derive the form of adiabatic exponents as a function of the parameter β introduced in Section 2.3.1 and the adiabatic index γ of the ideal gas, i.e., C_P/C_V (here referred to the gas only, not the whole system). Considering

$$\begin{aligned}
\mathrm{d}Q &= 0 \\
&= \mathrm{d}U + P\mathrm{d}V \\
&= C_V\mathrm{d}T + \mathrm{d}(aT^4V) + P\mathrm{d}V \\
&= C_V\mathrm{d}T + (4aT^3V\mathrm{d}T + aT^4\mathrm{d}V) + P\mathrm{d}V
\end{aligned}$$

and

$$C_V\mathrm{d}T = C_V\frac{\beta PV}{NkT}\mathrm{d}T = \frac{C_V}{C_P - C_V}\beta PV\mathrm{d}\ln T = \frac{\beta}{\gamma - 1}PV\mathrm{d}\ln T \ ,$$

$$4aT^3V\mathrm{d}T = 12\frac{aT^4}{3}V\mathrm{d}\ln T = 12(1-\beta)PV\mathrm{d}\ln T \ ,$$

and

$$aT^4\mathrm{d}V = 3(1-\beta)P\mathrm{d}V \ ,$$

we have

$$(\frac{\beta}{\gamma-1}+12(1-\beta))\mathrm{d}\ln T+(3(1-\beta)+1)\mathrm{d}\ln V=0\ .$$

Therefore, comparing this relation with Eq.(2.43), we can get

$$\begin{aligned}\Gamma_3-1 &= \frac{3(1-\beta)+1}{\frac{\beta}{\gamma-1}+12(1-\beta)}\\ &= \frac{(4-3\beta)(\gamma-1)}{\beta+12(1-\beta)(\gamma-1)}\ .\end{aligned} \tag{2.48}$$

In a similar manner we can also obtain

$$\Gamma_2=1+\frac{(4-3\beta)(\gamma-1)}{\beta^2+3(1-\beta)(4+\beta)(\gamma-1)} \tag{2.49}$$

and

$$\Gamma_1=\beta+\frac{(4-3\beta)^2(\gamma-1)}{\beta+12(1-\beta)(\gamma-1)}\ . \tag{2.50}$$

For $\gamma=\frac{5}{3}$, the case of monatomic molecules, the above expressions turn into

$$\Gamma_1=\frac{32-24\beta-3\beta^2}{24-21\beta}\ , \tag{2.51}$$

$$\Gamma_2=\frac{32-24\beta-3\beta^2}{24-18\beta-3\beta^2}\ , \tag{2.52}$$

and

$$\Gamma_3=\frac{32-27\beta}{24-21\beta}\ . \tag{2.53}$$

In the extreme case of $\beta=1$, that is, the pressure comes only from gas particles, ideal and monatomic for the current case, we have $\Gamma_1=\Gamma_2=\Gamma_3=\frac{5}{3}=\gamma$, just as expected. On the other hand, for the case of $\beta=0$, we have $\Gamma_1=\Gamma_2=\Gamma_3=\frac{4}{3}$ for a photon field in equilibrium. The latter result actually does not depend on the value of γ. It indicates that when the adiabatic exponents approach $\frac{4}{3}$, stars cannot exist, since they would not be gravitationally bound any more.

Adiabatic exponents in terms of thermodynamic variables

More generally, we may express these adiabatic exponents in terms of thermodynamic variables *of the whole system*, no matter whether the gas is ideal or not. Considering that

$$\begin{aligned}\mathrm{d}Q &= \mathrm{d}U + P\mathrm{d}V \\ &= \left(\frac{\partial U}{\partial T}\right)_V \mathrm{d}T + \left(\left(\frac{\partial U}{\partial V}\right)_T + P\right)\mathrm{d}V\end{aligned}$$

and then

$$C_P = \left(\frac{\partial Q}{\partial T}\right)_P = C_V + \left(\left(\frac{\partial U}{\partial V}\right)_T + P\right)\left(\frac{\partial V}{\partial T}\right)_P \ ,$$

we have

$$\left(\left(\frac{\partial U}{\partial V}\right)_T + P\right) = (C_P - C_V)\left(\frac{\partial T}{\partial V}\right)_P$$

and

$$\mathrm{d}Q = C_V\mathrm{d}T + (C_P - C_V)\left(\frac{\partial T}{\partial V}\right)_P \mathrm{d}V \ . \tag{2.54}$$

For adiabatic processes, $\mathrm{d}Q = 0$, so,

$$\frac{\mathrm{d}T}{T} + \frac{C_P - C_V}{C_V}\frac{V}{T}\left(\frac{\partial T}{\partial V}\right)_P \frac{\mathrm{d}V}{V} = 0 \ ,$$

and therefore

$$\Gamma_3 - 1 = \frac{C_P - C_V}{C_V}\frac{V}{T}\left(\frac{\partial T}{\partial V}\right)_P \ . \tag{2.55}$$

Similarly one can reach the following two equations:

$$\Gamma_1 = -\frac{C_P}{C_V}\frac{V}{P}\left(\frac{\partial P}{\partial V}\right)_T \tag{2.56}$$

and

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{C_P T}{(C_P - C_V)\left(\frac{\partial T}{\partial P}\right)_V P} \ . \tag{2.57}$$

One should note that the C_P and C_V here are that of the whole system, which is not necessarily ideal. These expressions are consistent with Eq.(2.47) and, in the case of an ideal gas, reduce to $\Gamma_1 = \Gamma_2 = \Gamma_3 = C_P/C_V = \gamma$. One can also obtain Eq.(2.55)-(2.57) directly from definitions of heat capacities and adiabatic exponents, i.e., Eq.(2.2), (2.6) and Eq.(2.38)-(2.40).

2.3.3 Entropy

Another important thermodynamic function is entropy, whose change plays an essential role in the equations of stellar structures for describing the evolution. We would like to write it down as a function of another two thermodynamic variables, the temperature T and the volume V.

For an ideal gas, we have

$$\begin{aligned} T\mathrm{d}S &= \mathrm{d}U + P\mathrm{d}V \\ &= C_V\,\mathrm{d}T + \frac{NkT}{V}\mathrm{d}V \ . \end{aligned}$$

Then,

$$\mathrm{d}S = C_V\,\mathrm{d}\ln T + Nk\,\mathrm{d}\ln V \ ,$$

and

$$\begin{aligned} S &= C_V \ln T + Nk\ln V + \text{constant} \\ &= Nk\ln\left(VT^{\frac{1}{\gamma-1}}\right) + \text{constant} \ . \end{aligned} \tag{2.58}$$

For the entropy of radiation, one can see that

$$T\mathrm{d}S = \mathrm{d}(aT^4V) + \frac{1}{3}aT^4\mathrm{d}V \ ,$$

$$\mathrm{d}S = 4aT^2V\mathrm{d}T + \frac{4}{3}aT^3\mathrm{d}V \ ,$$

and

$$S = \frac{4}{3}aT^3V \ . \tag{2.59}$$

The entropy of the system is the sum of these two,

$$S = Nk\ln\left(VT^{\frac{1}{\gamma-1}}\right) + \frac{4}{3}aT^3V + \text{constant} \ . \tag{2.60}$$

It can also be expressed in terms of other variables. If we consider the volume V to be the volume of a unit mass, it can be turned into $\frac{1}{\rho}$, where ρ is the mass density, and the entropy S is then entropy per unit mass s_m:

$$s_m = \frac{k}{\mu m_\mathrm{u}} \ln \left(\frac{T^{\frac{1}{\gamma - 1}}}{\rho} \right) + \frac{4}{3} a \frac{T^3}{\rho} + \text{constant} \ , \tag{2.61}$$

where μ is the mean molecular weight and m_u is one amu. Entropy is additive. It is easy to turn s_m into the entropy of the whole system. For all the processes that occur in the universe, in particular to our interest here the evolution of stars, the entropy always increases, that is, $\triangle S > 0$. When we assume, or approximate, a process as an adaibatic one, we have $\triangle S = 0$.

2.4 Statistical distribution functions

We very often assume the stellar materials, in particular in the stellar interior, to be in a thermal equilibrium state, at least locally, so that the concept of temperature still applies. In thermal equilibrium, particles are distributed in the energy space in a way depending on particles' nature. From the distribution function, one may easily compute the corresponding pressure. Fermions follow the Fermi-Dirac distribution and bosons follow the Bose-Einstein distribution. In the classical regime, that is, low density and high temperature, they both follow the Maxwell-Boltzmann distribution, which we will discuss first in the following.

2.4.1 The Maxwell-Boltzmann distribution

We may describe the distribution of particles in the momentum space as

$$n = \int f_p(p) \mathrm{d}^3 p \ , \tag{2.62}$$

with n being the number density. For classical particles, the distribution function $f_p(p)$ is the Maxwell-Boltzmann distribution,

$$f_p(p) = \frac{n}{(2\pi m k T)^{3/2}} \exp(-\frac{p^2}{2mkT}) \ . \tag{2.63}$$

The pressure of such a distribution is

$$\begin{aligned} P &= \frac{1}{3}\int v p f_p(p) \mathrm{d}^3 p && (2.64) \\ &= \frac{1}{3}\int_0^\infty \frac{p^2}{m}\frac{n}{(2\pi mkT)^{3/2}}\exp(-\frac{p^2}{2mkT})\, 4\pi p^2 \mathrm{d}p \\ &= nkT\ . \end{aligned}$$

The last equality is reached with $\int_0^\infty p^4 \exp(-ap^2)\mathrm{d}p = \frac{3}{8}\sqrt{\frac{\pi}{a^5}}$. Here we see that the equation of state of the ideal gas actually can be derived from the Maxwell-Boltzmann distribution. The energy density can also be obtained as

$$u = \int \epsilon f_p(p) \mathrm{d}^3 p = \frac{3}{2} nkT\ , \tag{2.65}$$

where $\epsilon = p^2/2m$ is the kinetic energy.

The Maxwell-Boltzmann distribution is also often expressed in the speed space. Let $f_v(v)\mathrm{d}v$ be the probability of finding a particle in the speed range of v to $v + \mathrm{d}v$, so that

$$\int_0^\infty f_v(v)\mathrm{d}v = 1\ . \tag{2.66}$$

Then the distribution function is

$$f_v(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} v^2 \exp(-\frac{mv^2}{2kT})\ . \tag{2.67}$$

With this distribution, we have the average speed $\langle v\rangle$, average speed square $\langle v^2\rangle$, and the most probable speed v_{mp} as

$$\langle v\rangle = \int_0^\infty f_v v \mathrm{d}v = \sqrt{\frac{8kT}{\pi m}}\ , \tag{2.68}$$

$$\langle v^2\rangle = \int_0^\infty f_v v^2 \mathrm{d}v = \frac{3kT}{m}\ , \tag{2.69}$$

and

$$v_{\mathrm{mp}} = \sqrt{\frac{2kT}{m}}\ . \tag{2.70}$$

The so-called root-mean-square (rms) speed, v_{rms}, is

$$v_{\mathrm{rms}} = \sqrt{\langle v^2\rangle} = \sqrt{\frac{3kT}{m}}\ . \tag{2.71}$$

2.4.2 Quantum statistics

The Maxwell-Boltzmann distribution is only valid for classical particles. In quantum statistics the distribution function f is such that

$$\frac{\mathrm{d}N}{\mathrm{d}^3x\mathrm{d}^3p} = \frac{g}{h^3} f \ , \tag{2.72}$$

where the left hand side is the particle number density in the phase space, that is, the number density n in the usual space is

$$n = \int \frac{\mathrm{d}N}{\mathrm{d}^3x\mathrm{d}^3p}\mathrm{d}^3p = \int \frac{g}{h^3} f \,\mathrm{d}^3p \ .$$

The g at the right-hand side is the **statistical weight**, or the number of states in that phase-space differential volume, which is

$$g = 2s + 1 \ , \tag{2.73}$$

where s is the spin quantum number. For fermions, s is a positive half integer, like 1/2, 3/2, etc., while for bosons, it's a non-negative integer. Note that for photons, $g = 2$, although they are spin 1 particles.

The distribution function for fermions is

$$f(\epsilon) = \frac{1}{\exp(\frac{\epsilon-\mu}{kT}) + 1} \quad \text{(Fermi-Dirac distribution)} \ , \tag{2.74}$$

and for bosons is

$$f(\epsilon) = \frac{1}{\exp(\frac{\epsilon-\mu}{kT}) - 1} \quad \text{(Bose-Einstein distribution)} \ . \tag{2.75}$$

The μ in the above equations is the chemical potential. For photons, $\mu = 0$. For classical particles, $-\mu/kT \gg 1$, the distribution reduces to the Maxwell-Boltzman distribution, and we have the Boltzmann relation:

$$\frac{n_i}{n_j} = \frac{g_i}{g_j} \exp(-\frac{\epsilon_i - \epsilon_j}{kT}) \ . \tag{2.76}$$

2.4.3 The Fermi-Dirac distribution and degeneracy

In this section we take electrons as a typical case for fermions. Electrons are fermions with spin $\frac{1}{2}$. The electron number density $n_{\rm e}$ is

$$n_{\rm e} = \int \frac{g_{\rm e}}{h^3} \frac{{\rm d}^3 p}{\exp(\frac{\epsilon-\mu}{kT})+1} = \int_0^\infty \frac{8\pi}{h^3} \frac{p^2 {\rm d}p}{\exp(\frac{\epsilon-\mu}{kT})+1} \ , \tag{2.77}$$

and its pressure is

$$P_{\rm e} = \int_0^\infty \frac{8\pi}{3h^3} \frac{vpp^2 {\rm d}p}{\exp(\frac{\epsilon-\mu}{kT})+1} \ . \tag{2.78}$$

This is the equation of state for an electron Fermi gas.

Defining $u \equiv p^2/2mkT$ and $\alpha \equiv -\mu/kT$, Eq.(2.77) and Eq.(2.78) can be written as

$$n_{\rm e} = \frac{4\pi}{h^3}(2mkT)^{\frac{3}{2}} \int_0^\infty \frac{u^{\frac{1}{2}}{\rm d}u}{\exp(\alpha+u)+1} \tag{2.79}$$

and

$$P_{\rm e} = \frac{8\pi kT}{3h^3}(2mkT)^{\frac{3}{2}} \int_0^\infty \frac{u^{\frac{3}{2}}{\rm d}u}{\exp(\alpha+u)+1} \ . \tag{2.80}$$

The equation of state is then parameterized by α and T. We can see that α depends on μ (the chemical potential) and T, so given μ and T, $P_{\rm e}$ and $n_{\rm e}$ are determined. On the other hand, actually if two among the four quantities (μ, T, $P_{\rm e}$ and $n_{\rm e}$) are given, the other two can be determined. The chemical potential μ can therefore be regarded as a function of $n_{\rm e}$ and T. Conventionally the above two integrals are denoted by

$$F_{\frac{1}{2}} = \int_0^\infty \frac{u^{\frac{1}{2}}{\rm d}u}{\exp(\alpha+u)+1} \tag{2.81}$$

and

$$F_{\frac{3}{2}} = \int_0^\infty \frac{u^{\frac{3}{2}}{\rm d}u}{\exp(\alpha+u)+1} \ . \tag{2.82}$$

The equation of state then reads

$$P_{\rm e} = n_{\rm e}kT\left(\frac{2}{3}\frac{F_{\frac{3}{2}}}{F_{\frac{1}{2}}}\right) \ . \tag{2.83}$$

The ratio $(2F_{\frac{3}{2}}/3F_{\frac{1}{2}})$ describes the extent to which the gas differs from an ideal Boltzmann gas. A plot depicting its behavior as a function of α can be found in Clayton (1983, page 98). That ratio approaches unity for $\alpha \gg 1$.

For the limiting case of $\mu/kT \to \infty$, we have a completely degenerate gas. The distribution function is only non-zero, and equal to unity, when the energy is lower than its chemical potential, $\epsilon < \mu$. In such a case, the electron number density and pressure are

$$n_{\rm e} = \int_0^{p_{\rm F}} \frac{8\pi}{h^3} p^2 {\rm d}p = \frac{8\pi}{3h^3} p_{\rm F}^3 \tag{2.84}$$

and

$$P_{\rm e} = \int_0^{p_{\rm F}} \frac{8\pi}{3h^3} vpp^2 {\rm d}p \ , \tag{2.85}$$

where $p_{\rm F}$ is the Fermi momentum, above which no particles are populated. The corresponding energy is called the **Fermi energy** $\epsilon_{\rm F}$, which is the chemical potential of the system at the extreme case of $T = 0$.

Since in some circumstances the Fermi momentum can be quite high and the particles can no longer be treated as non-relativistic ones, the expression of pressure can be complicated. For the two extreme cases, non-relativistic ($p = mv$) and ultra-relativistic ($v = c$), we have

$$\begin{aligned} P_{\rm e} &= \frac{8\pi}{3h^3} \int_0^{p_{\rm F}} \frac{p^4}{m} {\rm d}p \qquad \text{(non-relativistic)} && (2.86) \\ &= \frac{8\pi}{15h^3 m_{\rm e}} p_{\rm F}^5 && (2.87) \\ &= (\frac{3}{8\pi})^{2/3} \frac{h^2}{5m_{\rm e}} n_{\rm e}^{5/3} && (2.88) \end{aligned}$$

and

$$\begin{aligned} P_{\rm e} &= \frac{8\pi}{3h^3} \int_0^{p_{\rm F}} cp^3 {\rm d}p \qquad \text{(ultra-relativistic)} && (2.89) \\ &= \frac{2\pi c}{3h^3} p_{\rm F}^4 && (2.90) \\ &= (\frac{3}{\pi})^{1/3} \frac{hc}{8} n_{\rm e}^{4/3} \ . && (2.91) \end{aligned}$$

Readers should also check the energy density in these cases.

Corrections due to the effect of a finite temperature, that is, $T \neq 0$ but still $\mu \gg kT$, can be found in some other textbooks. For the pressure, the next order correction is

$$P_{\rm e}(T) = P_{\rm e}(0) + \frac{\pi}{4} n_{\rm e} kT \frac{kT}{\epsilon_{\rm F}} \ . \tag{2.92}$$

One can usually compare the typical thermal energy kT with the Fermi energy $\epsilon_{\rm F}$, which depends on the number density only, to check the degree of degeneracy. For readers' convenience, numerical values of the Fermi energy are shown below (all in cgs units):

$$\epsilon_{\rm F} = \frac{p_{\rm F}^2}{2m_{\rm e}} = 5.8 \times 10^{-27} n_{\rm e}^{\frac{2}{3}} = 4.2 \times 10^{-11} \left(\frac{\rho}{\mu_{\rm e}}\right)^{\frac{2}{3}} \text{(non-relativistic)} \tag{2.93}$$

and

$$\epsilon_{\rm F} = p_{\rm F} c = 9.8 \times 10^{-17} n_{\rm e}^{\frac{1}{3}} = 8.3 \times 10^{-9} \left(\frac{\rho}{\mu_{\rm e}}\right)^{\frac{1}{3}} \quad \text{(ultra-relativistic)} \tag{2.94}$$

2.4.4 The Bose-Einstein distribution and the Planck function

Bosons are particles with integer spin numbers. They obey the Bose-Einstein statistics. In this section we will show that the Planck funtcion is actually the Bose-Einstein distribution of photons. Photons are bosons with zero chemical potential, i.e., $\mu = 0$. The number density of photons with momentum in the range of p to $p + {\rm d}p$ is

$$n_p {\rm d}p = \frac{2}{h^3} \frac{4\pi p^2 {\rm d}p}{\exp(\epsilon/kT) - 1} \ . \tag{2.95}$$

Since $\epsilon = h\nu = pc$ and $n_\nu {\rm d}\nu = n_p {\rm d}p$, we have

$$n_\nu {\rm d}\nu = \frac{8\pi}{c^3} \frac{\nu^2 {\rm d}\nu}{\exp(h\nu/kT) - 1} \ . \tag{2.96}$$

With $n_\nu {\rm d}\nu \cdot h\nu \cdot c = I_\nu {\rm d}\nu \cdot 4\pi$, the specific intensity of such a photon field is

$$I_\nu = \frac{2\nu^2}{c^2} \frac{h\nu}{\exp(h\nu/kT) - 1} \ , \tag{2.97}$$

which is the Planck function B_ν mentioned in Eq.(2.27).

Exercises

1. What are the mean molecular weight and electron molecular weight of the following gases: (1) a completely ionized hydrogen gas (2) a gas composed of completely ionized hydrogen (mass fraction 0.75) and neutral helium (mass fraction 0.25) (3) a gas composed of hydrogen (mass fraction 0.75) and helium (mass fraction 0.25), both completely ionized.

