非線性震盪中的混沌現象

有時初始條件的微小差異，將造成最終現象的極大改變。前者的小誤差，會造成後者極大的錯誤。預測將成為不可能的事，我們面臨的是偶發現象。

Poincare,《科學方法》，1908 年

目的:

混沌(Chaos)指的是一非週期性、無法預測的現象，能在非線性震盪中產生。系統運動受初始狀態影響很大。有很多原因可以使規律的運動變成混沌，像是驅動頻率、驅動振幅、阻尼大小和初始條件。

原理:

本實驗的裝置如圖 1 所示，包含一個鋁製轉盤連接兩條彈簧，和一放在轉盤邊緣的質點，不均勻的質量使震盪成為非線性。可以改變正弦驅動力的頻率(調整輸入電壓的大小控制驅動頻率)，觀察由可預測運動變為混沌運動的過程。

非線性震盪的混沌運動，主要有四種觀察

(一)、阻尼震盪:

角位移(θ)與時間(t)關係:

週期性震盪為一正弦函數，而在混沌運動時為不規則曲線如下圖 2 所示。

(二)、Potential diagram

位能(U)與角位移關係

利用圓盤邊緣質點的位能與彈簧彈性位能的疊加，產生非線性的位能，質點質量為 m，圓盤外半徑為 R，內半徑為 r，令$\theta = 0$時彈簧伸長量為 0，重力位能為 0，

$$重力位能 = mgR\cos\theta$$

$$彈簧位能 = \frac{1}{2}k(r\theta)^2 + \frac{1}{2}k(r\theta - d)^2 \quad (1)$$

因此得到

$$U = \frac{1}{2}k(r\theta)^2 + \frac{1}{2}k(r\theta - d)^2 + mgR\cos\theta \quad (2)$$

由圖 1 中可看出，在圓盤的左右會各有一個平衡點。

而當小球從靜止釋放時，

$U_0 = \text{const.}$，$T_0 = 0$

且根據能量守恆

$U + T = \text{const.}$

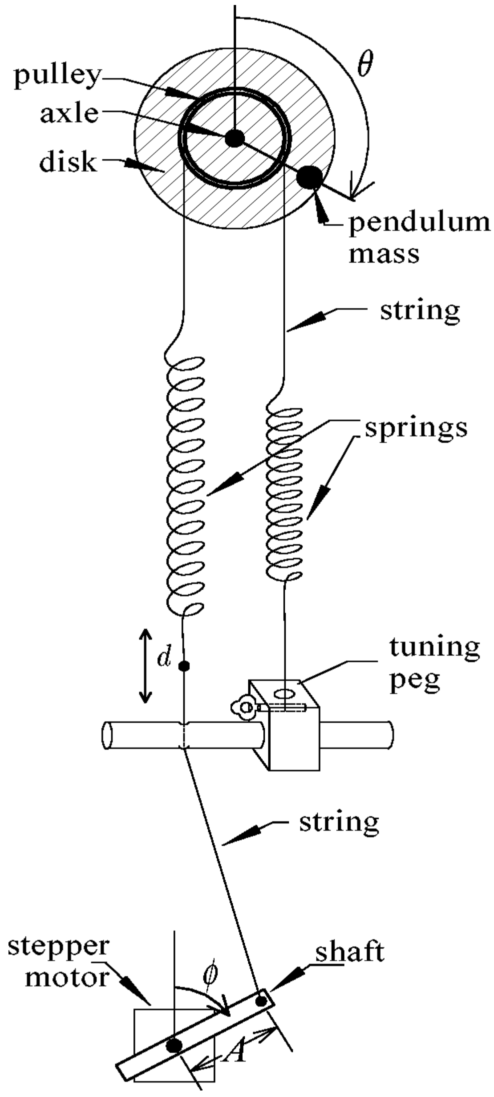

圖 1

整理可得

$U_0 = U + T$

$$U = U_0 - T = U_0 - \frac{1}{2}I\omega^2 \quad (3)$$

因此在實驗時，欲測得 U 與θ的關係圖如圖 3，可將 U 對$-\omega^2$作圖求得。

(三)、相圖(phase diagram):

角速度(ω)與角位移(θ)關係，其週期性運動為一橢圓，混沌運動為一封閉曲線如圖 4 所示。

在描述鐘擺(pendulum)的運動，我們可以寫下運動方程式，我們發現他還有阻尼(damper)，我們用$-b\omega$表示阻力矩隨角速度在變，我們知道力矩是

$$\tau = -\frac{\partial V}{\partial \theta} = I\alpha = I\frac{d\omega}{dt}$$

$$\alpha = \frac{d\omega}{dt} = -\Gamma\omega - \Gamma'\omega - \kappa\theta + \mu\sin\theta + \epsilon\cos\phi$$

其中$d = A\cos\phi$，

$$\epsilon = \frac{Akr}{I}, \mu = \frac{mgL}{I}, \kappa = \frac{2kr^2}{I}, \Gamma = \frac{b}{I}, \Gamma' = \frac{b'}{I},$$