2. Verify Eqs.(2.49) and (2.50).

3. Find the velocity mean square fluctuation, $\langle (v - \langle v \rangle)^2 \rangle$, of the Maxwell-Boltzmann distribution.

4. Show that the electron pressure P_e and the electron energy density u_e of a completely degenerate electron gas are, for the non-relativistic case, $P_\mathrm{e} = \frac{2}{5}\, n_\mathrm{e}\epsilon_\mathrm{F}$ and $u_\mathrm{e} = \frac{3}{5}\, n_\mathrm{e}\epsilon_\mathrm{F}$, and for the ultra-relativistic case, $P_\mathrm{e} = \frac{1}{4}\, n_\mathrm{e}\epsilon_\mathrm{F}$ and $u_\mathrm{e} = \frac{3}{4}\, n_\mathrm{e}\epsilon_\mathrm{F}$.

5. From Eqs.(2.81) and (2.82) show that when $\alpha \to \infty$, one has $F_{\frac{3}{2}}/F_{\frac{1}{2}} \to \frac{3}{2}$ and $P_\mathrm{e} = n_\mathrm{e} kT$, the pressure of a Maxwellian electron gas, and when $\alpha \to -\infty$, one has $F_{\frac{3}{2}}/F_{\frac{1}{2}} \to -\frac{3}{5}\alpha$ and one can, from Eq.(2.83), reach Eq.(2.88), the pressure of a completely degenerate non-relativistic electron gas.

6. With the equation of state of a completely degenerate electron gas expressed as

   $$P_\mathrm{e} = K_\mathrm{nr}(\rho/\mu_\mathrm{e})^{\frac{5}{3}} \tag{2.98}$$

   and

   $$P_\mathrm{e} = K_\mathrm{ur}(\rho/\mu_\mathrm{e})^{\frac{4}{3}} \tag{2.99}$$

   for the non-relativistic and ultra-relativistic case respectively, find the numerical values, in gaussian units, of K_nr and K_ur.

7. $F_{\frac{1}{2}}(\alpha = 0) = 0.678094$ (a table of Fermi-Dirac functions may be found in Clayton (1983, page 95); you may also get this value with a simple numerical integration of Eq.(2.81)). Find the electron number density for temperature $T = 10^6$ K and $\alpha = 0$. For a completely ionized hydrogen gas, what is the corresponding mass density? What is the ratio of thermal energy kT to the electron Fermi energy of such a system with a fixed volume?

Chapter 3

Static Stellar Structure

3.1 Equations of stellar structure and evolution

In this section we first summarize the basic equations governing the construction of models for stellar structure and evolution. With only the first two, that is, equations for pressure and for mass, it suffices to reveal many general properties of stellar structure. That is what we shall do in the other sections of this chapter. Note that with the approximation of spherical symmetry, stellar structure is a one-dimensional problem.

- pressure – hydrostatic equilibrium

  $$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho\frac{GM}{r^2} \tag{3.1}$$

  M is the total mass within the radius r. This equation describes the balance of the gravitational force and that due to pressure gradient when the hydrostatic equilibrium assumption is made. In fact, the equation of motion for a mass element is

  $$\rho\frac{\mathrm{d}^2 r_\rho}{\mathrm{d}t^2} = -\rho\frac{GM}{r^2} - \frac{\mathrm{d}P}{\mathrm{d}r} \ , \tag{3.2}$$

  if only forces of gravity and that due to pressure gradient are considered. r_ρ in the term at the left hand side of this equation is the distance from the stellar center to some mass element. This term is usually dropped when the hydrostatic equilibrium is assumed. It is valid if the evolution time scale (referred to the change of the configuration) is much longer than the free-fall time scale or, equivalently, the speed of mass-element motion is much lower than the local sound speed. The former can be seen from that if we have

$$T_{\mathrm{evo}}^2 \gg \frac{R^3}{GM} = t_{\mathrm{ff}}^2 \ ,$$

where T_{evo} is the evolution time scale, then

$$|\rho\ddot{r}| \sim \rho\frac{R}{T_{\mathrm{evo}}^2} \ll \rho\frac{GM}{R^2} \ .$$

The latter can be seen from

$$v_{\mathrm{s}}^2 = (\frac{\partial P}{\partial\rho})_S \sim \frac{P}{\rho} \ ,$$

$$|\rho\ddot{r}| \sim \frac{1}{2}\rho|\frac{\mathrm{d}v^2}{\mathrm{d}r}| \sim \rho\frac{v^2}{R} \ ,$$

and

$$\frac{\mathrm{d}P}{\mathrm{d}r} \sim \frac{P}{R} \ .$$

So, if we have $v^2 \ll v_{\mathrm{s}}^2 \sim \frac{P}{\rho}$, it follows that $|\rho\ddot{r}| \ll |\frac{\mathrm{d}P}{\mathrm{d}r}|$.

- mass – mass conservation

$$\frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2\rho \tag{3.3}$$

- luminosity – energy conservation

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2\rho\,(\varepsilon - T\frac{\mathrm{d}s_m}{\mathrm{d}t}) \tag{3.4}$$

ε is the energy produced (by nuclear fusions) per unit time per unit mass and s_m is the entropy per unit mass. The last term, which describes the evolution, may not be negligible in many circumstances. The presence of this term implies that the structure of a star does not only depend on the boundary conditions but also on its history of evolution.

- temperature – energy transportation

  $$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L}{4\pi r^2} \tag{3.5}$$

  for radiative equilibrium **or**

  $$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{\Gamma_2 - 1}{\Gamma_2}\frac{T}{P}\frac{\mathrm{d}P}{\mathrm{d}r} \tag{3.6}$$

  for convective equilibrium.
  κ is the opacity that we will discuss in more details in Chapter 4 and Chapter 5. Γ_2 is one of the adiabatic exponents discussed in Section 2.3.2.

There are four equations with four dependent variables, $P(r)$, $M(r)$, $L(r)$, and $T(r)$. The density $\rho(r)$ is determined with the equation of state $P = P(\rho, T, X_i)$, where X_i stands for compositions. All other quantities, $\kappa(r)$, $\varepsilon(r)$, $s_m(r)$, and $\Gamma_2(r)$, are known functions of ρ, T, and X_i.

With a given stellar radius R, boundary conditions can be chosen as

$$L(r=0) = 0, M(r=0) = 0,\ P(r=R) = 0, T(r=R) = 0\ . \tag{3.7}$$

The last two conditions at $r = R$ are called the zero boundary conditions. They are of course not rigorously valid in that they do not represent the observed surface properties. A more physical value may be found in, for example, Chiu (1968, page 26 and §8.6). Since the pressure and temperature at the surface are extremely low compared with those in the stellar interior, the solution of these differential equations is actually not sensitive to these small values at the outer boundary. A more detailed discussion about these boundary conditions can be found in Clayton (1983, §6.1).

For a star with given compositions and radius R, the solution of stellar structure can be found by solving the equations discussed above with appropriate boundary conditions. Since the boundary conditions are not at the same point – this is a two-point boundary-value problem – the uniquess of the solution is not guaranteed. It has been shown, nonetheless, the solution is 'locally unique', that is, solutions are not close to one another infinitesimally (Kahler 1972).

It may be more convenient to use the co-called **Lagrangian coordinate**, M, that is, the total encircled mass from the stellar center to the point in question, as the independent variable to re-formulate all these equations. In such a case, r is treated as a function of M, and the boundary conditions with a given stellar mass $\mathcal{M}$, are

$$L(M=0)=0, r(M=0)=0,\ P(M=\mathcal{M})=0, T(M=\mathcal{M})=0\ . \tag{3.8}$$

It is also more often desired to compute a stellar model with a given total mass instead of radius.

3.2 The equation of hydrostatic equilibrium

From the first two of the four equations, that is, the equations for pressure and for mass, one can infer some general properties. Further with an assumed relation between the pressure and density, the stellar structure ($P(r)$ and $\rho(r)$) can be determined.

As a first application, from Eq.(3.1) and (3.3) we have:

$$\frac{\mathrm{d}}{\mathrm{d}r}(P+\frac{GM^2}{8\pi r^4}) = -\frac{GM^2}{2\pi r^5} < 0\ . \tag{3.9}$$

It gives a lower limit to the pressure P_c at the stellar center:

$$P_\mathrm{c} > \frac{G\mathcal{M}^2}{8\pi R^4}\ , \tag{3.10}$$

since $\frac{GM^2}{8\pi r^4} \to 0$ when $r \to 0$ and $P \to 0$ when $r \to R$. This is actually a quite loose limit; check the case of the Sun (see Section 1.3.4), where the dimensional estimate ($P \sim \frac{G\mathcal{M}^2}{R^4}$) gives the pressure at about 1/3 radius from the center. Nonetheless, it gives a quick constraint. Actually one may prove this constraint can be more stringent by a factor of 3, that is,

$$P_{\rm c} > \frac{3}{8\pi}\frac{G\mathcal{M}^2}{R^4} \ , \tag{3.11}$$

since

$$\frac{\mathrm{d}}{\mathrm{d}r}(P + \frac{3}{8\pi}\frac{GM^2}{r^4}) = -2(\langle\rho\rangle - \rho)\frac{GM}{r^2} < 0 \ , \tag{3.12}$$

where $\langle\rho\rangle$ is the average density of the region from the stellar center to the point considered, where the density is ρ.

3.2.1 The virial theorem

From the equation of hydrostatic equlibrium, we have

$$4\pi r^3 \frac{\mathrm{d}P}{\mathrm{d}M} = -\frac{GM}{r} \ .$$

The left-hand side can be turned into

$$\frac{\mathrm{d}}{\mathrm{d}M}(4\pi r^3 P) - 4\pi r^2 \cdot 3P\frac{\mathrm{d}r}{\mathrm{d}M} = -\frac{GM}{r} \ ,$$

and then after integrating over M from 0 to the total stellar mass $\mathcal{M}$, we have

$$4\pi r^3 P|_0^{\mathcal{M}} - \int_0^{\mathcal{M}} \frac{3P}{\rho}\mathrm{d}M = -\int_0^{\mathcal{M}} \frac{GM}{r}\mathrm{d}M \ .$$

It follows that

$$\oint P\mathrm{d}V = -\frac{1}{3}\Omega \ , \tag{3.13}$$

where $\mathrm{d}V = \frac{\mathrm{d}M}{\rho}$ and Ω is the gravitational potential energy of the star.

For the non-relativistic case,

$$\begin{aligned} P &= \frac{1}{3}\int v \cdot mv \cdot f\mathrm{d}v \\ &= \frac{2}{3}K_V \ , \end{aligned} \tag{3.14}$$

where $\int f \mathrm{d}v = n$, n is the number density, and K_V is the kinetic energy per unit volume. It follows that

$$K = -\frac{1}{2}\Omega \tag{3.15}$$

and

$$E = K + \Omega = -K \ . \tag{3.16}$$

Eq.(3.15) is usually referred as **the virial theorem**. We note that it is obtained under the assumption of a *static, non-relativistic* system with gravitational interaction only.

For the ultra-relativistic case,

$$\begin{aligned} P &= \frac{1}{3}\int c \cdot p \cdot f \mathrm{d}v \\ &= \frac{1}{3}K_V \ . \end{aligned} \tag{3.17}$$

So,

$$K = -\Omega \tag{3.18}$$

and

$$E = K + \Omega = 0 \ . \tag{3.19}$$

This is of course only an idealized limiting case, for particles will never reach the speed of light and $E = 0$ in fact describes an unbounded system.

From the virial theorem, we have

$$\triangle K = -\frac{1}{2}\triangle\Omega \ ,$$

$$\triangle E = \frac{1}{2}\triangle\Omega \ ,$$

and

$$\triangle E \triangle K < 0 \ .$$

The above shows that between two hydrostatic-equilibrium states of a star, change in the gravitational energy is compensated by the change in the kinetic energy with only one half of the amount. The total mechanical energy is not conserved. There must occur at the same time other processes to account for the energy transformation.

One application of the virial theorem concerns the hydrostatic stability and the value of the adiabatic exponent Γ_1. Consider a small mass element of the star, for an adiabatic infinitesimal change of this mass element,

$$\mathrm{d}U = -P\mathrm{d}V \ ,$$

where U is the kinetic energy plus other possible degrees of freedom. From

$$\frac{\mathrm{d}P}{P} + \Gamma_1 \frac{\mathrm{d}V}{V} = 0$$

and

$$PV = \frac{2}{3}K$$

(so this is for the non-relativistic case; see Eq.(3.14) and consider an infinitesimally small mass element), we have

$$\mathrm{d}U = \frac{2}{3}\frac{\mathrm{d}K}{\Gamma_1 - 1} \ .$$

For a monatomic ideal gas, $\Gamma_1 = 5/3$, we have as expected $\mathrm{d}U = \mathrm{d}K$, that is, $U = K$ and there are no other degrees of freedom besides the kinetic energy in the internal energy. For the whole star we may consider

$$U = \frac{2}{3}\frac{K}{\Gamma_1 - 1} \ ,$$

if the adiabatic exponent Γ_1 is constant everywhere in the star, or it is understood in an average sense. With $K = -\frac{1}{2}\Omega$, the total energy $E = U + \Omega$ is then

$$E = \frac{\Gamma_1 - \frac{4}{3}}{\Gamma_1 - 1}\Omega \ . \tag{3.20}$$

If the total energy E is positive, the star can fly apart. It occurs, as shown in Eq.(3.20), when Γ_1 decreases to be smaller than $\frac{4}{3}$ (but still larger than 1). A pure photon gas has $\Gamma_1 = \frac{4}{3}$ (Section 2.3.2) and does not form a stable star. It also indicates that a radiation pressure dominated star has a more fragile hydrostatic structure. Ionization may make Γ_1 smaller, sometimes even smaller than $\frac{4}{3}$ (think of an ideal gas with a large C_V). Regions of ionization in the stellar interior could therefore be subject to hydrodynamical instability.

3.2.2 Scale heights in atmospheres

Again, from the equation of hydrostatic equilibrium and considering the situation at the stellar surface, we have

$$\frac{\mathrm{d}P}{\mathrm{d}z} = -\rho g \ , \ g = \frac{G\mathcal{M}}{R^2} \ .$$

Now for simplicity we assume a constant temperature T and $P = \frac{\rho k T}{\mu m_{\mathrm{u}}}$. The spatial variation of temperature in the atmosphere of a star is usually much smaller than that of density and pressure. It follows that

$$P = P_0 \exp(-\frac{\mu m_{\mathrm{u}} g z}{kT}) \tag{3.21}$$

and

$$\rho = \rho_0 \exp(-\frac{\mu m_{\mathrm{u}} g z}{kT}) \ . \tag{3.22}$$

The concept of the **scale height** can be realized in this case as

$$H \equiv \frac{kT}{\mu m_{\mathrm{u}} g} \ , \tag{3.23}$$

which is actually, in a more general form,

$$H \equiv -\frac{P}{\frac{\mathrm{d}P}{\mathrm{d}z}} \ . \tag{3.24}$$

The scale height is a characteristic scale in height over which the density or pressure varies significantly. It is generally also a function of height.

3.2.3 The Von Zeipel's theorem

If a star is not spherically symmetric, which is actually the reality, the **Von Zeipel's theorem** goes like the following. Because

$$\nabla P = -\rho \nabla \phi \ ,$$

the two vector fields, ∇P and $\nabla \phi$, are parallel. The pressure is therefore a function of the gravitational potential ϕ only, that is, $P = P(\phi)$. (Equi-potential surfaces coincide with equi-pressure surfaces.) From

$$\nabla P = \frac{\mathrm{d}P}{\mathrm{d}\phi} \nabla \phi \ ,$$

we have

$$\rho = -\frac{\mathrm{d}P}{\mathrm{d}\phi} = \rho(\phi) \ ,$$

that is, the density is also a function of ϕ only. It is obvious that the same is true for the temperature. So, all properties are constant in an equi-potential surface.

3.3 The linear stellar model

From the equations of hydrostatic equilibrium and mass conservation we have got some useful properties. To go further, but still not to get into the whole business involving the equations of energy conservation and transportation, some assumed knowledge about the density profile is needed. As an example, we consider a linear model, in which the density profile is

$$\rho = \rho_\mathrm{c}(1 - r/R) \ , \tag{3.25}$$

where ρ_c is the density at the stellar center and R is the stellar radius. The mass profile can be obtained from

$$\frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2 \rho_\mathrm{c}(1 - r/R)$$

to be

$$M(r) = (\frac{4r^3}{3} - \frac{r^4}{R})\pi\rho_\mathrm{c} \ . \tag{3.26}$$

The total mass of the star is then

$$\mathcal{M} = M(R) = \frac{\pi R^3}{3}\rho_\mathrm{c} \ . \tag{3.27}$$

Instead of characterizing the profiles with the parameter ρ_c, one can also use the total mass of the star. So, the mass profile is

$$M(r) = (\frac{4r^3}{R^3} - \frac{3r^4}{R^4})\mathcal{M} \ . \tag{3.28}$$

The pressure profile can be obtained likewise from

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM}{r^2}\rho_\mathrm{c}(1 - r/R)$$

to be

$$P(r) = P_{\rm c} - G(\frac{2r^2}{3} - \frac{7r^3}{9R} + \frac{r^4}{4R^2})\pi\rho_{\rm c}^2 \ . \tag{3.29}$$

The central pressure $P_{\rm c}$ can be determined by requiring

$$P(R) = P_{\rm c} - \frac{5\pi G}{36}\rho_{\rm c}^2 R^2 = 0 \ ,$$

which gives

$$P_{\rm c} = \frac{5\pi G}{36}\rho_{\rm c}^2 R^2 = \frac{5G}{4\pi}\frac{\mathcal{M}^2}{R^4} \ .$$

The pressure profile can thus also be described with either $\rho_{\rm c}$ or $\mathcal{M}$.

3.4 Polytropic stellar models

A star is called a polytrope of index n when the relation between the pressure and density is assumed to be

$$P = K\rho^{\frac{n+1}{n}} \ , \tag{3.30}$$

where K and n are constants. This relationship looks artificial, but in fact it may be a quite good approximation in many cases. For example, for $n = 3/2$, i.e., $P = K\rho^{5/3}$, it may well represent a non-relativistic degenerate star. For $n = 3$, $P = K\rho^{4/3}$, it describes an ultra-relativistic degenerate star, or a star with a constant radiation pressure fraction (see the Eddington standard model discussed at the end of this section). For $n = \infty$, $P = K\rho$, it is an imaginary isothermal star consisting of an ideal Maxwellian gas.

3.4.1 The Lane-Emden equation

From the equations of hydrostatic equlibrium and mass conservation we may get

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}) = -4\pi G\rho \ . \tag{3.31}$$

By defining two dimensionless variables, θ and ξ, in place of density ρ and radius r,

$$\rho = \rho_{\mathrm{c}}\theta^n \tag{3.32}$$

and

$$r = \alpha\xi \ , \tag{3.33}$$

where

$$\alpha = \sqrt{\frac{(n+1)K{\rho_{\mathrm{c}}}^{\frac{1-n}{n}}}{4\pi G}} \ , \tag{3.34}$$

Eq.(3.31) can be re-formulated, with Eq.(3.30), as

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}) = -\theta^n \ , \tag{3.35}$$

which is the **Lane-Emden equation** of index n. The Lane-Emden equation is a second-order differential equation of θ, which gives the density profile. With a given n, θ as a function of ξ is uniquely determined with two boundary conditions. Since $\rho(0) = \rho_{\mathrm{c}}$, we have $\theta(0) = 1$. We can also find that $\frac{\mathrm{d}\theta}{\mathrm{d}\xi}|_0 = 0$. With these two boundary conditions and a given polytropic index n, the solution $\theta(\xi)$ is determined and, when translated into the real density ρ and radius r, is parameterized only by ρ_{c} and K. In other words, for a given polytrope of index n and constant K, stellar models are parameterized by ρ_{c} only.

Analytic solutions to the Lane-Emden equation exist only for $n = 0$, 1, and 5:

$$n = 0, \quad \theta = 1 - \xi^2/6 \quad (\rho \text{ is a constant}) \tag{3.36}$$

$$n = 1, \quad \theta = \frac{\sin\xi}{\xi} \tag{3.37}$$

$$n = 5, \quad \theta = (1 + \xi^2/3)^{-1/2} \tag{3.38}$$

The outer boundary of the star corresponds to the ξ at which θ takes the first zero value. Denoting this ξ as ξ_1, i.e., $\theta(\xi_1) = 0$, we can see that for $n = 0$, $\xi_1 = \sqrt{6}$, for $n = 1$, $\xi_1 = \pi$, and for $n = 5$, $\xi_1 \to \infty$.

The Lane-Emden equation can also take other forms. If we define

$$\chi = \xi\theta(\xi) \tag{3.39}$$

then we have

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}\xi^2} = -\frac{\chi^n}{\xi^{n-1}} \ . \tag{3.40}$$

Or, if

$$x = \frac{1}{\xi} \ , \tag{3.41}$$

then

$$x^4\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = -\theta^n \ . \tag{3.42}$$

3.4.2 Properties of a polytropic star

Since the solution to the Lane-Emden equation can be uniquely obtained, either analytically or numerically, for a given polytropic index n, properties like the total mass $\mathcal{M}$ and the stellar radius R can also be determined with given K and ρ_c. Values of ξ_1 and $\frac{\mathrm{d}\theta}{\mathrm{d}\xi}|_{\xi_1}$ are also needed, which can be known from the solution for different n (Table 3.1).

- The stellar radius R is

$$R = \alpha\xi_1 = \sqrt{\frac{(n+1)K{\rho_\mathrm{c}}^{\frac{1-n}{n}}}{4\pi G}}\xi_1 \ . \tag{3.43}$$

- The mass of the star $\mathcal{M}$ is

$$\begin{aligned}
\mathcal{M} &= \int_0^R 4\pi r^2\rho\mathrm{d}r \\
&= 4\pi\alpha^3\rho_\mathrm{c}\int_0^{\xi_1}\xi^2\theta^n\mathrm{d}\xi \\
&= -4\pi\alpha^3\rho_\mathrm{c}\int_0^{\xi_1}\frac{\mathrm{d}}{\mathrm{d}\xi}(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi})\mathrm{d}\xi \\
&= 4\pi\sqrt{\left(\frac{(n+1)K}{4\pi G}\right)^3{\rho_\mathrm{c}}^{\frac{3-n}{n}}}\left(-\xi_1^2\left.\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right|_{\xi_1}\right) \ .
\end{aligned} \tag{3.44}$$

n	ξ_1	$-\xi_1^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi}\vert_{\xi_1}$	$\frac{\rho_\mathrm{c}}{\bar{\rho}}$
0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.25	8.01894	1.94980	88.153
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6189.47
4.9	169.47	1.7355	934800
5.0	∞	1.73205	∞

Table 3.1: Constants of the Lane-Emden equation (Chandrasekhar 1939, page 96).

- The average density $\bar{\rho}$ is

$$\bar{\rho} = \frac{4\pi\alpha^3\rho_\mathrm{c}\left(-\xi_1^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}|_{\xi_1}\right)}{\frac{4\pi}{3}\alpha^3\xi_1^3} = \rho_\mathrm{c}\left(-\frac{3}{\xi_1}\left.\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right|_{\xi_1}\right) . \tag{3.45}$$

- The central pressure P_c can be trivially expressed as $P_\mathrm{c} = K\rho_\mathrm{c}^{(n+1)/n}$. It can also be formulated in terms of the total mass $\mathcal{M}$ and radius R as

$$P_\mathrm{c} = \frac{1}{4\pi(n+1)(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}|_{\xi_1})^2}\frac{G\mathcal{M}^2}{R^4} . \tag{3.46}$$

- Now let's find the gravitational potential energy Ω of a polytropic star in terms of the polytropic index, total mass and the stellar radius. From

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = (n+1)\frac{\mathrm{d}(P/\rho)}{\mathrm{d}r} = -\frac{GM}{r^2}$$

and

$$\begin{aligned}
(n+1)\frac{P}{\rho} &= -\int_{\infty}^{r}\frac{GM(r')}{r'^2}\mathrm{d}r' + C \\
&= -\Phi(r) + \int_{\infty}^{R}\frac{GM(r')}{r'^2}\mathrm{d}r' \\
&= -\Phi(r) - \frac{G\mathcal{M}}{R}\ ,
\end{aligned}$$

where Φ is the gravitational potential as a function of r and the constant C is so chosen by considering $P/\rho = K\rho^{1/n} = 0$ at $r = R$, we have

$$\begin{aligned}
\Omega &= \frac{1}{2}\int_0^{\mathcal{M}}\Phi\mathrm{d}M \\
&= -\frac{1}{2}\left((n+1)\int_0^{\mathcal{M}}\frac{P}{\rho}\mathrm{d}M + \frac{G\mathcal{M}^2}{R}\right) \\
&= -\frac{1}{2}\left((n+1)\oint P\mathrm{d}V + \frac{G\mathcal{M}^2}{R}\right) \\
&= -\frac{1}{2}\left((n+1)(-\frac{\Omega}{3}) + \frac{G\mathcal{M}^2}{R}\right)\ .
\end{aligned}$$

Eq.(3.13) was used to reach the last equality. The factor $\frac{1}{2}$ at the beginning of the right hand side is needed since ϕ is the potential established by the whole configuration. So, in terms of $\mathcal{M}$, R, and n, the potential energy is

$$\Omega = -\frac{3}{5-n}\frac{G\mathcal{M}^2}{R}\ . \tag{3.47}$$

One can see that, for a polytropic star to have a negative gravitational potential energy, we must have $n < 5$.

From the above discussion, we can also see that $\frac{\mathrm{d}R}{\mathrm{d}\rho_\mathrm{c}} = 0$ for $n = 1$ and $\frac{\mathrm{d}\mathcal{M}}{\mathrm{d}\rho_\mathrm{c}} = 0$ for $n = 3$. The latter is particularly interesting because for that kind of model ($n = 3$) the stellar mass depends only on K. We have also seen the model of $n = 5$ corresponds to an infinite star, $\xi_1 \to \infty$. This is consistent with Eq.(3.47), which requires $n < 5$ for a polytropic star.

3.4.3 The Eddington standard model

Now consider a star whose pressure comes from radiation and an ideal gas, just like the one discussed in Section 2.3.1. From

$$P_{\mathrm{r}} = \frac{1}{3}aT^4 = (1-\beta)P$$

and

$$P_{\mathrm{g}} = \frac{\rho k T}{\mu m_{\mathrm{u}}} = \beta P \ ,$$

we can have

$$P = \left(\frac{k}{\mu m_{\mathrm{u}}}\right)^{4/3} \left(\frac{3(1-\beta)}{a\beta^4}\right)^{1/3} \rho^{4/3} \ , \tag{3.48}$$

which is just a polytropic relationship of $n = 3$, if the approximation of β and μ being constant is adopted. This approximation should not be pushed to the limiting cases of $\beta = 1$ or 0. Although this approximation cannot be really good, this model, called **the Eddington standard model**, provides a useful tool to estimate properties of many stars.

3.4.4 The Chandrasekhar limit

For an *ultra-relativistic* degenerate star we also have $P = K\rho^{4/3}$. From Eq.(3.44), Eq.(2.91) and solutions to the Lane-Emden equation (see Table 3.1) the total mass of such a star is

$$\mathcal{M} = \frac{5.80}{\mu_{\mathrm{e}}^2} M_{\odot} \ . \tag{3.49}$$

A white dwarf is a star supported by the electron degeneracy pressure against gravity. With mass getting higher, the star gets more and more relativistic. Such a configuration can only exist with mass less than that making the whole star ultra-relativistic. So the mass in Eq.(3.49) is the *upper limit* of the mass of white dwarfs, which is called **the Chandrasekhar limit**. The major composition of a white dwarf is carbon and oxygen. The corresponding electron mean molecular weight is $\mu_{\mathrm{e}} = 2$, and the Chandrasekhar limit is about $1.45M_{\odot}$.

Exercises

1. Verify Eq.(3.12).

2. Estimate typical scale heights for the Earth's atmosphere ($T \sim 300$ K), the solar atmosphere ($T \sim 6000$ K), the atmosphere of a white dwarf ($T \sim 20000$ K), and of a neutron star ($T \sim 10^6$ K). The mass of the latter two can be taken as one solar mass.

3. Find the gravitational potential energy for the linear stellar model in terms of G, $\mathcal{M}$ and R.

4. Show $\frac{\mathrm{d}\theta}{\mathrm{d}\xi}|_0 = 0$.

5. Derive Eq.(3.46).

6. Show that the mass of the Eddington standard model is given numerically by

$$\mathcal{M} = 18.3 \frac{\sqrt{1-\beta}}{\mu^2\beta^2} M_\odot \ . \tag{3.50}$$

7. Estimate P_c, ρ_c and T_c for a star of mass $1M_\odot$ and radiu $1R_\odot$, using the Eddington standard model and the linear model. For the former, what are the values of β and μ?

8. Verify Eq.(3.49).

Chapter 4

Energy Transport in Stellar Interiors

Stars shine so that we can detect their existence. Their shining also implies evolution through energy output. The conversion of energy among different forms and their transportation in stars are thus important to the understanding of stellar structure and evolution. In the previous chapter, the pressure profile and mass profile in a star are studied with an assumed polytropic relation, which is surely an idealization. In this chapter we will discuss the description of energy transport in stars, which is closely related to the temperature profile in stellar interiors, which through the equation of state and the coupled equations will give a more realistic pressure and density profile. Energy transport via radiation, convection and conduction are discussed in the following sections.

4.1 Radiation fields

In the following we shall introduce the notions that one uses to describe a radiation field before formulating its transportation. The axial symmetry is often a very good approximation when dealing with radiation transport in the stellar interiors and atmospheres. In such a case, the various quantities described below can be expressed as moments of the specific intensity.

- **the specific intensity** –

The specific intensity I_ν is defined as

$$I_\nu(\vec{r}, \hat{\Omega}, t)\, \mathrm{d}\nu \mathrm{d}\Omega \mathrm{d}t\, \hat{\Omega} \cdot \hat{n} \mathrm{d}A = \mathrm{d}E \ , \tag{4.1}$$

where $\mathrm{d}E$ is the energy passing through the area $\mathrm{d}A$, whose normal direction is $\hat{n}$, in the direction $\hat{\Omega}$ within the frequency range $\mathrm{d}\nu$, the solid angle $\mathrm{d}\Omega$, and the time interval $\mathrm{d}t$.

- **the mean intensity** –

The mean intensity (or, the mean specific intensity, to be more precise) is defined as

$$J_\nu \equiv \frac{1}{4\pi} \int I_\nu(\hat{\Omega}) \mathrm{d}\Omega \ . \tag{4.2}$$

If I_ν is isotropic, we have $J_\nu = I_\nu$.

The energy density can be obviously related to the mean intensity:

$$u_\nu = \frac{4\pi}{c} J_\nu \ . \tag{4.3}$$

When the polar axis is chosen, the integration over the solid angle can be written as

$$J_\nu = \frac{1}{4\pi} \int I_\nu \mathrm{d}\mu \mathrm{d}\phi \ ,$$

where μ is the cosine of the polar angle and ϕ is the azimuthal angle. If I_ν is axially symmetric, we have

$$J_\nu = \frac{1}{2} \int_{-1}^{1} I_\nu \mathrm{d}\mu \ , \tag{4.4}$$

which is the zeroth-order moment of I_ν.

- **the flux** –

The **flux density** $\mathcal{F}_\nu$ (or called the **specific flux**) is

$$\mathcal{F}_\nu(\hat{n}) = \int I_\nu(\hat{\Omega})\hat{\Omega}\cdot\hat{n}\mathrm{d}\Omega \ , \tag{4.5}$$

which is the energy per unit time per unit area per unit frequency *passing through an area* whose normal direction is $\hat{n}$. If I_ν is axially symmetric with respect to $\hat{n}$, we have

$$\mathcal{F}_\nu = 2\pi \int_{-1}^{1} I_\nu \mu \mathrm{d}\mu \tag{4.6}$$

The **Eddington flux** H_ν is defined as

$$H_\nu = \frac{1}{2}\int_{-1}^{1} I_\nu \mu \mathrm{d}\mu \ , \tag{4.7}$$

which is the first-order moment of I_ν. It is clear that

$$H_\nu = \frac{\mathcal{F}_\nu}{4\pi} \ . \tag{4.8}$$

Sometimes another quantity F_ν, the **astrophysical flux**, is used, which is simply $F_\nu = \frac{\mathcal{F}_\nu}{\pi} = 4H_\nu$.
One may also define the flux as a vector in the following way:

$$\vec{\mathcal{F}}_\nu = \int I_\nu(\hat{\Omega})\hat{\Omega}\mathrm{d}\Omega \ . \tag{4.9}$$

It is clear that $\mathcal{F}_\nu(\hat{n})$ define earlier is simply the magnitude of the projection of $\vec{\mathcal{F}}$ in the $\hat{n}$ direction.

- **The K-integral** –

The projected *magnitude sum* of momentum transport per unit time per unit area in the radiation field of frequency range $\mathrm{d}\nu$ through an area with normal direction $\hat{n}$ is

$$(\int_{\mu=-1}^{\mu=1} \frac{I_\nu}{c} \mu^2 \mathrm{d}\mu \mathrm{d}\phi) \, \mathrm{d}\nu \ . \tag{4.10}$$

If the radiation field is isotropic, it is the same as the radiation pressure discussed in Section 2.3.1, since

$$P_\nu \mathrm{d}\nu = 2 \times (\int_{\mu=0}^{\mu=1} \frac{I_\nu}{c} \mu^2 \mathrm{d}\mu \mathrm{d}\phi) \, \mathrm{d}\nu = (\int_{\mu=-1}^{\mu=1} \frac{I_\nu}{c} \mu^2 \mathrm{d}\mu \mathrm{d}\phi) \, \mathrm{d}\nu \ . \tag{4.11}$$

If again I_ν is axially symmetric with respect to $\hat{n}$, we have

$$P_\nu = 2\pi \int_{\mu=-1}^{\mu=1} \frac{I_\nu}{c} \mu^2 \mathrm{d}\mu \ ,$$

which is often expressed as $P_\nu = \frac{4\pi}{c} K_\nu$, where

$$K_\nu = \frac{1}{2} \int_{-1}^{1} I_\nu \mu^2 \mathrm{d}\mu \tag{4.12}$$

is the K-integral, the second-order moment of I_ν.

4.2 The equation of radiation transfer

In stellar atmospheres, the spacial range considered is often very much smaller than the stellar radius, and therefore the plane-parallel approximation is adequate. Even in stellar interiors, to associate each location at a certain radial distance from the center with a certain optical depth, similar notations used in the plane-parallel approximation is also adopted. In a plane-parallel medium with $\hat{z}$ in the normal direction, the equation of radiation transfer reads

$$\cos\theta \frac{\mathrm{d}I_\nu}{\mathrm{d}z} = -k_\nu I_\nu + j_\nu \ , \tag{4.13}$$

where θ is the angle between the z-axis and the direction of radiation propagation, k_ν is the attenuation coefficient and j_ν is the specific **emissivity**, both of which in general include contribution from scattering. In a general case without the plane-parallel symmetry, one may simply drop out $\cos\theta$ and consider z as the actual path, or equivalently, consider $\mathrm{d}\ell = \mathrm{d}z/\cos\theta$ as the actual path. The attenuation coefficient k_ν is also often expressed as $k_\nu = \rho\kappa_\nu$, where ρ is the mass density and κ_ν is the **opacity**. The subscript ν used for k_ν and κ_ν is different from our convention in that it emphasizes the frequency dependence but does not denote a quantity per unit frequency. The opacity may include contribution from the so-called bound-bound, bound-free, free-free opacities, and the loss due to scattering. It is strongly frequency-dependent. Some more details will be discussed in the next chapter. Defining the **optical depth** τ_ν as

$$\mathrm{d}\tau_\nu = -k_\nu \mathrm{d}z$$

and the source function S_ν as

$$S_\nu = j_\nu / k_\nu \ ,$$

the radiative transfer equation turns into

$$\mu \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = I_\nu - S_\nu \ . \tag{4.14}$$

The source function S_ν, as being the ratio of emissivity and the attenuation coefficient, is an intrinsic physical property of the medium, that is, the matter through which the radiation propagates. If $S_\nu > I_\nu$, I_ν will increase along the path in the direction indicated by μ. A formal solution can be found by noting that the above equation can be turned into

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\nu}(I_\nu e^{-\tau_\nu/\mu}) = -\frac{S_\nu}{\mu} e^{-\tau_\nu/\mu}$$

and then, integrating from τ_2 to τ_1,

$$I_\nu(\tau_1) = I_\nu(\tau_2) e^{-(\tau_2-\tau_1)/\mu} + \frac{1}{\mu}\int_{\tau_1}^{\tau_2} S_\nu(t) e^{-(t-\tau_1)/\mu} \mathrm{d}t \ . \tag{4.15}$$

Considering $\tau_2 \to \infty$ and τ_1 assigned to be 0, the emergent specific intensity is

$$I_\nu(\tau_1 = 0, \mu) = \frac{1}{\mu}\int_0^\infty S_\nu(t) e^{-t/\mu} \mathrm{d}t = \frac{1}{\mu}\tilde{S}_\nu(1/\mu) \ , \tag{4.16}$$

where $\tilde{S}_\nu$ is the Laplace transform of the source function S_ν.

Radiative transfer equation can also be written in terms of flux and radiation pressure. Integrating Eq.(4.14) over μ, we have

$$\frac{\mathrm{d}H_\nu}{\mathrm{d}\tau_\nu} = J_\nu - \frac{1}{2}\int_{-1}^{1} S_\nu \mathrm{d}\mu \ . \tag{4.17}$$

With Eq.(4.14) $\times\mu$ and then integrated over μ, one has

$$\frac{\mathrm{d}K_\nu}{\mathrm{d}\tau_\nu} = H_\nu - \frac{1}{2}\int_{-1}^{1} S_\nu \mu \mathrm{d}\mu \ . \tag{4.18}$$

If S_ν is isotropic, we have

$$\frac{\mathrm{d}K_\nu}{\mathrm{d}\tau_\nu} = H_\nu \ . \tag{4.19}$$

4.3 Radiative equilibrium

4.3.1 The diffusion approximation

In the deep interior of a star, the radiation field is almost isotropic and approaches thermal equilibrium. Matter there is also close to thermal equilibrium with the radiation field. In such a case, the source function will be very close to the Planck function, i.e., $S_\nu \sim B_\nu$. This can be understood by placing an element of such matter in a cavity containing a radiation field at a certain temperature. Since we assume that matter is in thermal equilibrium at that temperature, from Eq.(4.14), we can see that S_ν must be equal to B_ν at that temperature so that the radiation field, as described by I_ν, can remain in that equilibrium with $I_\nu = B_\nu$ in this gedanken experiment. The condition that $S_\nu = B_\nu$ is called the local thermal equilibrium (LTE) assumption. Expanding the source function at optical depth t_ν as a power series with respect to a certain point (or layer) at optical depth $\tau_{\nu,0}$ where the LTE assumption is made,

$$S_\nu(t_\nu) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\mathrm{d}^n B_\nu}{\mathrm{d}{\tau_\nu}^n}(t_\nu - \tau_{\nu,0})^n \ , \tag{4.20}$$

and from Eq.(4.15), one can find that (e.g. Mihalas 1978, p.50), considering $\tau_1 = \tau_{\nu,0}$ and $\tau_2 = \infty(\mu > 0)$ or $\tau_2 = 0(\mu < 0)$, we have

$$I_\nu(\tau_{\nu,0}, \mu) = \sum_{n=0}^{\infty} \mu^n \frac{\mathrm{d}^n B_\nu}{\mathrm{d}{\tau_\nu}^n} = B_\nu(\tau_{\nu,0}) + \mu \frac{\mathrm{d}B_\nu}{\mathrm{d}\tau_\nu} + \mu^2 \frac{\mathrm{d}^2 B_\nu}{\mathrm{d}{\tau_\nu}^2} + \cdots \ . \tag{4.21}$$

All the above derivatives of the Planck function are evaluated at the depth $\tau_{\nu,0}$. The **diffusion approximation** is that, at a very large optical depth ($\tau_{\nu,0} \gg 1$), one may keep only terms up to the first derivative so that the specific intensity and its moments can be written as

$$\begin{aligned} I_\nu(\tau_\nu, \mu) &\approx B_\nu(\tau_\nu) + \mu \frac{\mathrm{d}B_\nu}{\mathrm{d}\tau_\nu} \ , &(4.22)\\ J_\nu(\tau_\nu) &\approx B_\nu(\tau_\nu) \ , &(4.23)\\ H_\nu(\tau_\nu) &\approx \frac{1}{3}\frac{\mathrm{d}B_\nu}{\mathrm{d}\tau_\nu} \ , &(4.24) \end{aligned}$$

and

$$K_\nu(\tau_\nu) \approx \frac{1}{3} B_\nu(\tau_\nu) \ . \tag{4.25}$$

The Eddington flux can then be expressed in terms of the temperature gradient:

$$H_\nu = \frac{1}{3}\frac{\mathrm{d}B_\nu}{\mathrm{d}\tau_\nu} = -\frac{1}{3}\left(\frac{1}{\rho\kappa_\nu}\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\right)\frac{\mathrm{d}T}{\mathrm{d}r} \ , \tag{4.26}$$

where $\mathrm{d}\tau_\nu = -\rho\kappa_\nu \mathrm{d}r$ is used. Recall that $\mathcal{F}_\nu = 4\pi H_\nu$ and for the total flux we have

$$\mathcal{F} = \int \mathcal{F}_\nu \mathrm{d}\nu = \frac{L}{4\pi r^2} = -\frac{4\pi}{3}\left(\int \frac{1}{\rho\kappa_\nu}\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\mathrm{d}\nu\right)\frac{\mathrm{d}T}{\mathrm{d}r} \ . \tag{4.27}$$

Defining the **Rosseland mean opacity** $\langle\kappa\rangle$ as

$$\frac{1}{\langle\kappa\rangle} = \frac{\int \frac{1}{\kappa_\nu}\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\mathrm{d}\nu}{\int \frac{\mathrm{d}B_\nu}{\mathrm{d}T}\mathrm{d}\nu} \tag{4.28}$$

and noting that

$$\int \frac{\mathrm{d}B_\nu}{\mathrm{d}T}\mathrm{d}\nu = \frac{4\sigma T^3}{\pi} \ ,$$

Eq.(4.27) leads to

$$L = -\frac{16\pi}{3}\frac{acr^2T^3}{\rho\langle\kappa\rangle}\frac{\mathrm{d}T}{\mathrm{d}r} \ , \tag{4.29}$$

which describes the magnitude of *radiative* luminosity at a radius with given temperature gradient and other local physical properties. The Rosseland mean opacity $\langle\kappa\rangle$ is an average of opacities with emphasis on the frequency ranges in which the Planck function's temperature derivative is the largest ($\frac{\mathrm{d}B_\nu}{\mathrm{d}T}$ peaks at about $\nu = 2.8kT/h$) or in which the medium is more transparent (therefore a smaller κ_ν).