I 是轉動慣量。

現在要在 phase space 畫出軌跡，我們可以由式(3)，

$\omega = \sqrt{\frac{2[U_o - U]}{I}}$ (此時 Uo 為總能量，U 為位能)，然後ω對θ作圖就可以得到 phase diagram 如圖 4 所示。一開始的起始點決定了能量跟接下來的運動軌跡 phase space。如果是有周期性的振盪軌跡的話，phase space 就會是封閉的圖形，而箭頭是其運動方向，但是是非線性的振盪，在一開始的狀態變化很快，在 phase space 就不會是封閉的曲線。

(四)、Poincare plot:

角速度(ω)與角位移(θ)關係，每一個驅動週期記錄一次，在週期運動中 Poincare plot 為一點在有阻泥的週期性運動，其圖形為一個向原點移動的曲線。如圖 5 所示

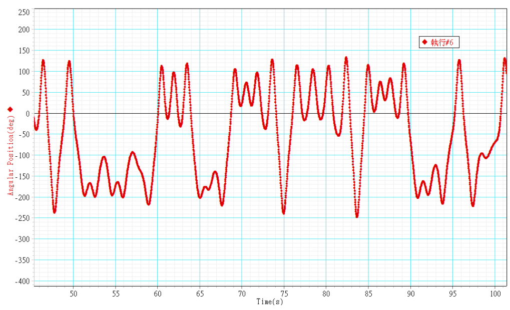

圖 2

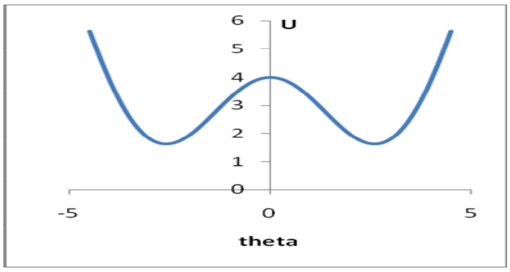

圖 3 Potential diagram

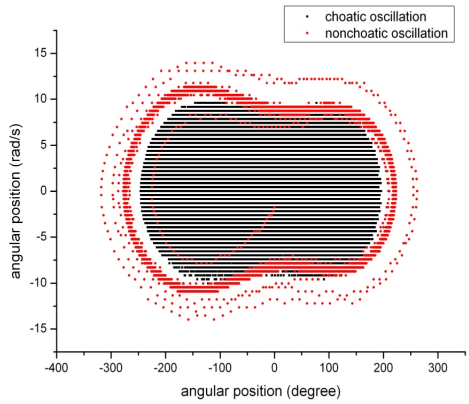

圖 4 phase diagram 相圖

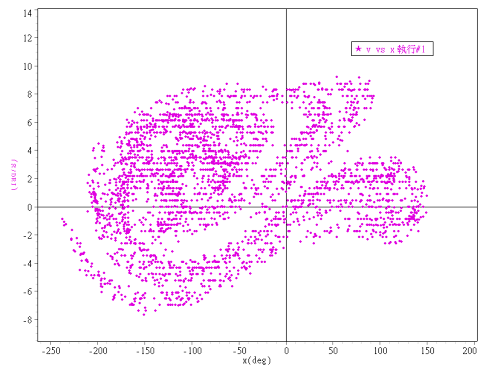

圖 5 Poincare plot

相圖和 Poincare plot 對混沌運動的辨識很有效，因為在混沌運動的圖形中，軌跡不會重複，也就是當時間無限長，圖會被塗黑。

儀器及附件:

非線性震盪裝置一組、鋁盤(有加質點)一個、彈簧兩個、棉線、Rotary Motion Sensor 一個、Photogate Head 一個、DC 電源供應器(SE-9720)一個、Science Workshop 750 Interface 一台

步驟:

(一)、Potential diagram

1.拿下磁阻。點選 DataStudio 的程式 "potential well"之檔案 (若要手動設定的話，先選擇轉動感應器，然後再測量的部分：需先計算-(角速度)^2，在對角位置作圖)。

2. 不開啟電源供應器的情況下，讓質點(point mass)在兩個平衡點之間來回震盪。質點停在一平衡點時，用手將質點轉過上方，繼續轉，超過另一邊平衡點約 90°時先不要放開。(每轉一圈觀察質點的平衡位置，約 2-3 次可找到兩質點平衡位置)。

3. 點選" DataStudio"圖表的"potential well" ，啟動後就可放開質點，記錄一次完整振盪。(由圖表的" potential wel" 觀察及記錄)

4.繪出位能圖($-\omega^2$ 與θ)，並換算成 U 與θ位能圖，注意單位(SI 制)