When the energy is transported by radiation only (the so-called **radiative equilibrium**), the temperature gradient should obey the following:

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{4ac}\frac{\rho\langle\kappa\rangle}{T^3}\frac{L}{4\pi r^2} \ , \tag{4.30}$$

which is the equation of temperature gradient in radiative equilibrium, here derived with the diffusion approximation, that is, only valid at a very large optical depth. We note that in Eq.(4.28) the concept of reduced opacity is not yet included; see the next chapter.

4.3.2 The mass-luminosity relation

From Eq.(4.29) one can make an order-of-magnitude estimate:

$$\mathcal{L} \sim \frac{4\pi}{3}\frac{ac}{\rho\langle\kappa\rangle}\frac{R^2T^4}{R} \propto \frac{RT^4}{\rho\langle\kappa\rangle} \ . \tag{4.31}$$

Considering that $P \sim \frac{G\mathcal{M}^2}{R^4}$, $P \approx \frac{\rho kT}{\mu m_\mathrm{p}}$ and $\rho \sim \mathcal{M}/R^3$, we have $T \propto \mu\mathcal{M}/R$ for a gas-dominated star. Therefore,

$$\mathcal{L} \propto \frac{\mu^4}{\langle\kappa\rangle}\mathcal{M}^3 \ , \tag{4.32}$$

which gives a rough relation between stellar luminosity and its mass. The mass dependence also exists in μ and $\langle\kappa\rangle$, however. If instead, we consider a radiation-dominated star, $P \approx \frac{1}{3}aT^4 \sim \frac{G\mathcal{M}^2}{R^4}$, we have $T \propto \mathcal{M}^{0.5}/R$. We then have

$$\mathcal{L} \propto \mathcal{M}/\langle\kappa\rangle \ . \tag{4.33}$$

This is consistent with the $\mathcal{M}$-$\mathcal{L}$ relationship inferred from observations in that the dependence weakens towards the high mass end. The Rosseland mean opacity can be quite well approximated by the Kramer opacity, that is, $\langle\kappa\rangle \propto \rho T^{-3.5}$. For gas-dominated stars, we then have $\mathcal{L} \propto \mu^{7.5}\mathcal{M}^{5.5}/R^{0.5}$. The dependence on mass gets stronger in this case. We should also note, however, for main sequence stars the radius and mass are positively correlated. The mass dependence of luminosity is therefore reduced to some extent. For radiation-dominated stars, we will have $\mathcal{L} \propto \mu^{3.5}\mathcal{M}^{3.5}/R^{0.5}$. Again, we see a weaker luminosity dependence on stellar masses for higher-mass stars.

4.4 Non-radiative energy transport

4.4.1 Convective instability

Convection may occur when the heat flux is so large that radiation alone is not efficient enough to transport the flux. To explore the condition for the convective instability to occur, let's consider a mass element undergoing an adiabatic, perturbative rising. We assume that the pressure inside the element and in its surroundings keeps equal to each other all the time because the time scale for reaching pressure balance, to the order of the free-fall time scale, is usually much shorter than that for reaching thermal equilibrium. If the density drop during the rising is larger than the surroundings, which is computed based on the assumption of radiative equilibrium, the mass element will continue rising, i.e. the perturbation is not damped. In such a case, convection occurs, and the assumption of radiative equilibrium is no longer justified. In other words, if

$$\left|\frac{\mathrm{d}\rho}{\mathrm{d}r}\right|_{\mathrm{ad}} > \left|\frac{\mathrm{d}\rho}{\mathrm{d}r}\right|_{\mathrm{rad}}$$

then we have convective instability. The above condition is the same as

$$\left|\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\mathrm{rad}} > \left|\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\mathrm{ad}} , \tag{4.34}$$

since the pressure drop is the same for the element and its surroundings. The quantity with subscript 'rad' is that computed from radiative equilibrium.

In computing a stellar model, one has to check whether convection occurs or not at all the points, that is, to compare the temperature gradient obtained from Eq.(4.30) with $\left|\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\mathrm{ad}}$. It is thus desirable to express $\left|\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\mathrm{ad}}$ in terms of other local thermodynamic functions.

From Eq.(2.42) one can easily see that

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{ad}} = (1-\frac{1}{\Gamma_2})\frac{T}{P}\left(\frac{\mathrm{d}P}{\mathrm{d}r}\right) \ . \tag{4.35}$$

So the condition for convective instability can be stated as

$$\left|\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\mathrm{rad}} > \left|(1-\frac{1}{\Gamma_2})\frac{T}{P}\left(\frac{\mathrm{d}P}{\mathrm{d}r}\right)\right| \ . \tag{4.36}$$

A general form of Γ_2 can be found in Eq.(2.57). If Γ_2 is closer to unity, it is easier for convection to occur.

For the case of an ideal gas, $PV = NkT$ and $1-\frac{1}{\Gamma_2} = (C_P - C_V)/C_P$, one can find

$$\begin{aligned}\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{ad}} &= -\frac{c_P - c_V}{c_P}\frac{\mu m_{\mathrm{u}} g}{k} \\ &= -\frac{g}{c_P} \ ,\end{aligned} \tag{4.37}$$

where c_P is the heat capacity at constant pressure per unit mass and g is the gravitational acceleration. The last equality is obtained by noting that $C_P - C_V = Nk$ and $c_P - c_V = Nk/M = k/(\mu m_{\mathrm{u}})$. In a region where c_P is large (therefore c_V is also large and Γ_2 is closer to unity) it is more likely to have convective instability. This may occur in an ionization region.

The maximal luminosity that radiation can transfer without causing convection is

$$L_{\max}(r) = -\frac{16\pi}{3}\frac{acr^2T^3}{\rho\langle\kappa\rangle}\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{ad}} \ , \tag{4.38}$$

which, with Eqs.(2.31), (3.1) and (4.35), can be turned into

$$\frac{L_{\max}(r)}{M(r)} = \frac{16\pi Gc}{\langle\kappa\rangle}(1-\frac{1}{\Gamma_2})\frac{P_{\mathrm{r}}}{P} \ . \tag{4.39}$$

The luminosity $L_{\max}$ can be relatively small in regions with large opacity and convection is likely to happen. Again, regions of ionization are possible examples. On the other hand, high above the photosphere, the opacity, as well as the pressure, gets smaller and smaller because density drops very quickly (think of Kramer opacity and note that temperature does not change much). The maximal heat flux that can be carried by radiation therefore becomes larger. Depending on the actual structure, convection can occur in different regions in a star, but at the outer atmosphere, radiative equilibrium is usually a good description. Close to the stellar core, if the energy generation per unit mass is larger than the maximum value described in Eq.(4.39), convection will occur. This is the case for massive stars at whose central core the hydrogen burning is mainly through the CNO cycle, which is very much temperature sensitive and therefore is concentrated in a small core region, where the large luminosity of massive stars is already achieved. Therefore the energy generation per unit mass is large and convection tends to occur. In contrast, the p-p chain hydrogen burning, like that going on in the Sun, happens in a larger core region. The luminosity per unit mass is therefore smaller and radiative equilibrium is established there.

4.4.2 The mixing-length theory

In principle, when convection occurs, energy transportation is still carried out by the two mechanisms: convection and radiation. However, it will be demonstrated in the following that the convection is so efficient a mechanism that almost all the heat is transported by convection and transportation by radiation can be neglected, as long as convection occurs. It is, nonetheless, not simple at all to develop a satisfying theory of energy transportation rate for convection. In the following we will describe the basic picture of the so-called **mixing length theory**, and demonstrate at the same time the the actual temperature gradient is very close to the adiabatic one. So, for the concern of computing a stellar model, the adiabatic temperature gradient can be employed for convection zones.

Now, define **the superadiabatic gradient** as

$$\frac{\mathrm{d}\triangle T}{\mathrm{d}r} \equiv \left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{ad}} - \frac{\mathrm{d}T}{\mathrm{d}r}\ , \tag{4.40}$$

which is the difference between the adiabatic temperature gradient and the actual one, and consider a 'mixing length' Λ, over which the mass elements travel adiabatically and then exchange heat with the surroundings. In the following we will try to express convection flux in terms of the superadiabatic gradient, the mixing length and other local physical properties.

The convective heat flux can be written as

$$\mathcal{F}_{\rm conv} = \frac{1}{2}\rho\bar{v} \cdot c_P \triangle T \ , \tag{4.41}$$

where $\bar{v}$ is a typical average speed, $c_P \triangle T = \triangle q$ is the heat excess per unit mass carried by the convective material, and the factor $\frac{1}{2}$ takes into account that only half of the mass is rising upwards. The temperature excess $\triangle T$ can be written in terms of the mixing length and the superadiabatic gradient as

$$\triangle T \approx \Lambda \frac{{\rm d}\triangle T}{{\rm d}r} \ ,$$

and the average speed $\bar{v}$ can be estimated from the upward motion of the convecting mass element as in the following.

Since the pressure, which keeps equal all the time in the mass element and in its surroundings, is proportional to ρT (so we are considering an ideal gas system dominated by gas pressure), we have

$$\triangle\rho \approx -\frac{\rho}{T}\triangle T \ , \tag{4.42}$$

where $\triangle\rho$ and $\triangle T$ are the density and temperature excess in the element over its surroundings, and ρ and T should be understood as taking a certain average value in the traveling path. The *net* force (difference of the buoyant force and the gravitational one) per unit volume can be estimated as

$$f = -g\triangle\rho = g\frac{\rho}{T}\triangle T \ ,$$

which gives an acceleration of

$$\frac{{\rm d}^2 r_\rho}{{\rm d}t^2} = g\frac{\triangle T}{T} \ ,$$

where r_ρ is the position of the mass element in question. The average speed can then be estimated as the product of the acceleration and a typical time scale:

$$\bar{v} \approx g\frac{\triangle T}{T}t$$

with t being the time scale of the adiabatic travel, that is, the time scale for completing heat exchange. This time scale is about

$$t \approx \sqrt{\frac{\Lambda T}{g \triangle T}} \ , \tag{4.43}$$

as can be seen from the acceleration expression. Now we have

$$\bar{v} \approx \sqrt{g\Lambda \frac{\triangle T}{T}}$$

and the convective heat flux is

$$\mathcal{F}_{\mathrm{conv}} = \frac{1}{2}\rho c_P \left(\frac{g}{T}\right)^{\frac{1}{2}} \Lambda^2 \left(\frac{\mathrm{d}\triangle T}{\mathrm{d}r}\right)^{\frac{3}{2}} \ . \tag{4.44}$$

Correspondingly, the luminosity transported by convection is

$$L_{\mathrm{conv}} = \frac{4\pi r^2}{2}\rho c_P \left(\frac{g}{T}\right)^{\frac{1}{2}} \Lambda^2 \left(\frac{\mathrm{d}\triangle T}{\mathrm{d}r}\right)^{\frac{3}{2}} \ . \tag{4.45}$$

The determination of an adequate mixing length is not trivial. The adiabatic assumption is probably not good for a mixing length larger than the local scale height, beyond which properties of the surroundings are very different from the original environments of the convective cell. In the stellar interior, unlike in the transition region around the atmosphere, the scale height is not much shorter than the whole stellar dimension. To demonstrate the claim made at the beginning of this section, let's take the Sun as an example and denote Λ with $\alpha R_\odot$. From the solar model computed by Bahcall & Ulrich (1988; see also Guzik & Lebreton 1991), at $r = 0.75R_\odot$ other quantities are $M = 0.985M_\odot$, $T = 1.82 \times 10^6$ K, and $\rho = 0.122$ g/cm^3. The local gravitational acceleration is $g = GM/r^2 \approx 4.7 \times 10^4$ cm/sec/sec and the heat capacity is about $c_P \approx \frac{5}{2}\frac{k}{\mu m_{\mathrm{u}}} \approx 10^{8.6}$ erg/K/g, assuming a monatomic ideal gas and $\mu \approx \frac{1}{2}$. Inserting these values into Eq.(4.45) and assuming all the solar flux in transported by convection, $L_{\mathrm{conv}} = 10^{33.6}$ erg/sec, we have

$$\frac{\mathrm{d}\triangle T}{\mathrm{d}r} \approx 10^{-11}\alpha^{-\frac{4}{3}}\mathrm{K/cm} \ . \tag{4.46}$$

With $\alpha = 0.1$, the superadiabatic gradient is about 10^{-10} K/cm, which is 10^{-6} times the typical temperature gradient in the Sun. It implies that in the convection zone of the Sun, the deviation of the actual temperature gradient from the adiabatic one is really negligible. So one may use the adiabatic temperature gradient, Eq.(3.6), to describe the real temperature gradient when convection occurs. The time scale for the convective cell to reach thermal equilibrium with its surroundings, from Eq.(4.43), is about $2 \times 10^6\, \alpha^{\frac{2}{3}}$ sec, which is indeed much longer than the dynamic time scale.

The mixing length theory is limited by the lacking of a good estimation for the mixing length. Furthermore, closer to the stellar surface the scale heights get shorter and shorter. The superadiabatic gradient is then not necessarily small. Unfortunately, convection does occur in the outer part of a star like our Sun. For efforts in establishing a sound convection theory, readers are referred to review articles in Cox, Livingston & Matthews (1991).

4.4.3 Thermal conduction

Thermal conduction is the phenomena that heat is transfered by particles as their kinetic energy from higher-temperature regions to lower-temperature ones via successive collisions. In the sense of a diffusion process, it is similar to the radiation transfer in which heat is carried by photons. Its efficiency depends basically on the energy excess of the carriers over their environment at each collision. Since the thermal speed of electrons is much larger than that of ions, it is pretty well to consider conduction by electrons only. Except for degenerate gases, the mean free path of a photon is usually much longer than those of electrons. Photons can travel farther and the energy difference between them and the new surroundings is larger. Therefore, in usual cases, radiation is much more efficient than thermal conduction for energy transportation. However, thermal conduction may play important roles, for example, in constructing the structure of compact stars.

To explore the efficiency of thermal conduction, one is interested in knowing the conductive flux for a given temperature gradient. In this regard, we will outline a general way and then present a simplified, approximate approach to obtain an approximate expression for a non-relativistic non-degenerate gas.

A general approach to estimate the conduction flux

The particle current density and the energy flux they carry can be written in terms of the distribution function as

$$J_r = \frac{2m^3}{h^3} \int v_r f(\vec{v}) \mathrm{d}^3 v \tag{4.47}$$

and

$$\mathcal{F} = \frac{2m^3}{h^3} \int K(v) v_r f(\vec{v}) \mathrm{d}^3 v \ , \tag{4.48}$$

where the subscript r stands for the radial direction, which is what we are interested in, f is the electron distribution function that $\int \frac{2}{h^3} f \mathrm{d}^3 p = n_\mathrm{e}$, and K is the kinetic energy of a particle at speed v. The distribution function also depends on location, which is not expressed explicitly for simplicity. When we consider the thermal conduction with the spherical symmetry of the star or in a plane-parallel atmosphere, the whole problem is basically one dimensional, i.e., in the radial direction. With a given temperature gradient, we want to find $\mathcal{F}$ subject to the condition of $J_r = 0$ for a static star. The distribution function $f(\vec{v})$ is no longer the equilibrium one in the presence of a temperature gradient. Otherwise J_r and $\mathcal{F}$ will both be zero.

The **Boltzman transport equation** reads

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla_{\vec{r}} + \vec{a} \cdot \nabla_{\vec{v}} \right) f = \left(\frac{\partial f}{\partial t} \right)_\mathrm{col} \ . \tag{4.49}$$

We may consider the distribution function as

$$f(\vec{v}) = f_0(v) + g(\vec{v}) \ , \tag{4.50}$$

where $g(\vec{v})$ is a perturbation to the equlibrium distribution f_0. With terms of zeroth-order in perturbation, the Boltzman transport equation can be written as

$$\left(\frac{\partial f}{\partial t} \right)_\mathrm{col} = v_r \frac{\partial f_0}{\partial r} + a_r \frac{\partial f_0}{\partial v_r} \ . \tag{4.51}$$

The right-hand side of the above equation can be expressed in terms of local thermodynamic properties, the temperature gradient and the local acceleration field by noting that

$$\begin{aligned}
\frac{\partial f_0}{\partial r} &= \frac{\partial f_0}{\partial \mu}\frac{\mathrm{d}\mu}{\mathrm{d}r} + \frac{\partial f_0}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}r} \\
&= -\frac{\partial f_0}{\partial \epsilon}\frac{\partial \mu}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}r} + (-\frac{\partial f_0}{\partial \epsilon}\frac{\epsilon-\mu}{T})\frac{\mathrm{d}T}{\mathrm{d}r} \\
&= -\frac{\partial f_0}{\partial \epsilon}(\frac{\partial \mu}{\partial T} + \frac{\epsilon-\mu}{T})\frac{\mathrm{d}T}{\mathrm{d}r} \ ,
\end{aligned}$$

and

$$a_r\frac{\partial f_0}{\partial v_r} = a_r\frac{\partial f_0}{\partial \epsilon}\frac{\partial \epsilon}{\partial v_r} = v_r\frac{\partial f_0}{\partial \epsilon}(ma_r) \ .$$

So we have

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{col}} = v_r\frac{\partial f_0}{\partial \epsilon}\left(ma_r - (\frac{\partial \mu}{\partial T} + \frac{\epsilon-\mu}{T})\frac{\mathrm{d}T}{\mathrm{d}r}\right) \ . \tag{4.52}$$

One then usually take the **relaxation time** approach, that is, from the details of scattering processes to express the collision term in such a form that

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{col}} = -\frac{g(\vec{v})}{\tau(v)} \ , \tag{4.53}$$

where $\tau(v)$ is the relaxation time. It is the time scale that the distribution function would need to relax to its equilibrium value via collisions only, that is,

$$f(\vec{v},t) = f_0(v) + g_0(\vec{v})\exp(-t/\tau(v)) \ . \tag{4.54}$$

If the relaxation time can be found from consideration of a suitable model involving relevant scattering processes, the perturbation $g(\vec{v})$ can be expressed, from Eq.(4.52), in terms of $\frac{\mathrm{d}T}{\mathrm{d}r}$ and a_r. For example, the relaxation time can take the form of

$$\frac{1}{\tau(v)} = 2\pi v n_{\mathrm{i}}\int_0^{\pi}\frac{\mathrm{d}\sigma(\theta)}{\mathrm{d}\Omega}(1-\cos\theta)\sin\theta\,\mathrm{d}\theta \ , \tag{4.55}$$

for a model of uncorrelated ions and electron-ion scattering only, where $n_{\rm i}$ is the number density of ions and $\frac{{\rm d}\sigma}{{\rm d}\Omega} = (\frac{Ze^2}{mv^2})^2(1-\cos\theta)^{-2}$ is the **Rutherford cross section** (Clayton 1983, p.244). This $g(\vec{v})$ is then employed to compute J_r and $\mathcal{F}$, and by requiring $J_r = 0$, $\mathcal{F}$ can be written in terms of $\frac{{\rm d}T}{{\rm d}r}$. A detailed calculation, which applies to any degree of degeneracy for a non-relativistic gas as treated here, can be found, for example, in Clayton (1983, §3-4). A relativistic treatment can be found in Chiu (1968, p.178).

A simplified approach for a non-relativistic, non-degenerate gas

Because electrons have a much larger speed than ions, we may consider the conductive flux at a location r towards positive-r direction as

$$\mathcal{F} = \frac{1}{6} n_{\rm e}\bar{v}\varepsilon_{r-\ell_{\rm e}} - \frac{1}{6} n_{\rm e}\bar{v}\varepsilon_{r+\ell_{\rm e}} \ , \tag{4.56}$$

where ε_r is the typical energy of a single electron at position r, which is about $\frac{3}{2}kT$, and $\ell_{\rm e}$ is the mean free path of electron-ion collisions. Because of the Z^2 dependence of the collision cross section and the degeneracy of electrons, collisions between electrons and electrons are usually ignored (Clayton 1983, p.238). The above equation can be turned into

$$\begin{aligned} \mathcal{F} &= -\frac{1}{2}kn_{\rm e}\bar{v}\ell_{\rm e}\frac{{\rm d}T}{{\rm d}r} && (4.57) \\ &\equiv -\lambda_{\rm c}\frac{{\rm d}T}{{\rm d}r} \ , && (4.58) \end{aligned}$$

where the thermal conductivity $\lambda_{\rm c}$ is

$$\lambda_{\rm c} = \frac{1}{2}kn_{\rm e}\bar{v}\ell_{\rm e} \ . \tag{4.59}$$

The mean speed $\bar{v}$ is about $\sqrt{3kT/m_{\rm e}}$ and the mean free path $\ell_{\rm e}$ is about $(\sigma n_{\rm i})^{-1}$, where $n_{\rm i}$ is the ion number density and σ is the scattering cross section. This cross section can be estimated as $\sigma \sim \pi r_0^2$ with r_0 being the closest approaching distance, $m_{\rm e}\bar{v}^2/2 = Ze^2/r_0$. Noting that $n_{\rm e} = \frac{\rho}{m_{\rm u}}\frac{1+X}{2}$ (Eq.(2.23) and Eq.(2.25), thus with the assumption of complete ionization) and $n_{\rm i} = \frac{\rho}{Am_{\rm u}}$, the conductivity turns into this form:

$$\lambda_{\rm c} = \frac{9\sqrt{3}}{16\pi}\frac{k^{\frac{7}{2}}}{m_{\rm e}^{\frac{1}{2}}e^4}\frac{A}{Z^2}(1+X)\,T^{\frac{5}{2}} \ . \tag{4.60}$$

The above equation deals with only one species of ions. X can only takes the value 1 (with $A = Z = 1$) or 0. For mixed species of ions, one needs only to replace Z^2/A with $(\sum_{Z=1} \frac{\rho_Z Z^2}{A_Z})/\rho = (\sum_{Z=1} n_Z Z^2) m_{\rm u}/\rho$, where ρ_Z, A_Z and n_Z are the mass density, atomic weight and number density of the ion with atomic number Z, respectively. Therefore, we have a more general expression:

$$\lambda_{\rm c} = \frac{9\sqrt{3}}{16\pi} \frac{k^{\frac{7}{2}}}{m_{\rm e}^{\frac{1}{2}} e^4} \frac{\rho}{(\sum_{Z=1} \rho_Z Z^2/A_Z)} (1+X)\, T^{\frac{5}{2}} \ . \tag{4.61}$$

To compare the efficiency of conduction with radiation, it is common to define a **conductive opacity** $\kappa_{\rm c}$ as

$$\kappa_{\rm c} = \frac{4acT^3}{3\rho\lambda_{\rm c}} \tag{4.62}$$

so that the flux is

$$\mathcal{F} = -\frac{4ac}{3\rho\kappa_{\rm c}} T^3 \frac{{\rm d}T}{{\rm d}r} \ , \tag{4.63}$$

similar to Eq.(4.29). When the two mechanisms, radiation and conduction, are both considered, the combined opacity κ is

$$\frac{1}{\kappa} = \frac{1}{\kappa_{\rm r}} + \frac{1}{\kappa_{\rm c}} \ . \tag{4.64}$$

The conductive opacity for a non-relativistic non-degenerate gas is, from Eq.(4.61),

$$\kappa_{\rm c} \sim 1.6 \times 10^5 \ \frac{(\sum_{Z=1} \rho_Z Z^2/A_Z)}{\rho(1+X)} \frac{T_7^{\frac{1}{2}}}{\rho_0} \ \ {\rm cm}^2/{\rm g} \ . \tag{4.65}$$

For an extremely degenerate gas, but still non-relativistic, the conductive opacity is

$$\kappa_{\rm c} \sim 5 \times 10^{-3} \ \frac{(\sum_{Z=1} \rho_Z Z^2/A_Z)}{\rho(1+X)} \frac{T_7^2}{\rho_5^2} \Theta \ \ {\rm cm}^2/{\rm g} \ , \tag{4.66}$$

where Θ is a factor of order unity (Cox & Giuli 1968, p.390). The factor $(\sum_{Z=1} \rho_Z Z^2/A_Z)/\rho$ is kept in this current form without being spelt out to join the numerical factor so that it is easier to consider the mass fraction ρ_Z/ρ of different species. These approximate expressions can be used to compare with those radiative opacities discussed in Section 5.3. Usually one finds that conduction is more important than radiation only in a degenerate gas.

Radiation as conduction

From Eq.(4.56) one may easily replace $n_e\varepsilon$ with photon energy density u and $\bar{v}$ with c to have

$$\begin{aligned}\mathcal{F} &= \frac{1}{6}cu_{r-\ell} - \frac{1}{6}cu_{r+\ell} && (4.67)\\ &= \frac{c\ell}{3}\frac{\mathrm{d}u}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}r} && (4.68)\\ &= -\frac{4}{3}ac\ell T^3\frac{\mathrm{d}T}{\mathrm{d}r}\ , && (4.69)\end{aligned}$$

where ℓ is the mean free path of a photon and $u = aT^4$ is used. We note that $\ell = \frac{1}{\rho\kappa}$, where ρ is the mass density of matter and κ is a certain kind of average opacity. We then have

$$\mathcal{F} = -\frac{4}{3}\frac{acT^3}{\rho\kappa}\frac{\mathrm{d}T}{\mathrm{d}r}\ , \tag{4.70}$$

which is equivalent to Eq.(4.30). It is clear that the Rosseland mean opacity should be used for κ from the consideration that $\mathcal{F} = \int\mathcal{F}_\nu \mathrm{d}\nu$. Here, the condition that $u = aT^4$ and an isotropic photon number flux is assumed.

4.4.4 Circulation

When the rotation of a star is taken into account, the hydrostatic equilibrium equation can still hold if the potential is an equivlent one involving contribution from the gravitational force and the centrifugal force. Such a star will not be spherical, but oblate to some extent. Recall the Von Zeipel's theorem that all the thermodynamic quantities are functions of the potential only. With $\mathrm{d}T/\mathrm{d}r = (\mathrm{d}T/\mathrm{d}\phi)\nabla\phi$, the radiative flux can be written as

$$\vec{\mathcal{F}} = f(\phi)\nabla\phi\ . \tag{4.71}$$

On an surface of constant potential, the radiative flux is not constant since the magnitude of $\nabla\phi$ is larger in the polar direction than in the direction of the equatorial plane. If the thermodynamic properties of a mass element remain static, it should hold that

$$\nabla\cdot\vec{\mathcal{F}} = 0\ .$$

From Eq.(4.71) we have

$$\nabla \cdot \vec{\mathcal{F}} = \frac{\mathrm{d}f}{\mathrm{d}\phi} |\nabla\phi|^2 + f(\phi)\nabla^2\phi \ . \tag{4.72}$$

For a rigid rotation with angular velocity ω,

$$\nabla^2\phi = 4\pi G\rho - 2\omega^2 \ , \tag{4.73}$$

which is also a function of ϕ only. Now that $\nabla\phi$ is not a constant over an equipotential surface, the condition $\nabla \cdot \vec{\mathcal{F}} = 0$ cannot be true everywhere on equipotentials. Some stellar material must be heated and get warmer.

The radiative flux is larger in the polar direction, so the effect of heating is stronger there. Material may rise in the polar direction and sink in the equatorial plane. Such a circulation gives rise to the so-called **meridional flow**. The reality is much more complicated than what is discussed here. Predictions of the flow speed and detections of such flows are still controversial, even for the case of the Sun, which is probably the only star whose meridional flow can be measured (see, e.g. Howard et al. 1991, Zhao et al. 2013).

Exercises

1. Using Eq.(4.15) verify Eq.(4.21).

2. Estimate the scale height at 0.75 $R_\odot$ in the Sun. Compare this with the choice of a mixing length of 0.1 $R_\odot$ and with the scale height at the solar surface.

3. If somewhere in a star's interior, $\rho = 1$ g/cm^3, $T = 10^6$ K, and the radiative opacity is mainly due to the bremsstrahlung κ_{ff}, which is about $10^{23}\rho T^{-3.5}$ in gaussian units (see Section 5.3), estimate the mean free path of a photon and that of an electron for electron-ion scattering. For simplicity, assume the composition is a completely ionized hydrogen gas.

Chapter 5

Opacities and Line Profiles

To solve the equations governing the structure and evolution of stars, as discussed in Section 3.1, physics input is needed. That includes equation of state, nuclear energy generation rate, and opacities. Opacities are block-out of photon propagation due to interactions between photons and matter. Photons may be absorbed by atoms or ions, which may undergo a transition from a lower energy state to a higher one (a *bound-bound* process, excitation) or release an ionized electron (a *bound-free* process, ionization, the photoelectric effect). The former happens for photons of certain specific energies and therefore results in a line opacity, while the latter makes a continuous one. Free electrons may also absorb photons when passing through a Coulomb field, e.g. in the vicinity of an ion or a nucleus. This is the inverse process of bremsstrahlung. It is a *free-free* process and also makes a continuous opacity. Photons may also be scattered by electrons into a direction different from the original propagation. Scattering therefore serves as a mechanism to reduce the radiation intensity in a certain direction. It can be treated as a continuous opacity. Given composition, density, and temperature, the magnitude of the aforementioned opacities depend on occupation in different excitation states and degree of ionization, whose determination is what Saha's equation is about.

5.1 Saha's equation

We shall first review the spectroscopic notations of energy levels and the partition function. To denote a certain atomic energy level, a format like $n\,^{2s+1}\ell_j$ is often used, where n is the principle quantum number for the energy level, s is the spin quantum number, ℓ is the orbital quantum number, which runs from 0 to $n-1$, and j is the total angular momentum. The statistical weight, g_n, of a certain level denoted by n is $g_n = 2n^2$. This is of course treating the level n as one level, neglecting all the detailed energy-level splitting. In this sense, the statistical weight can be understood as the number of degenerate states for a certain energy level.

From Boltzmann's law, Eq.(2.76), we have

$$N = \sum_{n=1}^{\infty} N_n = A \sum_{n=1}^{\infty} g_n\, e^{-E_n/kT} \equiv AZ(T) \ , \tag{5.1}$$

where N is the total particle number of the system and $Z(T)$ is the partition function:

$$Z(T) = \sum_{n=1}^{\infty} g_n\, e^{-E_n/kT} \ . \tag{5.2}$$

The series summation in the partition function apparently does not converge. In a real physical system, however, there is a suppression of continuum due to the presence of other particles. The summation is therefore truncated at a finite term and the ionization energy is replaced by an effective one. The total number of particles at energy level n can be expressed as

$$N_n = \frac{N}{Z(T)} g_n\, e^{-E_n/kT} \ . \tag{5.3}$$

Now, to derive Saha's equation, let's consider the ratio of particle numbers at two different states,

$$\frac{N_{i+1,m}(v)\mathrm{d}v}{N_{i,n}} = \frac{g(v)}{g_{i,n}} \frac{e^{-(E_{i,\infty}+E_{i+1,m}+\frac{1}{2}m_e v^2)/kT}}{e^{-E_{i,n}/kT}} \ , \tag{5.4}$$

where $N_{i+1,m}(v)\mathrm{d}v$ is the number of those $(i+1)$-times ionized ions which are at energy level m and whose $(i+1)$th-times ionized electron is at the speed v. The notation for energies used in the above equation is that $E_{i,j}$ is the energy of the i times ionized ion at energy level j, the ground state is denoted by $j = 1$, all ground state energy of different ionization states, $E_{i,1}$, is set to zero, and $E_{i,\infty}$ is the energy required for the ionization from state $(i, 1)$ to $(i + 1, 1)$. *Please particularly pay attention to the meaning of notations used here.* The state energies in the numerator and denominator in Eq.(5.4) are both referenced with respect to $E_{i,1}$, which is set to be zero. The statistical weight $g(v)$ is the statistical weight of the $(i + 1)$-times ionized ion at energy level m, $g_{i+1,m}$, times the number of states available to one electron of speed v, that is,

$$g(v) = g_{i+1,m} \times g_{\mathrm{e}} \times \frac{4\pi p^2 \mathrm{d}p}{h^3 \times n_{\mathrm{e}}} , \tag{5.5}$$

where the electron degeneracy g_{e} will be taken as 2. Defining the ionization energy of energy level n as $\chi_{i,n} = E_{i,\infty} - E_{i,n}$, we have

$$\begin{aligned} \frac{N_{i+1,m}}{N_{i,n}} &= \frac{\int N_{i+1,m}(v)\mathrm{d}v}{N_{i,n}} \\ &= \frac{2g_{i+1,m}}{g_{i,n}} \frac{\int_0^\infty 4\pi p^2 e^{-\frac{p^2}{2m_{\mathrm{e}}kT}}\mathrm{d}p}{n_{\mathrm{e}}h^3} e^{-E_{i+1,m}/kT}\, e^{-\chi_{i,n}/kT} . \end{aligned} \tag{5.6}$$

Noting that the integration over momentum p is equal to $(2\pi m_{\mathrm{e}}kT)^{3/2}$, we then have

$$\frac{n_{\mathrm{e}}N_{i+1,m}}{N_{i,n}} = \frac{2g_{i+1,m}}{g_{i,n}} \frac{(2\pi m_{\mathrm{e}}kT)^{3/2}}{h^3} e^{-E_{i+1,m}/kT}\, e^{-\chi_{i,n}/kT} . \tag{5.7}$$

With the above equation and summing up all the energy level m, we have

$$N_{i+1} = \sum_{m=1}^{\infty} N_{i+1,m} = 2Z_{i+1} \frac{(2\pi m_{\mathrm{e}}kT)^{3/2}}{h^3}\, e^{-\chi_{i,n}/kT} \frac{N_{i,n}}{n_{\mathrm{e}}g_{i,n}} .$$

Then similarly, we can turn the above equation into

$$N_{i,n} = \frac{n_{\mathrm{e}}N_{i+1}g_{i,n}e^{(E_{i,\infty}-E_{i,n})/kT}h^3}{2Z_{i+1}(2\pi m_{\mathrm{e}}kT)^{3/2}}$$

and by summing up all the energy level n to get

$$N_i = \sum_{n=1}^{\infty} N_{i,n} = \frac{n_{\mathrm{e}}N_{i+1}Z_i h^3 e^{+E_{i,\infty}/kT}}{2Z_{i+1}(2\pi m_{\mathrm{e}}kT)^{3/2}} .$$

With this, finally we reach **Saha's equation**:

$$\frac{n_\mathrm{e} n_{i+1}}{n_i} = \frac{2Z_{i+1}}{Z_i} \frac{(2\pi m_\mathrm{e} kT)^{3/2}}{h^3} e^{-\chi_i/kT} \ , \tag{5.8}$$

where the ionization energy is $\chi_i = \chi_{i,1} = E_{i,\infty}$. In fact, from Eq.(5.7), we have already got

$$\frac{n_\mathrm{e} n_{i+1,m}}{n_{i,n}} = \frac{2(g_{i+1,m} e^{-E_{i+1,m}/kT})}{(g_{i,n} e^{-E_{i,n}/kT})} \frac{(2\pi m_\mathrm{e} kT)^{3/2}}{h^3} e^{-\chi_i/kT} \ , \tag{5.9}$$

from which Eq.(5.8) can also be derived. The degree of ionization of a certain species in a system is decribed by Saha's equation, Eq.(5.8). Different species are coupled to one another through n_e, the number density of free electrons. In addition to the temperature dependence, which is the dominant one, density also plays a role in a complicated way.

As an example, for an H-He system, the degree of ionization can be defined as

$$\eta_1 = \frac{n_\mathrm{p}}{n_\mathrm{H} + n_\mathrm{p}} \ ; \ \eta_2 = \frac{n_{\mathrm{He}^+}}{n_\mathrm{He} + n_{\mathrm{He}^+} + n_\alpha} \ ; \ \eta_3 = \frac{n_\alpha}{n_\mathrm{He} + n_{\mathrm{He}^+} + n_\alpha} \ . \tag{5.10}$$

These ionization degrees are linked to Saha's equation in the following way:

$$\frac{n_\mathrm{p}}{n_\mathrm{H}} = \frac{\eta_1}{1-\eta_1} \ ; \quad \frac{n_{\mathrm{He}^+}}{n_\mathrm{He}} = \frac{\eta_2}{1-\eta_2-\eta_3} \ ; \quad \frac{n_\alpha}{n_{\mathrm{He}^+}} = \frac{\eta_3}{\eta_2} \tag{5.11}$$

and

$$n_\mathrm{e} = \rho\left(\frac{X}{m_\mathrm{H}}\eta_1 + \frac{Y}{m_\mathrm{He}}(\eta_2 + 2\eta_3)\right) \ . \tag{5.12}$$

Then, the ionization degrees η_1, η_2 and η_3 can be solved. The partition function of free protons and free α particles can be taken as simply 1 and very often only a few terms are needed to compute partition functions of other bound-system species to a very high accuracy.

5.2 The reduced opacity

Consider the natural decay from a higher level n to a lower level m. The associated, emitted photon has an energy of $h\nu = E_n - E_m$. We shall now discuss transition probabilities of these micro-processes in an equilibrium state. The Einstein coefficient A_{nm} is the probability per unit time for such a decay, that is,

$$N_n(t) = N_n(0)e^{-A_n t} \tag{5.13}$$

and

$$A_n = \sum_{m=1}^{m<n} A_{nm} \ . \tag{5.14}$$

A_{nm} is about 10^8 s^{-1} for H$_\alpha$ line.

The reverse process of the natural decay happens when adequate photons are available. Denoting the probability per unit time for the transition from level m to level n as $u_\nu B_{mn}$, where u_ν is the specific energy density of the photon field at frequency ν, we have in the equilibrium state

$$N_n A_{nm} = N_m u_\nu B_{mn} = N_m B_{mn} \frac{8\pi}{c^3} \frac{h\nu^3}{e^{h\nu/kT} - 1} \ . \tag{5.15}$$

This is, however, inconsistent with Boltzmann's law, Eq.(2.76), since A_{nm} and B_{mn}, as intrinsic properties of the matter, may depend on ν but not on T, the temperature. There is no way to make the above equation into the form of Boltzmann's law. Something is still missing.