(二)、阻尼震盪、相圖與共振頻率

1.裝置磁阻。點選 DataStudio 的程式"chaos"之檔案。

2.不開啟電源供應器的情況下，讓質點在兩個平衡點之間來回震盪。質點停在一平衡點時，用手將質點轉過上方，繼續轉，超過另一邊平衡點約 90°先不要放開。

3.點選" DataStudio"圖表後啟動開始記錄之，放手使其震盪，直至停止。

3.檢視角度與時間的關係圖。此震盪是否為正弦曲線？是否有阻尼？
(將角度與時間拖至圖表並觀察及記錄)

4.檢視角速度與角度的相圖(phase diagram)。嘗試指出阻尼如何對相圖產生影響？(由圖表的" phase plot" 觀察及記錄)

5.點選" DataStudio"功能的 FFT, 將角度與時間數據拖至 FFT 由圖形找出共振頻率

(三)、非混沌震盪

***注意由現在開始的實驗(三)、(四)前要給相同的初始條件：**
質點平衡點接近最高點，驅動臂一開始位於最低點。

1.將驅動臂力臂調整約 3.3 cm。確定每次旋轉 driver arm 都有經過光閘(有通過的話光閘會顯示紅色)。

2.點選 DataStudio 的程式"chaos"之檔案。

3.開啟電源並將電壓調至約 1.4-1.5 V 使得系統做**簡單的來回震盪**。

4.幾分鐘之後，點選" DataStudio"圖表後啟動開始記錄之，時間約 3-5 分鐘

5.繪出該非混沌震盪的 phase diagram。

6.逐漸增加驅動頻率改變轉盤運動，其運動不完全是正弦函數(由角度與時間的關係圖觀察)，但仍為週期性運動後，關閉電源，調整初始條件後，開啟電源並重覆步驟 4、5。

(四)、混沌震盪:

1.調整驅始頻率至共振頻率，並調整磁鐵和轉盤的距離，使轉盤的運動變得非常複雜。

2.關閉電源，調回初始條件後，開啟 DataStudio 的程式”chaos”之檔案。

3.開啟電源後，點選” DataStudio”圖表啟動開始記錄之，時間約 1 小時。

4.繪出 Poincare plot 圖及 phase diagram。

問題:

1.在位能圖中有兩個位能井，他們一樣深嗎？為什麼？如果只有一個位能井，對運動會有什麼影響？

2 步驟(三)中.檢視角度對時間的關係圖是否為正弦曲線?週期是?此週期和 driving 週期相等嗎?為什麼圖形會和步驟(二)中的不同?

3.步驟(三)中檢視角速度與角度的關係圖，試說明為何圖形形狀會是如此。此關係圖和步驟(二)中的 phase diagram 有何不同?

參考資料:

Appendix: Chaos Theory

Chaos theory is very much a 20th century development, but the man who probably best deserves the title "Father of Chaos Theory" was a great French mathematician of the 19th century named Henri Poincaré. As discussed on the Dynamical Systems page, Isaac Newton had given the world what seemed to be the final word on how the solar system worked. But Poincaré made the observation that Newton's beautiful model was posited on the basis of the interaction between just two bodies. That is all Newton's differential equations allow. It was natural for anyone with an inquiring mind to wonder what would happen if three or more bodies were allowed in the model. In fact, the question became so famous that a prize was offered for its solution and it was given a name -- "The Three Body Problem." Poincaré, being one of the preeminent mathematicians of the time, tried his hand at solving the Three Body Problem. Ironically, he ended up winning the prize by writing a paper showing that he could not solve it. The problem was that, while Newton's differential equations for two bodies have nice clean "closed form" solutions, the equations for three bodies do not. They must be "solved" by approximate numerical techniques, which effectively change the modeling process from continuous to discrete. The two-body solution gives analytic confirmation of the great Johannes Kepler's empirically derived laws of planetary motion. Poincare found that the numerical "solution" of the three-body problem revealed orbits "so tangled that I cannot even begin to draw them." In addition, Poincaré discovered a very disturbing fact: when the three bodies were started from slightly different initial positions, the orbits would trace out drastically different paths. He wrote, **"It may happen that small differences in the initial positions may lead to enormous differences in the final phenomena. Prediction becomes impossible."** This is the statement which gives Poincare the claim to the title "Father of Chaos Theory." This is the first known published statement of the property now known as "sensitivity to initial conditions", which is one of the defining properties of a chaotic dynamical system.

Poincaré's conclusions about the three-body problem were undeniably correct, but also unwelcome in Newton's perfectly ordered universe. Science is very much about making predictions of future events based upon laws derived from the observation of past events. Deterministic models, so it was thought, must yield perfect or near-perfect predictability. Yet Poincaré's model for three bodies is just an extension of Newton's two-body model and is, therefore, also deterministic. But Poincare says that in the context of this model "prediction becomes impossible" in some instances. From the time of Newton until the time of Poincaré, scientists had experienced too many spectacular successes to simply jettison the comfortable clockwork predictability of Newton's two-body calculus in favor of Poincaré's disturbing uncertainties. Besides they had no palatable way to deal with the numerical drudgery involved in calculating Poincaré's discretely computed orbits. Thus Poincare's monumental discovery of deterministic chaos was destined to be placed on the scientific back burner.