In fact, in the presence of ambient photons, not only absorption transition will occur, but also emission transition will be induced. Denoting the probability per unit time of the stimulated (induced) emission as $u_\nu B_{nm}$, we now have in the equilibrium state

$$N_n A_{nm} + N_n u_\nu B_{nm} = N_m u_\nu B_{mn} \ , \tag{5.16}$$

from which u_ν can be expressed as

$$u_\nu = \frac{A_{nm}/B_{nm}}{\frac{N_m}{N_n}\frac{B_{mn}}{B_{nm}} - 1} \ . \tag{5.17}$$

Since this is in equilibrium, $u_\nu = \frac{8\pi}{c^3}\frac{h\nu^3}{e^{h\nu/kT}-1}$ and $N_m/N_n = \frac{g_m}{g_n}e^{h\nu/kT}$, one can find that the relation between Einstein coefficients is

$$A_{nm} = \frac{8\pi}{c^3} h\nu^3 B_{nm} \tag{5.18}$$

and

$$g_n B_{nm} = g_m B_{mn} \ . \tag{5.19}$$

We note that Einstein coefficients are intrinsic properties of an atom, which depend on the atomic structure only. Therefore, the above relation, although obtained by considering the thermal equilibrium case, is universally valid.

In considering radiation transfer, the stimulated emission can be regarded as 'negative absorption'. The net absorption becomes

$$\begin{aligned} N_m u_\nu B_{mn} - N_n u_\nu B_{nm} &= N_m u_\nu B_{mn}(1 - \frac{N_n B_{nm}}{N_m B_{mn}}) \\ &= N_m u_\nu B_{mn}(1 - e^{-h\nu/kT}) \ , \end{aligned} \tag{5.20}$$

in which LTE is assumed so that N_n/N_m follows Boltzmann's law. It is therefore very important to note that for the radiative opacities, that is, b-b, b-f and f-f opacities, a 'reduced' one should be used to account for the induced emission. The reduced opacity, κ^*, is

$$\kappa^* = \kappa(1 - e^{-h\nu/kT}) \ . \tag{5.21}$$

5.3 Continuous opacities

The attenuation of radiation during its propagation is described by the attenuation coefficient k_ν in Eq.(4.13), or equivalently by the opacity κ_ν, which is $\kappa_\nu = k_\nu/\rho = n\sigma_\nu/\rho$, where ρ is the mass density, n is the number density of the particles causing the attenuation, and σ_ν is the corresponding cross section. Since there are several absorption processes, including excitation (the bound-bound absorption), ionization (bound-free), and inverse bremsstrahlung (free-free), which cause attenuation, the opacity κ_ν should be understood as the sum of opacities due to these processes. Furthermore, electron scattering also removes photons from the original propagation direction, and thus causes attenuation on photons propagating in that direction. It can also be treated as one kind of absorption and a 'scattering opacity' $\kappa_{\rm sc}$ can be defined. The Rosseland mean opacity is then

$$\frac{1}{\langle\kappa\rangle_{\rm R}} = \frac{1}{\int \frac{{\rm d}B_\nu}{{\rm d}T}{\rm d}\nu} \int \frac{1}{(\kappa_{\rm bb} + \kappa_{\rm bf} + \kappa_{\rm ff})(1 - e^{-h\nu/kT}) + \kappa_{\rm sc}} \frac{{\rm d}B_\nu}{{\rm d}T}{\rm d}\nu \ . \tag{5.22}$$

5.3.1 Electron scattering

We shall now discuss the 'scattering opacity' first. The cross section of photons scattering off electrons is much larger than that of scattering off ions, so we will consider electron scattering only. Depending on energetics, the scattering between photons and electrons is customarily divided into three types: Thomson scattering, Compton scattering, and inverse Compton scattering. When the photon energy is much smaller than electron's rest energy, the energy of the photon before and after the scattering is about the same. This is the Thomson scattering. When the photon energy is not small, photons will lose energy in the scattering. This is the Compton scattering. When the electron is more energetic, photons may gain energy from electrons. This is the inverse Compton scattering. For optical light, Thomson scattering is a good approximation.

One may find in many textbooks (e.g. Jackson 1975; Rybicki & Lightman 1979) the derivation of the Thomson scattering cross section. Its differential form is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{e^2}{m_\mathrm{e}c^2}\right)^2 \frac{1}{2}(1+\cos^2\theta) \ , \tag{5.23}$$

where θ is the scattering angle. The dependence on mass indicates that ion scattering is not important. The total cross section σ_T is call the **Thomson cross section**,which is

$$\sigma_\mathrm{T} = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega = \frac{8\pi}{3}\left(\frac{e^2}{m_\mathrm{e}c^2}\right)^2 = 0.667\times 10^{-24}\mathrm{cm}^2 \ . \tag{5.24}$$

Thomson cross section is very often used when estimating the significance of interaction between photons and electrons. We note that it is equal to $\frac{8\pi}{3}r_\mathrm{e}^2$, where r_e is the classical radius of the electron.

The scattering opacity is $\kappa_\mathrm{sc} = n_\mathrm{e}\sigma_\mathrm{T}/\rho$. Noting that $\mu_\mathrm{e} = \frac{\rho}{n_\mathrm{e}m_\mathrm{u}}$ and $\mu_\mathrm{e} = \frac{2}{1+X}$ for a completely ionized gas, we have

$$\kappa_\mathrm{sc} = \frac{1+X}{2}\frac{\sigma_\mathrm{T}}{m_\mathrm{u}} \approx 0.2\ (1+X)\,\mathrm{cm}^2/\mathrm{g} \ . \tag{5.25}$$

5.3.2 Free-free opacity

An exact treatment of the free-free opacity requires quantum mechanics computation. Here we will take a simple classical approach and then later add modifications from results which are more accurate. We consider the Bremsstrahlung emission first. The energy emitted by an accelerated electron is, from Larmor's formula,

$$E = \int_{-\infty}^{\infty} \frac{\mathrm{d}E}{\mathrm{d}t}\mathrm{d}t = \frac{2e^2}{3c^3}\int_{-\infty}^{\infty} a^2 \mathrm{d}t \ , \tag{5.26}$$

where a is the acceleration. For an electron passing through the vicinity of an ion of charge Ze with an initial speed v and impact parameter b, considering the major interaction time period to be b/v and in that time period the acceleration is about $\frac{Ze^2}{b^2 m_\mathrm{e}}$, we have the total emitted energy

$$E \approx \frac{2e^2}{3c^3}\left(\frac{Ze^2}{b^2 m_\mathrm{e}}\right)^2 \frac{b}{v} \ . \tag{5.27}$$

The major frequency ν of the emitted photons is such that $2\pi\nu\frac{b}{v} \approx 1$. Therefore we have $b \approx \frac{v}{2\pi\nu}$ and $\mathrm{d}b = -\frac{v}{2\pi}\frac{\mathrm{d}\nu}{\nu^2}$. For a single encounter, the differential energy emitted per unit time at ν by electrons at v is the product of the differential electron number flux at v, the differential area of the encounter for emission at frequency ν, and the emitted energy, that is,

$$\begin{aligned} & n_\mathrm{e} v f(v)\mathrm{d}v \times 2\pi b \mathrm{d}b \times E \\ &= n_\mathrm{e} f(v)\mathrm{d}v \times \left(\frac{8\pi^2}{3}\frac{Z^2 e^6}{m_\mathrm{e}^2 c^3}\frac{1}{v}\right)\mathrm{d}\nu \\ &= c u_\nu \mathrm{d}\nu \times \mathrm{d}\sigma_\nu(v) \times (1 - e^{-h\nu/kT}) \ , \end{aligned}$$

where the last equality comes from the requirement that the emitted energy should be equal to the absorbed energy when the system is in equilibrium and $\mathrm{d}\sigma_\nu(v)$ is the differential absorption cross section due to electrons at v. This expression is the photon energy flux multiplied by the cross section with the factor $(1 - e^{-h\nu/kT})$ to take into account the stimulated emission.

Although we consider an equilibrium state here, the cross section is an intrinsic property of the system. Therefore the validity of the result obtained does not depend on this assumption. Noting that

$$u_\nu = \frac{4\pi}{c}\frac{2\nu^2}{c^2}\frac{h\nu}{e^{h\nu/kT} - 1} \ ,$$

we then have the absorption cross section to be

$$\begin{aligned} \mathrm{d}\sigma_\nu(v) &= n_\mathrm{e} f(v)\mathrm{d}v \times \left(\frac{8\pi^2}{3}\frac{Z^2e^6}{m_\mathrm{e}^2c^3}\frac{1}{v}\right)(4\pi\frac{2\nu^2}{c^2}h\nu)^{-1}e^{h\nu/kT} \\ &\approx n_\mathrm{e} f(v)\mathrm{d}v \times \frac{\pi}{3}\frac{Z^2e^6}{hcm_\mathrm{e}^2 v\nu^3}\ ,\ \text{for } h\nu \ll kT\ . \end{aligned} \qquad (5.28)$$

The free-free opacity can be obtained by integrating over electron's speed v, that is,

$$\rho\kappa_\mathrm{ff}(\nu) = n_\mathrm{i}\sigma_\mathrm{ff} = n_\mathrm{i}\int n_\mathrm{e} f(v)\mathrm{d}v \times \frac{\pi}{3}\frac{Z^2e^6}{hcm_\mathrm{e}^2 v\nu^3} \times \frac{4}{\sqrt{3}} g_\mathrm{ff}(v,\nu)\ , \qquad (5.29)$$

where correction from quantum mechanics computation is added at the end, and $g_\mathrm{ff}(v,\nu)$ is the Gaunt factor for free-free processes. Then, after the integration we get

$$\rho\kappa_\mathrm{ff}(\nu) = \frac{4}{3}(\frac{2\pi}{3m_\mathrm{e}kT})^{1/2}\frac{Z^2e^6}{hcm_\mathrm{e}^2\nu^3} \times n_\mathrm{i}n_\mathrm{e}\bar{g}_\mathrm{ff}(\nu)\ , \qquad (5.30)$$

where $\bar{g}_\mathrm{ff}(\nu)$ is the speed-averaged Gaunt factor, which is basically of order of unity.

Let's now consider the Rosseland mean of the free-free opacity. Since the dependence on ν in $\bar{g}_\mathrm{ff}(\nu)$ is not strong, we adopt the approximation to treat it separately from the ν^{-3} dependence in κ_ff when performing the average over ν. Keeping relevant factors, we have the Rosseland mean of κ_ff to be

$$\langle\kappa_\mathrm{ff}\rangle_\mathrm{R} \propto \frac{\int \frac{\mathrm{d}B}{\mathrm{d}T}\mathrm{d}\nu}{\int \frac{\nu^3}{(1-e^{-h\nu/kT})}\frac{\mathrm{d}B}{\mathrm{d}T}\mathrm{d}\nu} \times \frac{Z^2}{T^{1/2}}\frac{n_\mathrm{i}n_\mathrm{e}}{\rho}\langle\bar{g}_\mathrm{ff}\rangle_\nu\ . \qquad (5.31)$$

From $\int \frac{\mathrm{d}B}{\mathrm{d}T}\mathrm{d}\nu = \frac{4\sigma}{\pi}T^3$ and $n_\mathrm{e} = \frac{\rho}{\mu_\mathrm{e}m_\mathrm{u}}$, it follows that

$$\langle\kappa_\mathrm{ff}\rangle_\mathrm{R} \propto \frac{Z^2n_\mathrm{i}}{\mu_\mathrm{e}}T^{-3.5}\ . \qquad (5.32)$$

The Z^2n_i factor should be replaced by $\sum_Z Z^2n_Z$ to include the case of multiple species. Since

$$\begin{aligned} \sum_Z Z^2n_Z &= \frac{\rho}{m_\mathrm{u}}(X+Y+z\langle\frac{Z^2}{A}\rangle) \\ &\approx \frac{\rho}{m_\mathrm{u}}(1-z)\ , \end{aligned} \qquad (5.33)$$

in which the metal mass fraction is denoted by z to avoid the confusion with the atomic number Z and the last approximation is valid for $z \ll 1$, and $\mu_\mathrm{e} \approx \frac{2}{1+X}$ for complete ionization, we have

$$\langle \kappa_\mathrm{ff} \rangle_\mathrm{R} \propto (1+X)(1-z)\rho T^{-3.5} \ . \tag{5.34}$$

The Rosseland mean of the free-free opacity can therefore be expressed as

$$\langle \kappa_\mathrm{ff} \rangle_\mathrm{R} = 3.68 \times 10^{22} \langle \bar{g}_\mathrm{ff} \rangle_\nu (1+X)(1-z)\rho T^{-3.5}\,\mathrm{cm}^2/\mathrm{g} \ , \tag{5.35}$$

where $\langle \bar{g}_\mathrm{ff} \rangle_\nu$ only weakly depends on temperature and all the quantities are in gaussian units. An opacity with density and temperature dependence of $\rho T^{-3.5}$ is called the **Kramer opacity**.

5.3.3 Bound-free opacity

The bound-free opacity is far more complicated than the free-free one. It involves ionization of different species and from different states. For more detailed treatment, readers are referred to Clayton (1983, p.205), Chiu (1968, p.230) and Cox & Giuli (1968, p.380). An approximate expression of the Rosseland mean bound-free opacity, valid for a certain range of density and temperature, is

$$\langle \kappa_\mathrm{bf} \rangle_\mathrm{R} \approx 4.34 \times 10^{25} \frac{\langle g_\mathrm{bf} \rangle_\nu}{t} (1+X) z \rho T^{-3.5}\,\mathrm{cm}^2/\mathrm{g} \ , \tag{5.36}$$

where $\langle g_\mathrm{bf} \rangle_\nu$ is the frequency-averaged Gaunt for the bound-free process, which is of order of unity, and t is the so-called 'guillotine factor', which is comparable or larger than one.

This opacity, just like the Rosseland mean free-free opacity, also has the form of the Kramer opacity. Its temperature dependence can be understood by considering that the ionization cross section for a hydrogenic electron at a state n is (Clayton 1983, Eq.(3-151))

$$\sigma_\mathrm{bf} = \frac{64\pi^4 m_\mathrm{e} e^{10} Z^4}{3\sqrt{3} c h^6 n^5} \frac{g_\mathrm{bf}(\nu, n, \ell, Z)}{\nu^3} \ . \tag{5.37}$$

The ν^{-3} factor will lead to a T^{-3} dependence when taking the Rosseland mean. The opacity in fact depends on not only the cross section but also the

	Core	Surface
ρ	10^2	10^{-7}
T	10^7	$10^{3.7}$
X	0.5	0.75
$\kappa_{\rm sc}$	0.3	0.35
$\kappa_{\rm ff}$	~ 2	~ 600
$\kappa_{\rm bf}$	~ 50	~ 15000
$\kappa_{\rm c}$	~ 1000	$\sim 10^{10}$

Table 5.1: Opacities in the core and near the surface of the sun. All quantities are in gaussian units. Note that the larger the less important for $\kappa_{\rm c}$, which apparently does not play a significant role in a star like the sun.

number of available atoms/ions with bound electrons. A certain temperature dependence will come in when the ionization degree is considered. It is clear that the number of bound electrons increases with decreasing temperature. Furthermore, although the individual ionization cross section does not depend on density either, the number of bound electrons actually increases with increasing density, as can be seen from Saha's equation as long as the density is not too high. It turns out that in a limited parameter space the bound-free opacity also behaves like Kramer's opacity.

Taking the sun as an example, we have some rough estimate listed in Table 5.1. One can see that in the core of the sun, radiative opacities are all comparable. Near the surface, $\kappa_{\rm bf}$ and $\kappa_{\rm ff}$ dominate. Conduction is negligible in both cases. When we move further out from the photosphere of the sun, since density drops quickly, $\kappa_{\rm sc}$ will become more and more dominant. Readers are referred to Figure 3-15 in Clayton (1983, p.223) for the relative importance of these opacities at different density and temperature.

5.4 Bound-bound opacity and line broadening

5.4.1 Bound-bound opacity and natural broadening

Bound-bound opacity is determined by, besides the availability of atoms/ions with bound electrons, the absorption cross section of an atom to absorb a photon of a specific energy which is the energy difference between two electronic states of the atom. It in turn depends on the transition probability of that atom between the two specific states. These are complicated, because there are many, if not countless, possible transitions, but these absorption cross sections can be computed in Quantum Mechanics with the time-dependent perturbation theory, which we won't elaborate further and readers should consult quantum mechanics textbooks or Clayton (1983) and Rose (1998) for more details. We will only try to illustrate some major features here. The importance of bound-bound opacity is not easy to assess. It depends on composition and local properties like density and temperature. Of course neutral or partially ionized species are needed for this opacity to play some role. A figure to illustrate relative importance among different opacities in energy transportation can be found in Figure 3-15 in Clayton (1983).

Bound-bound opacity is a line opacity, in contrast to other continuum ones. It causes spectral absorption lines, superposed on a continuum. These lines are not sharply at a certain frequency, but are broadened. To understand line broadening, we start with the transition cross section. The absorption cross section for transition between states k and s can be expressed as

$$\sigma_{ks}(\omega) = \frac{2\pi^2 e^2}{m_\mathrm{e} c} f_{ks} \left(\frac{\Gamma/2\pi}{(\omega - \omega_{ks})^2 + (\Gamma/2)^2} \right) , \tag{5.38}$$

where $\hbar\omega_{ks}$ is the energy gap between states k and s, Γ is the FWHM of the profile function, and f_{ks} is the so-called **oscillator strength** of this transition, which depends on ω_{ks} and the wave functions of states k and s. It varies a lot for different transitions, but is roughly to the order of unity for some major (often seen) ones. For example, the Lyα line, which corresponds to the transition between 2p and 1s states of a hydrogen atom, has an oscillator strength equal to 0.42. The Lorentzian profile comes from the fact that the time-dependent wave function of state k can be expressed as

$$\Psi_k(t) = u_k \mathrm{e}^{-\frac{t}{2\tau_k}} \mathrm{e}^{i\omega_k t} \quad (t > 0) \ , \tag{5.39}$$

where u_k and ω_k are the time-independent eigen function and the energy level of state k, respectively, and τ_k is the mean radiative life time of state k, which can be computed with transition probabilities of all possible transitions. Note that the reason to have a factor of 2 in front of τ_k is that $\Psi_k(t)$ is the wave function, rather than the probability, which is to be evaluated as $\Psi_k^*\Psi_k$. Taking the 2p state of a hydrogen atom as an example again, its life time is found to be 0.16×10^{-8} sec. The probability that this state has an energy $E = \hbar\omega$ is given by its Fourier transform $\tilde{\Psi}_k(\omega)$. The Fourier *power* of such a (damped oscillation) wave function is exactly a Lorentzian profile with $\Gamma_k = \frac{1}{\tau_k}$ (Rose 1998, page 128; Rybicki & Lightman 1979, page 97). Since in general there is uncertainty in the energy at both the lower (k) and upper (s) levels, the width Γ in the cross section $\sigma(\omega)$, Eq.(5.38), is actually the sum of that at those two levels, that is, $\Gamma = \Gamma_k + \Gamma_s$ (Rose 1998, page 138-139). This direct summation may be understood from the fact that, when calculating the transition probability, a certain product of wave functions of the two states is considered. The exponential terms in the wave function lead to the summation of the individual width and also to the difference of the energies at the two energy levels.

Different notations and formalisms may be used by different authors for this cross section. To make clear the preference used here, we note the following normalization condition:

$$\int_{-\infty}^{\infty} \frac{\Gamma/2\pi}{(\omega-\omega_0)^2 + (\Gamma/2)^2} \mathrm{d}\omega = \frac{1}{\pi} \tan^{-1}\left(\frac{2(\omega-\omega_0)}{\Gamma}\right)|_{-\infty}^{\infty} = 1 \ . \tag{5.40}$$

Since this line width is due to the natural lifetime of energy states, it is called the **natural broadening** of spectral lines.

5.4.2 Doppler and collisional broadening

Atoms are in random thermal motion. Photons emitted or absorbed by these moving atoms therefore have an energy different from that in the rest frame of an atom, because of the Doppler effect. So, a spectral line will be broadened, beside its natural broadening. This is called the Doppler broadening.

The change in frequency associated with an atom of speed v_z (taking the z-axis as the line of sight and the observer is in the $+z$ direction) is

$$\triangle\omega = \omega - \omega_0 = \omega_0 \frac{v_z}{c} \ , \tag{5.41}$$

where ω_0 is the rest-frame frequency. The number of atoms distributed in the speed range from v_z to $v_z + \mathrm{d}v_z$ is proportional to $\exp(-\frac{m_\mathrm{a} v_z^2}{2kT})\mathrm{d}v_z$, where m_a is the mass of the atom. Replacing v_z with ω (Eq.(5.41)), we will have the profile function to be

$$\phi(\omega) = \frac{1}{\triangle\omega_\mathrm{D}\sqrt{\pi}} \mathrm{e}^{-(\frac{\omega-\omega_0}{\triangle\omega_\mathrm{D}})^2} \ , \tag{5.42}$$

where $\triangle\omega_\mathrm{D}$ is the Doppler width, defined as

$$\triangle\omega_\mathrm{D} = \frac{\omega_0}{c}\sqrt{\frac{2kT}{m_\mathrm{a}}} \ . \tag{5.43}$$

We note that the profile function ϕ is so defined that $\int \phi(\omega)\mathrm{d}\omega = 1$. **Doppler broadening** leads to a Gaussian line profile.

Natural broadening always exists. It appears together with the Doppler broadening. Furthermore, collisions may make the lifetime of an energy state shorter if the collisional frequency is high, which is often the case. We therefore have a Lorentzian line profile due to the collisional broadening, which, together with the natural broadening, has a width Γ as

$$\Gamma = \Gamma_k + \Gamma_s + 2\nu_\mathrm{c} \ , \tag{5.44}$$

where $\Gamma_k + \Gamma_s$ is the FWHM due to natural broadening and ν_c is the collisional frequency (Rybicki & Lightman 1979, page 290). Recalling that $\Gamma_k = \frac{1}{\tau_k}$, we can see that ν_c, which leads to a mean life time $\frac{1}{\nu_\mathrm{c}}$ for each energy state, is the width of the corresponding Lorentzian profile due to collision life time; there is an additional factor like $e^{-\frac{\nu_\mathrm{c}}{2}t}$ in Eq. (5.39). For each state, the uncertainty in energy due to finite life time is therefore characterized by the width $\Gamma_k + \nu_\mathrm{c}$. When considering the transition between states k and s, we reach the above Γ. Note that ν_c here is *not* the angular frequency ω_c, but Γ is still the FWHM in angular frequency.

Now we should consider the combined effect of such Lorentzian and Gaussian profiles, because all these phenomena, natural decay, thermal motion, and collision, are present at the same time. We may proceed by considering the profile as an average of the Lorentzian profile over the speed states of the atom, that is, taking ω_0 to be $\omega_0 + \omega_0 \frac{v_z}{c}$,

$$\phi(\omega) = \frac{\Gamma}{2\pi} \int_{-\infty}^{\infty} \frac{(\frac{m_\mathrm{a}}{2\pi kT})^{\frac{1}{2}} \exp(-\frac{m_\mathrm{a} v_z^2}{2kT})}{(\omega - \omega_0 - \omega_0 \frac{v_z}{c})^2 + (\frac{\Gamma}{2})^2} \mathrm{d}v_z \quad . \tag{5.45}$$

By employing the *Voigt function* $H(a,u)$,

$$H(a,u) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\exp(-y^2)\mathrm{d}y}{a^2 + (u-y)^2} \quad , \tag{5.46}$$

we may have $\phi(\omega)$ in a more compact form:

$$\phi(\omega) = \frac{1}{\triangle\omega_\mathrm{D}\sqrt{\pi}} H(a,u) \tag{5.47}$$

with

$$a = \frac{\Gamma}{2\triangle\omega_\mathrm{D}} \tag{5.48}$$

and

$$u = \frac{\omega - \omega_0}{\triangle\omega_\mathrm{D}} \quad . \tag{5.49}$$

If we consider $u \ll 1$, that is, ω very close to the central frequency ω_0, the integrant in the Voigt function, Eq.(5.46), is large when $y \approx u$ if $a \ll 1$. We then have

$$\int_{-\infty}^{\infty} \frac{\exp(-y^2)\mathrm{d}y}{a^2 + (u-y)^2} \approx \mathrm{e}^{-u^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{a^2 + x^2} = \frac{\pi}{a} \mathrm{e}^{-u^2} \quad , \tag{5.50}$$

which means close to the central frequency the profile is approximately a Gaussian; the Doppler broadening dominates. The above result does not apply if we do not have $a \ll 1$. In fact if $a \gg 1$, ϕ is approximately a constant close to the central frequency. On the other hand, if we consider $u \gg 1$, we have

$$\int_{-\infty}^{\infty} \frac{\exp(-y^2)\mathrm{d}y}{a^2 + (u-y)^2} \approx \frac{1}{a^2 + u^2} \int_{-\infty}^{\infty} \mathrm{e}^{-y^2} \mathrm{d}y = \frac{\sqrt{\pi}}{a^2 + u^2} \quad , \tag{5.51}$$

which means at the wing of the line the profile is more like a Lorentzian.

Exercises

1. For a pure hydrogen gas at a certain temperature, how does the ionization degree change with the electron number density? With the mass density? Neglect the density dependence of the partition function and of the ionization energy.

2. Compare the f-f and b-f opacities for population I stars ($Z = 0.02$) and population II stars ($Z = 0.001$) using Kramer's approximation with $\langle \bar{g}_{\text{ff}} \rangle = \langle g_{\text{bf}} \rangle = 1$ and the guillotine factor t equal to 10.

Chapter 6

Nuclear Energy Sources

Stars are powered by nuclear fusion reactions. For the case of the Sun, it is easy to argue against the gravitational and chemical energy as the origin of its luminosity. Nuclear energy is not only a viable option, but also a rich and powerful resort, which leads to the construction of consistent stellar models, to the understanding of stellar evolution, and to the theory of nucleosynthesis. Observations of supernova explosions, helioseismology, and solar neutrinos all provide support to this concept.

6.1 The ignition of thermonuclear reactions

6.1.1 The Coulomb barrier

For nuclei to fuse, the Coulomb barrier must be overcome. Typically, at a distance of one fermi (10^{-13} cm), the Coulomb energy is $E_{\mathrm{C}} \sim e^2/r \sim 1$ MeV. In the environment near the stellar center, temperature may be of order of 10^7 K or 10^8 K, which corresponds to a typical thermal energy of several thousands of electron volt (keV) only. Therefore, the **tunneling effect** plays an important role in triggering efficient reactions. The barrier height is proportional to the product of electrical charges of the two paticipating nuclei. It is obvious that lighter species should fuse first.

6.1.2 The minimum stellar mass for ignition

We shall now consider a very much simplified picture to estimate the minimum mass required for a star to ignite nuclear fusion. We first try to find a relation between the stellar mass and its central temperature. Assuming a star with a uniform mass distribution, the hydrostatic equilibrium equation reads

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho\frac{GM_r}{r^2} = -\frac{4\pi}{3}G\rho^2 r \ ,$$

and the pressure is

$$P(r) = \frac{2\pi}{3}G\rho^2(R^2 - r^2) \ ,$$

where R is the stellar radius. Further assuming an ideal-gas equation of state, we then have the temperature at the stellar center, T_c, as

$$T_\mathrm{c} = \frac{Gm_\mathrm{u}\mu M}{2kR} \ , \qquad (6.1)$$

where M is the total stellar mass. In this equation, to achieve a certain central temperature for nuclear burning to be effective, a certain stellar mass is required. It depends, however, still on the stellar radius, implying that density plays a role. We need to find a way to eliminate that dependence.

At the stellar center, if degeneracy pressure dominates, temperature is irrelevant to the structure stability against gravity. In case the temperature and density are high enough for nuclear fusion to proceed in such a degenerate environment, explosive nuclear burning will happen, since degeneracy pressure maintains the structure and temperature can go higher and higher. Examples are the so-called helium flash and thermal pulses (i.e. helium shell flashes). On the other hand, if the temperature and density are not high enough, but electron degeneracy can already hold the structure (i.e. degeneracy still dominates), the temperature and density will not increase further. No nuclear reaction will happen. Let's therefore set the condition for stable nuclear burning to happen at the stellar center to be that the electron degeneracy does not dominate there when the desired temperature is reached, that is,

$$\begin{aligned} kT_\mathrm{c} &> \varepsilon_{\mathrm{F,e}} \qquad (6.2) \\ &= \frac{\hbar^2}{2m_\mathrm{e}}(\frac{3\pi^2}{\mu_\mathrm{e}m_\mathrm{u}})^{\frac{2}{3}}(\frac{M}{\frac{4\pi}{3}R^3})^{\frac{2}{3}} \ . \end{aligned}$$

Table 6.1: Estimated minimum stellar mass for igniting nuclear burning based on simplified arguments discussed in the text. The central temperature T_c given here is the typical temperature at which corresponding reactions are efficient. For the case of hydrogen fuel, $X = 0.75$ and $Y = 0.25$ are assumed.

fuel	T_c	μ_e	μ	$M/M_\odot$
H	10^7 K	1.14	0.6	0.14
He	10^8 K	2	1.3	0.18
C	6×10^8 K	2	1.7	0.48

This is of course not a strict, rigorous condition but serves as a guidance only. We are interested in the constraint on mass for a given central temperature T_c, which under the simplification of a constant density is proportional to M/R. Eliminating R in the above equation with Eq.(6.1), we finally reach the condition that

$$M > \frac{3.9 \times 10^{-7}}{\mu_\mathrm{e}^{1/2} \mu^{3/2}} (T_\mathrm{c}/\mathrm{K})^{\frac{3}{4}} \, M_\odot \quad . \tag{6.3}$$

For a given T_c, there is a mass *lower* limit for degeneracy *not* to dominate at the stellar center. One can realize this point by considering $\rho \propto M/R^3 \propto T_\mathrm{c}^3/M^2$, that is, to reach a certain T_c, a larger mass requires only a smaller density so that degeneracy does not dominate. In Table 6.1 typical values are listed according to Eq.(6.3). These estimates are understandably rough and different from those obtained with detailed modeling. For example, many models indicate that, to ignite hydrogen burning, the minimum stellar mass is about 0.08 $M_\odot$. Nonetheless, the above discussion indicates that, if the stellar mass is small, degeneracy may be reached first before nuclear reaction can take place.

6.2 Nuclear energy generation rates

6.2.1 Q-value of a nuclear reaction

The energy released by a nuclear reaction is denoted as the **Q-value** of that reaction, which is the rest-energy change in the reaction. For a reaction like

$$X + a \to Y + b \ , \tag{6.4}$$

the Q-value is defined as

$$Q \equiv \triangle E = (M_X + M_a - M_Y - M_b)c^2 \ . \tag{6.5}$$

Since charge is conserved in the reaction, we may replace the nucleus mass with the atomic mass. Small errors due to difference in electron binding energy are introduced.

An often used expression for the Q value is, with the definition of **the atomic mass excess** $\triangle M$ of a certain element as

$$\triangle M = M - Am_{\mathrm{u}} \ , \tag{6.6}$$

where M is the atomic mass and A is the atomic mass number,

$$Q = (\triangle M_X + \triangle M_a - \triangle M_Y - \triangle M_b)c^2 \ . \tag{6.7}$$

The reason to use this definition is that the atomic mass is usually what is measured in laboratories. Tables of atomic mass excess can be found in many textbooks (e.g. Clayton (1983), Table 4-1). Some most-often-referred cases are listed here in Table 6.2. The atomic mass excess of C^{12} is by definition zero. The released energy may increase the thermal energy of the reaction products, be carried by photons (sometimes via the annihilation of electron-positron pairs), or escape as neutrinos.

6.2.2 Energy generation rate and lifetime

The nuclear energy generation rate per unit mass in the stellar interior, ε, as appearing in Eq.(3.4), can be written as

$$\varepsilon = \frac{Qr_{aX}}{\rho} \ , \tag{6.8}$$

when referred to a process involving the fusion of a and X, where r_{aX} is the rate per unit volume for this process. It can be expressed as

$$\begin{aligned} r_{aX} &= \frac{n_a n_X}{1+\delta_{aX}} \int v\sigma(v)\phi(v)\mathrm{d}v && (6.9) \\ &= \frac{n_a n_X}{1+\delta_{aX}} \langle v\sigma \rangle \ , && (6.10) \end{aligned}$$

in which δ_{aX} is the Kronecker symbol, v is the relative velocity, and $\phi(v)$ is the probability distribution. When the distributions of the two species a and X are both Maxwellian, it can be shown that $\phi(v)$ is also Maxwellian:

$$\phi(v)\mathrm{d}v = 4\pi \left(\frac{\mu}{2\pi kT}\right)^{\frac{3}{2}} \exp\left(-\frac{\mu v^2}{2kT}\right) v^2 \mathrm{d}v \ , \tag{6.11}$$

where μ is the reduced mass, $\mu = \frac{m_\mathrm{a} m_\mathrm{X}}{m_\mathrm{a}+m_\mathrm{X}}$. The nuclear energy generation rate per unit mass, ε, can therefore be computed.

The lifetime of species a against the reaction between a and X, τ_{aX}, can be defined as

$$\left(\frac{\partial n_a}{\partial t}\right)_X = -\frac{n_a}{\tau_{aX}} = -r_{aX}(1+\delta_{aX}) \ . \tag{6.12}$$

The factor $(1+\delta_{aX})$ appears in the above equation since if a and X are the same the number change rate is twice of the reaction rate. It is obvious that

$$\tau_{aX} = \frac{1}{n_X \langle v\sigma \rangle} \ , \tag{6.13}$$

and when there are other processes, for example, involving species Y, we have

$$\frac{1}{\tau_a} = \frac{1}{\tau_{aX}} + \frac{1}{\tau_{aY}} \ . \tag{6.14}$$

These time scales are closely related to the stellar evolution time scales.

6.2.3 The most effective energy

For given species participating the nuclear reaction and given temperature of the system, reactions happen most effectively among particles of a certain energy, which is determined mainly by the factors of the tunneling effect to overcome the Coulomb barrier in the reaction cross section and the thermal distribution to ensure the presence of enough particles. To find the energy dependence of the cross section, we start with the penetration factor of the tunneling effect (**the Gamow factor**), which is proportional to $\exp(-\frac{2\pi Z_1 Z_2 e^2}{\hbar v})$ from quantum mechanics calculations. It may be understood by considering the ratio of the Coulomb energy to the kinetic energy:

$$\frac{E_{\rm C}}{E_{\rm K}} = \frac{Z_1 Z_2 e^2/r_0}{\mu v^2/2} = \frac{2 Z_1 Z_2 e^2}{\hbar v} \ , \tag{6.15}$$

where $r_0 \approx \hbar/p = \hbar/\mu v$ is adopted. It is conventional to write the Gamow factor in terms of the kinetic energy in the center-of-mass frame $E = \frac{1}{2}\mu v^2$ as $\exp(-b/\sqrt{E})$, with

$$b \equiv 2\pi\alpha Z_1 Z_2 \sqrt{\frac{\mu c^2}{2}} \ , \tag{6.16}$$

where $\alpha = \frac{e^2}{\hbar c}$ is the fine structure constant.

Considering another 'area' factor $\pi\lambda^2 \propto (h/p)^2 \propto 1/E$, the reaction cross section is usually written as

$$\sigma(E) = \frac{S(E)}{E} \exp(-\frac{b}{\sqrt{E}}) \ . \tag{6.17}$$

The function $S(E)$ is then a slowly varying function, which can be determined by experiments.

Now, for the Maxwellian velocity distribution, with $E \propto v^2$, we have

$$\phi(E){\rm d}E = \phi(v){\rm d}v = \frac{2}{\sqrt{\pi} kT} \sqrt{\frac{E}{kT}} \exp(-\frac{E}{kT}) \, {\rm d}E \ . \tag{6.18}$$

The average reaction rate per pair of particles is then

$$\langle v\sigma \rangle \ = \ \int v(E)\sigma(E)\phi(E) \, {\rm d}E$$

$$
\begin{aligned}
&= \int \sqrt{\frac{2E}{\mu}} \frac{S(E)}{E} \exp(-\frac{b}{\sqrt{E}}) \frac{2}{\sqrt{\pi} kT} \sqrt{\frac{E}{kT}} \exp(-\frac{E}{kT}) \,\mathrm{d}E \\
&= \sqrt{\frac{8}{\pi\mu}} \frac{1}{(kT)^{3/2}} \int S(E) \exp(-\frac{E}{kT} - \frac{b}{\sqrt{E}}) \,\mathrm{d}E \ . \qquad (6.19)
\end{aligned}
$$

In the above integral, the exponent is composed of contributions from the thermal distribution and the tunneling effect. The most effective energy is that high enough to have significant tunneling and low enough to have more particles. The maximum of that exponetial occurs at the most effective energy E_0, which is

$$
E_0 = (\frac{bkT}{2})^{\frac{2}{3}} \ . \qquad (6.20)
$$

6.2.4 Approximate expressions

Since $S(E)$ is a slowly varying function and the integrant in Eq.(6.19) is only significant near the most effective energy E_0, an approximate expression of Eq.(6.19) can be obtained with the replacement of $S(E)$ by $S(E_0)$ and the exponential by a gaussian form. We approximate the exponential as

$$
\begin{aligned}
\exp(-\frac{E}{kT} - \frac{b}{\sqrt{E}}) &\approx \exp(-\frac{E_0}{kT} - \frac{b}{\sqrt{E_0}}) \exp(-(\frac{E - E_0}{\triangle})^2) \\
&= \exp(-\frac{3E_0}{kT}) \exp(-(\frac{E - E_0}{\triangle})^2) \ , \qquad (6.21)
\end{aligned}
$$

where $\triangle = 2\sqrt{E_0 kT/3}$ is so set to match the second derivatives. The rate can be approximated as

$$
\begin{aligned}
\langle v\sigma \rangle &\approx \sqrt{\frac{8}{\pi\mu}} \frac{S(E_0)}{(kT)^{3/2}} \exp(-\frac{3E_0}{kT}) \int_{-\infty}^{\infty} \exp(-(\frac{E - E_0}{\triangle})^2) \,\mathrm{d}E \\
&= \sqrt{\frac{8}{\pi\mu}} \frac{S(E_0)}{(kT)^{3/2}} \triangle \sqrt{\pi} \exp(-\frac{3E_0}{kT}) \ . \qquad (6.22)
\end{aligned}
$$

The error introduced due to the lower-limit change in the integration is negligible. We may now write down the rate in terms of b and temperature. With the definition that

$$
\eta \equiv \frac{3E_0}{kT} = 3(\frac{b}{2})^{\frac{2}{3}} (kT)^{-\frac{1}{3}} \ , \qquad (6.23)
$$

we have

$$\langle v\sigma\rangle \approx \frac{8\sqrt{2}}{9\sqrt{3}}\frac{S(E_0)}{\sqrt{\mu}b}\eta^2 e^{-\eta} \quad . \tag{6.24}$$

This is a quite clean expression. For a given reaction (b is then given), with the knowledge of S, it is a simple function of temperature. However, at least three more corrections should be called for attention. First, one more term in approximating $S(E)$ may be included, that is, $S(E) \approx S(E_0) + \frac{\mathrm{d}S}{\mathrm{d}E}(E - E_0)$, instead of $S(E_0)$ only. Second, the replacement of the exponential by a gaussian may be better treated. Third, electron screening, which we shall discuss later, may play some role.

The reaction rate is very sensitive to temperature. We may explore its dependence on the variation of temperature around a typical temperature at which the reaction efficiently takes place in the following way. At a temperature T close to the typical temperature T_*, we have

$$\frac{T}{T_*} = 1 + \xi \quad , \quad \xi \ll 1 \tag{6.25}$$

and

$$\frac{\eta}{\eta_*} = (\frac{T_*}{T})^{\frac{1}{3}} \approx 1 - \frac{\xi}{3} \quad . \tag{6.26}$$

The rate can be expanded as

$$\begin{aligned} \frac{r}{r_*} &= \frac{S(E_0(T))}{S(E_0(T_*))}(\frac{\eta}{\eta_*})^2 e^{-\eta+\eta_*} \\ &\approx (\frac{\eta}{\eta_*})^2 e^{-\eta_*(\frac{\eta}{\eta_*}-1)} \\ &\approx (1 - \frac{2}{3}\xi)(1 + \eta_*\frac{\xi}{3}) \\ &\approx 1 + \frac{(\eta_* - 2)}{3}\xi \quad . \end{aligned} \tag{6.27}$$

Considering that

$$(\frac{T}{T_*})^n \approx 1 + n\xi \quad ,$$

we may have

$$\frac{r}{r_*} \approx (\frac{T}{T_*})^n \quad , \tag{6.28}$$

with

$$n \equiv \frac{\eta_* - 2}{3} \quad . \tag{6.29}$$

Now the energy generation rate can be expressed as a power law in temperature T with power index n:

$$\begin{aligned} \varepsilon &= \frac{Qr}{\rho} = \frac{Q}{\rho} \frac{n_a n_X}{1+\delta_{aX}} \langle v\sigma \rangle \\ &\approx \frac{Q}{\rho^2} \frac{n_a n_X}{1+\delta_{aX}} \frac{\langle v\sigma \rangle_*}{T_*^n} \rho T^n \quad . \end{aligned} \tag{6.30}$$

For the reaction p+p $\to$ D+e$^+$+ν_{e} inside the Sun, $T_* \approx 10^7$ K, the index n is about 4.5. Other reactions may have even higher indices.

We did not show what is the so-called ‘typical’ temperature for a certain reaction. Those in Table 6.1 come from most model computations. Although local density also matters, from the discussion in the previous paragraph, we can see that the reaction rate is very sensitive to temperature and therefore those typical temperatures are pretty much universal to a large extent in stellar interiors.

6.3 Electron screening

In a plasma, although the Coulomb energy between particles is by definition much smaller than the typical kinetic energy, charges will still tend to collectively concentrate toward charges of different signs. It is a manifestation of the collective effect of the long-range interaction. The result is an effective shielding surrounding a charge and an effective binding of a charge with the whole plasma. Such a shielding can reduce the Coulomb barrier and enhance the rate of nuclear reaction inside stars.

6.3.1 The Debye-Hückel model

Consider a particle with charge $Z_i e$. The number density of particles with charge Ze at a distance r from the Z_i particle can be written as

$$n_Z(r) = \bar{n}_Z \, e^{-\frac{ZeV_i(r)}{kT}} \quad , \tag{6.31}$$

where $V_i(r)$ is the potential around particle Z_i due to *all* charged particles in the plasma, and $\bar{n}_Z$ is the average number density of particle Z, which should be approximately found when r is very large. The Poisson equation reads

$$\nabla^2 V_i = -4\pi\rho_Q = -4\pi e \sum_Z Z n_Z \tag{6.32}$$

with $Z_{\rm e} = -1$ for electrons. Under the condition of

$$\frac{ZeV_i(r)}{kT} \ll 1 \quad , \tag{6.33}$$

we have

$$\begin{aligned} \nabla^2 V_i(r) &\approx -4\pi e \sum_Z Z\bar{n}_Z \left(1 - \frac{ZeV_i(r)}{kT}\right) \\ &= \frac{4\pi e^2}{kT}\left(\sum_Z Z^2 \bar{n}_Z\right) V_i(r) \\ &= \kappa^2 V_i(r) \quad , \end{aligned} \tag{6.34}$$

which defines the notation κ. The solution of the above equation can be easily found to be

$$\begin{aligned} V_i(r) &= \frac{Z_i e}{r} e^{-\kappa r} \\ &= \frac{Z_i e}{r} e^{-r/R_{\rm D}} \quad , \end{aligned} \tag{6.35}$$

which satisfies $V_i \to \frac{Z_i e}{r}$ at $r \to 0$. The **Debye-Hückel radius** $R_{\rm D}$ is defined as

$$R_{\rm D} \equiv \frac{1}{\kappa} = \left(\frac{4\pi e^2}{kT} \sum_Z Z^2 \bar{n}_Z\right)^{-\frac{1}{2}} \quad . \tag{6.36}$$

The charge of particle Z_i is shielded by other charges in the plasma and the potential surrounding Z_i drops off exponentially. Introducing the parameter

$$\zeta \equiv \sum_{Z>0} (Z^2 + Z) \frac{X_Z}{A_Z} \tag{6.37}$$

the Debye-Hückel radius turns into

$$R_\mathrm{D} = \sqrt{\frac{kT}{4\pi e^2 \frac{\rho}{m_\mathrm{u}} \zeta}} \ . \tag{6.38}$$

The potential at $r \ll R_\mathrm{D}$ is

$$V_i \approx \frac{Z_i e}{r} - \frac{Z_i e}{R_\mathrm{D}} \ . \tag{6.39}$$

The effect of shielding is that as if there is equal amount of charge with opposite sign surrounding a charge at the Debye-Hückel radius. Near the charge the potential is modified by $-Q/R_\mathrm{D}$, where $Q = Z_i e$ is the charge of the particle. This is the potential experienced by charged particles of charge Q due to the contribution from the whole plasma.

A free electron in a plasma, therefore, is actually moving at a potential of $+e/R_\mathrm{D}$, instead of zero potential as usually set. This results in a **continuum depression** of energy $-e^2/R_\mathrm{D}$ for free electrons. Furthermore, the potential of an electron bound to a charge Ze is shifted by $-(Z-1)e/R_\mathrm{D}$ (the charge of the whole bound system is $(Z-1)e$, and note that the Z here is not the atomic number, but the charge number of the ion in question), and the corresponding energy level is shifted by $(Z-1)e^2/R_\mathrm{D}$. The ionization energy is then modified to be an **effective ionization energy** E':

$$E' = E + (-\frac{e^2}{R_\mathrm{D}}) - (\frac{(Z-1)e^2}{R_\mathrm{D}}) = E - \frac{Ze^2}{R_\mathrm{D}} \ , \tag{6.40}$$

where E is the original ionization energy. Besides, the combined effect of such a continuum depression and energy level up-shift, in addition to resulting in an effective ionization energy, also destroys higher energy states of an atom (or an ion). The problem of infinite summation in partition functions in the Saha equation therefore no longer exists.

Since we are considering interactions between particles, the gas is not assumed to be perfectly ideal. The deviation from an ideal gas can be expressed as

$$U = \frac{3}{2} NkT + U_\mathrm{C} \tag{6.41}$$

and

$$P = \frac{N}{V} kT + P_\mathrm{C} \ , \tag{6.42}$$

where U_C and P_C are those due to the Coulomb interaction. It can be shown that $P_\mathrm{C} = \frac{1}{3}\frac{U_\mathrm{C}}{V}$ (e.g. Clayton (1983), page 143, which is somewhat lengthy) and therefore

$$\begin{aligned} P_\mathrm{C} &= \frac{1}{3} \times \frac{1}{2} \sum_Z \bar{n}_Z Z e \frac{(-Ze)}{R_\mathrm{D}} \\ &= -\frac{1}{6} \frac{e^2}{R_\mathrm{D}} \sum_Z Z^2 \bar{n}_Z \ . \end{aligned} \tag{6.43}$$

The pressure of such a gas is modified (the magnitude is actually reduced) by this amount, which is a very small correction, otherwise the Debye-Hückel model will not be valid. This is the case of a nearly ideal gas. For systems with stronger internal interaction, other methods must be invoked.

6.3.2 Weak screening

Recall that the nuclear reaction cross section is energy dependent. For the nuclear reaction rate per pair with electron screening, we may consider an effective (kinetic-equivalent) energy $E_\mathrm{s} = E - U_\mathrm{s}$ in computing the cross section, where U_s is the potential energy of the incident particle due to the shielding potential in the vicinity of the target particle. We note that $U_\mathrm{s} = -Z_1 Z_2 e^2 / R_\mathrm{D} < 0$, where $Z_1 e$ and $Z_2 e$ are the charge of the two interacting nuclei. Now we have, with Eq.(6.18),

$$\begin{aligned} \langle \sigma v \rangle_\mathrm{s} &= \int \sigma(E_\mathrm{s}) v(E) \phi(E) \, \mathrm{d}E \\ &\propto \int \sigma(E_\mathrm{s}) E e^{-\frac{E}{kT}} \, \mathrm{d}E \\ &= \int_{-U_\mathrm{s}}^{\infty} \sigma(E_\mathrm{s})(E_\mathrm{s} + U_\mathrm{s}) e^{-\frac{E_\mathrm{s}+U_\mathrm{s}}{kT}} \, \mathrm{d}E_\mathrm{s} \\ &\approx e^{-\frac{U_\mathrm{s}}{kT}} \int_0^{\infty} \sigma(E_\mathrm{s}) E_\mathrm{s} e^{-\frac{E_\mathrm{s}}{kT}} \, \mathrm{d}E_\mathrm{s} \ , \end{aligned} \tag{6.44}$$

Table 6.2: Some values of the atomic mass excess. (Clayton 1983, page 289)

Z	element	A	$\triangle Mc^2$ (MeV)
0	n	1	8.07144
1	H	1	7.28899
1	D	2	13.13591
2	He	3	14.93134
2	He	4	2.42475
4	Be	8	4.94420
6	C	12	0
8	O	16	-4.73655
10	Ne	20	-7.04150
12	Mg	24	-13.93330

where in the last approximation we have considered $\sigma \approx 0$ for $E_\mathrm{s} \leq -U_\mathrm{s}$ and also $-U_\mathrm{s} \ll E_0$, the most effective energy, so that the U_s term in the integrant will make only a very small contribution and therefore can be dropped. Finally we have

$$\langle \sigma v \rangle_\mathrm{s} \approx e^{-\frac{U_\mathrm{s}}{kT}} \langle \sigma v \rangle \approx (1 - \frac{U_\mathrm{s}}{kT}) \langle \sigma v \rangle \ , \tag{6.45}$$

where the shielding exponent is

$$-\frac{U_\mathrm{s}}{kT} = \frac{Z_1 Z_2 e^2}{R_\mathrm{D} kT} = 0.188\, Z_1 Z_2 \rho^{\frac{1}{2}} \zeta^{\frac{1}{2}} T_6^{-\frac{3}{2}} \ . \tag{6.46}$$

This is of course only valid when it is much less than unity. For $Z_1 = Z_2 = 1$, $\rho = 150$ g/cm^3, $T_6 = 15$, $X = 0.75$, and $Y = 0.25$, the above expression gives a number of about 0.06. The nuclear reaction rate is increased by electron screening.

6.4 Major nuclear reactions in stars

6.4.1 The proton-proton chain and the CNO cycle

In main sequence stars, the major nuclear reaction is the so-called ‘hydrogen burning’, that is, four protons making an α particle. It is accomplished

Table 6.3: The proton-proton chain (Clayton 1983, page 380; Bahcall 1989, page 67). Q is the total Q value of the reaction, which includes $\langle Q_\nu \rangle$, the average Q carried by neutrinos.

reaction	Q (MeV)	$\langle Q_\nu \rangle$ (MeV)
(pp-I chain)		
$\mathrm{p} + \mathrm{p} \rightarrow \mathrm{D} + \mathrm{e}^+ + \nu$	1.442	0.263
$\mathrm{D} + \mathrm{p} \rightarrow \mathrm{He}^3 + \gamma$	5.494	
$\mathrm{He}^3 + \mathrm{He}^3 \rightarrow \mathrm{He}^4 + 2\ \mathrm{p}$	12.859	
(pp-II chain)		
$\mathrm{He}^3 + \mathrm{He}^4 \rightarrow \mathrm{Be}^7 + \gamma$	1.586	
$\mathrm{Be}^7 + \mathrm{e}^- \rightarrow \mathrm{Li}^7 + \nu$	0.862	0.862
$\mathrm{Li}^7 + \mathrm{p} \rightarrow 2\ \mathrm{He}^4$	17.347	
(pp-III chain)		
$\mathrm{Be}^7 + \mathrm{p} \rightarrow \mathrm{B}^8 + \gamma$	0.135	
$\mathrm{B}^8 \rightarrow \mathrm{Be}^8 + \mathrm{e}^+ + \nu$	17.979	6.710
$\mathrm{Be}^8 \rightarrow 2\ \mathrm{He}^4$	0.095	
(pep) $\mathrm{p} + \mathrm{e}^- + \mathrm{p} \rightarrow \mathrm{D} + \nu$	1.442	1.442
(hep) $\mathrm{He}^3 + \mathrm{p} \rightarrow \mathrm{He}^4 + \mathrm{e}^+ + \nu$	19.795	9.625

through various channels of reactions. In the proton-proton chain, as listed in Table 6.3, most α particles are produced through pp-I chain (about 85%) for the case of the Sun, and about 15% via pp-II chain. The reaction rates depend on the cross sections, compositions, density and temperature. The proton lifetime regarding the first reaction in the proton-proton chain is about 10^{10} years for the Sun. This is the key time scale which affects the evolution of the Sun.

The CNO cycle, listed in Table 6.4, also plays an important role in converting four protons into an α particle. For the Sun, the pp chian makes about 90% contribution and the CNO cycle about 10%. The CNO cycle is more important for upper main sequence stars because of the higher Coulomb barrier.

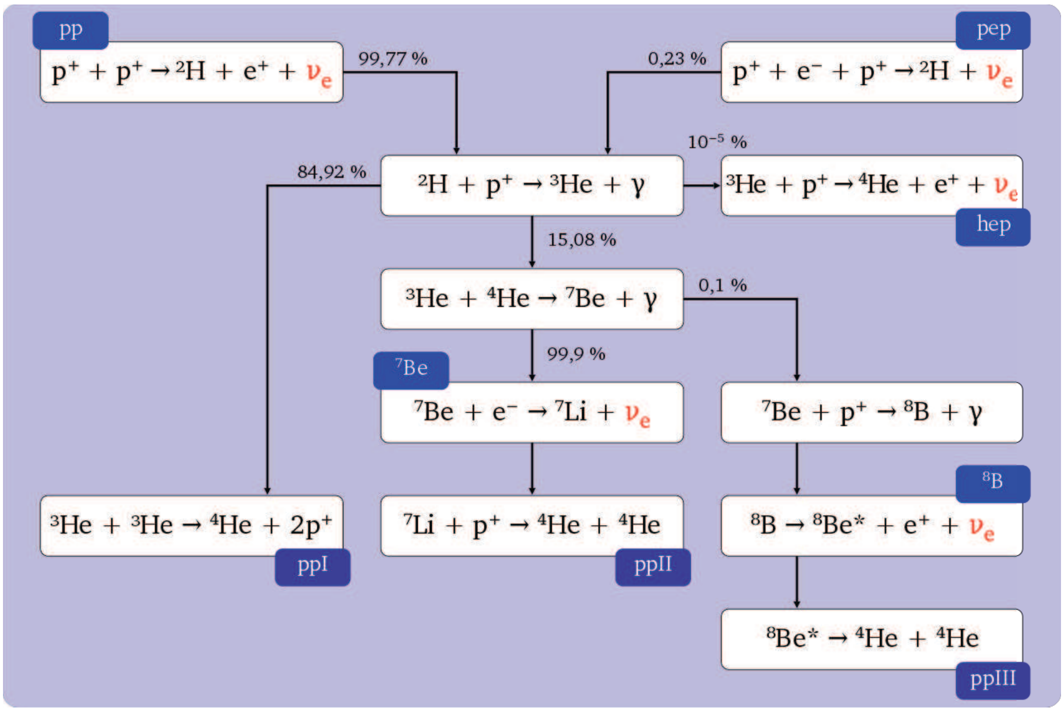

Figure 6.1: The proton-proton chain and its branching ratios for the current sun.

6.4.2 Helium burning

When the hydrogen fuel at the center of a star is exhausted, a helium core forms, which contracts gravitationally. This is the red giant stage. When the temperature at the center increases to about 10^8 K, the helium burning via the so-called **triple-α process** to make heavier elements becomes possible. Roughly speaking, only stars with a main-sequence mass larger than about 0.4 $M_\odot$ can reach this point.

The triple-α process is

$\mathrm{He}^4 + \mathrm{He}^4 \rightarrow \mathrm{Be}^8$

followed almost immediately by

$\mathrm{Be}^8 + \mathrm{He}^4 \rightarrow \mathrm{C}^{12} + \gamma.$

Table 6.4: The CNO cycle (Clayton 1983, page 392; Bahcall 1989, page 72)

reaction	Q (MeV)	$\langle Q_\nu \rangle$ (MeV)
$C^{12} + p \rightarrow N^{13} + \gamma$	1.944	
$N^{13} \rightarrow C^{13} + e^+ + \nu$	2.221	0.7067
$C^{13} + p \rightarrow N^{14} + \gamma$	7.550	
$N^{14} + p \rightarrow O^{15} + \gamma$	7.293	
$O^{15} \rightarrow N^{15} + e^+ + \nu$	2.760	0.9965
$N^{15} + p \rightarrow C^{12} + He^4$	4.965	
or ($\sim 4 \times 10^{-4}$)		
$N^{15} + p \rightarrow O^{16} + \gamma$	12.126	
$O^{16} + p \rightarrow F^{17} + \gamma$	0.601	
$F^{17} \rightarrow O^{17} + e^+ + \nu$	2.762	0.9994
$O^{17} + p \rightarrow N^{14} + He^4$	1.193	

Since Be^8 is unstable (lifetime about 2.6×10^{-16} sec against disintegration into two α particles; the atomic mass excess of Be^8 is 4.94420 MeV), a third α particle must be immediately available to ensure the formation of C^{12}. The total energy output of the triple-α process is 7.274 MeV.

In this stage, the reaction of α-capture

$C^{12} + He^4 \rightarrow O^{16} + \gamma$,

with $Q = 7.161$ MeV, also occurs. Depending on the availability of He^4 and the high temperature to overcome higher Coulomb barrier, reactions like

$O^{16} + He^4 \rightarrow Ne^{20} + \gamma$ ($Q = 4.73$ MeV)

and

$Ne^{20} + He^4 \rightarrow Mg^{24} + \gamma$ ($Q = 9.31$ MeV)

may also occur.

6.4.3 Carbon/Oxygen burning and beyond

After the exhaustion of helium at the stellar center, a C/O core forms. The core again contracts. If the stellar mass is large enough, say, above 4 $M_\odot$, carbon burning may occur when the temperature is about 6×10^8 K. There are multiple channels for carbon burning with different rates:

$$\begin{array}{lcll}
\mathrm{C}^{12}+\mathrm{C}^{12} & \rightarrow & \mathrm{Mg}^{24}+\gamma & Q = 13.930 \text{ MeV} \\
 & \rightarrow & \mathrm{Na}^{23}+\mathrm{p} & Q = 2.238 \text{ MeV} \\
 & \rightarrow & \mathrm{Ne}^{20}+\alpha & Q = 4.616 \text{ MeV} \\
 & \rightarrow & \mathrm{Mg}^{23}+\mathrm{n} & Q = -2.605 \text{ MeV} \\
 & \rightarrow & \mathrm{O}^{16}+2\alpha & Q = -0.114 \text{ MeV.}
\end{array}$$

The production of Mg^{23} requires too much energy and therefore is of a negligible rate. In the mean time, α-capture by C^{12}, O^{16}, Ne^{20}, and Mg^{24} may proceed efficiently. By the end of this **carbon burning** stage, the initial C^{12} nuclei have been converted primarily to O^{16}, Ne^{20}, Na^{23}, Mg^{24}, and Si^{28}.

For stars of a main-sequence mass larger than about 8 $M_\odot$, the core temperature can reach 10^9 K after the continued contraction of the oxygen core. At this high temperature, **photo-disintegration** becomes very important. A major one is

$$\mathrm{Ne}^{20} + \gamma \rightarrow \mathrm{O}^{16} + \alpha.$$

With α-capture by Ne^{20},

$$\mathrm{Ne}^{20} + \alpha \rightarrow \mathrm{Mg}^{24} + \gamma,$$

the net effect is to convert two Ne^{20} into O^{16} and Mg^{24}, plus a net release of 4.583 MeV in γ-rays. This is sometimes referred to as the **neon burning**.

At the same time, **oxygen burning** happens efficiently. The energetically accessible channels are

$$\begin{array}{lcll}
\mathrm{O}^{16}+\mathrm{O}^{16} & \rightarrow & \mathrm{S}^{32}+\gamma & Q = 16.539 \text{ MeV} \\
 & \rightarrow & \mathrm{P}^{31}+\mathrm{p} & Q = 7.676 \text{ MeV} \\
 & \rightarrow & \mathrm{S}^{31}+\mathrm{n} & Q = 1.459 \text{ MeV} \\
 & \rightarrow & \mathrm{Si}^{28}+\alpha & Q = 9.593 \text{ MeV} \\
 & \rightarrow & \mathrm{Mg}^{24}+2\alpha & Q = -0.393 \text{ MeV.}
\end{array}$$

The major product is Si^{28}.

At the end of oxygen burning, the stellar core continues to heat up. Subsequent reactions are basically re-arrangement by capturing a certain particle ejected from a nucleus due to photo-disintegration. The effect of such re-arrangement is to convert nuclear particles to their most stable forms, which are close to Fe^{56}, which has the maximum binding energy per nucleon. The final stage will be an iron core and eventually a supernova event during which many heavy elements are synthesized. There are more details in determining the mass ranges for different nuclear burning stages to happen. The possible existence of neutrino magnetic moments also matters (See, e.g., Heger et al. 2009).

Exercises

1. Show that Eq.(6.11) is indeed the case.
2. What are the Q and $\langle Q_\nu \rangle$ values for a complete pp-I, pp-II, and pp-III chain respectively?
3. Verify Eq.(6.38).
4. Suppose the thermal energy is much larger than the typical Coulomb energy between particles. Demonstrate that $R_{\mathrm{D}} \gg r$, where r is the typical inter-particle distance.

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